Narrow Systems and the Singular Cardinals Hypothesis

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Two-cardinal tree properties

Def Let $\kappa \le \lambda$ be uncountable cardinals, with κ regular.

- $P_\kappa \lambda := \{x \subseteq \lambda \mid |x| < \kappa\}$
- A (κ, λ)-list is a structure $D = \langle d_x \mid x \in P_\kappa \lambda \rangle$ such that $d_x \subseteq x$ for all $x \in P_\kappa \lambda$
- A (κ, λ)-tree is a structure $T = \langle T_x \mid x \in P_\kappa \lambda \rangle$ such that
  - $\emptyset \ne T_x \subseteq P(x)$
  - $\forall x \subseteq y \in P_\kappa \lambda \ \forall t \in T_y \ t \cap x \in T_x$

Note

- Given a (κ, λ)-list $D = \langle d_x \mid x \in \mathcal{P}_\kappa \lambda \rangle$, its "downward closure" $T = \langle T_x \mid x \in \mathcal{P}_\kappa \lambda \rangle$ is a (κ, λ)-tree, where

$$T_x := \boxed{\{ d_y \cap x \mid x \subseteq y \in \mathcal{P}_\kappa \lambda \}}$$

- Given a (κ, λ)-tree $T = \langle T_x \mid x \in \mathcal{P}_\kappa \lambda \rangle$, one can form a (κ, λ)-list by choosing an element $d_x \in T_x$ for each $x \in \mathcal{P}_\kappa \lambda$.

Def • Given a (κ, λ)-tree $T = \langle T_x \mid x \in P_\kappa \lambda \rangle$, a **cofinal branch** through T is a set $b \subseteq \lambda$ such that

$b \cap x \in T_x$ for all $x \in P_\kappa \lambda$

• Given a (κ, λ)-list $D = \langle d_x \mid x \in P_\kappa \lambda \rangle$, an **ineffable branch** through D is a set $b \subseteq \lambda$ such that

$\{x \in P_\kappa \lambda \mid b \cap x = d_x\}$ is stationary in $P_\kappa \lambda$.

These notions were used by Jech and Magidor in the 1970s to characterize strongly compact and supercompact cardinals.

Thm (Jech '73) Let κ be an uncountable cardinal. TFAE

1) κ is strongly compact
2) $\forall \lambda \geq \kappa$ every (κ, λ)-list has a cofinal branch

Thm (Magidor '74) Let κ be an uncountable cardinal. TFAE

1) κ is supercompact
2) $\forall \lambda \geq \kappa$ every (κ, λ)-list has an ineffable branch

If we want similar tree properties to consistently hold at small cardinals, we must restrict the class of trees/lists under consideration.

Def • A $(\underline{\kappa}, \lambda)$-tree T is **thin** if $\forall x \in P_\kappa \lambda \; |T_x| < \kappa$

• A (κ, λ)-list D is **thin** if its downward closure is thin

• A (κ, λ)-list D is **slender** if, for all large regular θ, there is a club $C \subseteq P_\kappa H(\theta)$ s.t. $\forall M \in C$ $\forall z \in M \cap P_{\omega_1} \lambda \;\; d_{M \cap \lambda} \cap z \in M$

Def (Weiß) Let $\kappa \le \lambda$ be uncountable cardinals, with κ regular.

- $(I)TP(\kappa, \lambda) \equiv$ Every thin (κ, λ)-list has a cofinal (ineffable) branch
- $(I)SP(\kappa, \lambda) \equiv$ Every slender (κ, λ)-list has a cofinal (ineffable) branch
- $(I)TP_\kappa \equiv \forall \lambda \ge \kappa \; (I)TP(\kappa, \lambda)$

$(I)SP_\kappa \equiv \forall \lambda \ge \kappa \; (I)SP(\kappa, \lambda)$

Thm (Weiß)

- $\mathrm{Con}(ZFC + \exists$ supercompact cardinal$)$
  $\Rightarrow \mathrm{Con}(ZFC + ISP_{\omega_2})$
- $\mathrm{Con}(ZFC + \exists$ strongly compact cardinal$)$
  $\Rightarrow \mathrm{Con}(ZFC + TP_{\omega_2})$

$\hookrightarrow TP(\omega_2, \lambda)$ for all $\lambda \geq \omega_2$

Thm (Viale - Weiß)

$PFA \Rightarrow ISP_{\omega_2}$

The Singular Cardinals Hypothesis

In its simplest form, the Singular Cardinals Hypothesis (SCH) says:

> For every singular strong limit μ, $2^\mu = \mu^+$.

Thm (Solovay) If κ is strongly compact, then SCH holds above κ.

