Generalized Mittag-Leffler conditions and the $\aleph_n$'s

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