Thm (Viale, Krueger) If ISP_κ holds, then SCH holds above κ.

Weiß wrote of the principles (I)SP and (I)TP as capturing the "combinatorial essence" of supercompactness and strong compactness.

This naturally raises the following question:

Q Does ITP_κ (or TP_κ) imply that SCH holds above κ?

Narrow $P_\kappa\lambda$-systems

The classical notion of narrow system was introduced by Magidor and Shelah in their study of the tree property at successors of singular cardinals.

There is a natural generalization to the setting of two-cardinal tree properties.

Def Let $\kappa \leq \lambda$ be uncountable cardinals, with κ regular.

A (concrete) $P_\kappa \lambda$**-system** is a structure $S = \langle S_x \mid x \in Y \rangle$ such that

- Y is a $\subseteq$-cofinal subset of $P_\kappa \lambda$
- $\emptyset \neq S_x \subseteq P(x)$
- $\forall x \subseteq y \in P_\kappa \lambda \; \exists s \in S_y$ s.t. $s \cap x \in S_x$

A **cofinal branch** through S is a set $b \subseteq \lambda$ such that

$\{x \in Y \mid b \cap x \in S_x\}$ is $\subseteq$-cofinal in Y

The **width** of S is

$$\sup\{|S_x| \mid x \in Y\}$$

S is **narrow** if $\text{width}(S)^+ < \kappa$

The **$P_\kappa\lambda$-narrow system property** ($P_\kappa\lambda$-NSP) is the assertion that every narrow $P_\kappa\lambda$-system has a cofinal branch.

$\mathrm{NSP}_\kappa \equiv P_\kappa\lambda\text{-NSP}$ holds for all $\lambda \geq \kappa$

Thm Suppose that μ is a singular limit of strongly compact cardinals. Then TP_{μ^+} holds.

$\mathrm{TP}(\mu^+, \lambda)$ for all $\lambda \geq \mu^+$

Proof Structure

- First, prove that for every $\lambda \geq \mu^+$ and every thin (μ^+, λ)-tree $T = \langle T_x \mid x \in \mathcal{P}_{\mu^+}\lambda \rangle$, there is a cofinal $Y \subseteq \mathcal{P}_{\mu^+}\lambda$ and, for all $x \in Y$, a nonempty $S_x \subseteq T_x$ such that $\langle S_x \mid x \in Y \rangle$ is a narrow $\mathcal{P}_{\mu^+}\lambda$-system.

• Then, prove that NSP_{μ^+} holds. The resulting cofinal branch through S is also a cofinal branch through T.

Comparing TP_κ to NSP_κ

thin (κ, λ)-tree	narrow $P_\kappa \lambda$-system
$T = \langle T_x \mid x \in P_\kappa \lambda \rangle$	$S = \langle S_x \mid x \in Y \rangle$
• $\forall x \subseteq y \ \forall t \in T_y \ t \cap x \in T_x$	• $\forall x \subseteq y \ \exists s \in S_y \ s \cap x \in S_x$
• $\forall x \ \lvert T_x \rvert < \kappa$	• $\exists \mu \ (\mu^+ < \kappa \wedge \forall x \ \lvert S_x \rvert \leq \mu)$

In practice, NSP_κ feels weaker than TP_κ.

- NSP_κ holds in all known models of TP_κ.
- All of the currently conceivable (to me) methods for verifying TP_κ will inevitably yield NSP_κ.

More concretely:

Thm Suppose that there is a proper class of supercompact cardinals. Then there is a (class) forcing extension in which NSP_κ holds for all regular uncountable κ.

Proof Method Cut off the universe below the first inaccessible limit of supercompact cardinals. Take a full-support iteration of Levy collapses turning the former supercompact cardinals into all double successor cardinals.

Nonetheless, the following remains open:

Q Does $\underline{TP_\kappa}$ (or ITP_κ) imply NSP_κ?

Thm $ISP_\kappa \Rightarrow NSP_\kappa$.

Thm If $\kappa \geq \omega_2$ is regular, then NSP_κ implies SCH above κ.

The proof builds on ideas from Viale's proof that PFA implies SCH.

We actually get that NSP_κ implies Shelah's Strong Hypothesis above κ (with the correct formulation), e.g.

Thm $NSP_{\omega_2} \Rightarrow SSH$

Thank you for your attention!

$NSP_\varkappa \Rightarrow \neg\square(\lambda)$

for all $^{\text{regular}}$ $\lambda \geq \varkappa$

$NSP_\varkappa \Rightarrow \nexists$ strongly unbdd subadditive functions

$c : \lambda \to \omega$ for

any $\lambda \geq \kappa$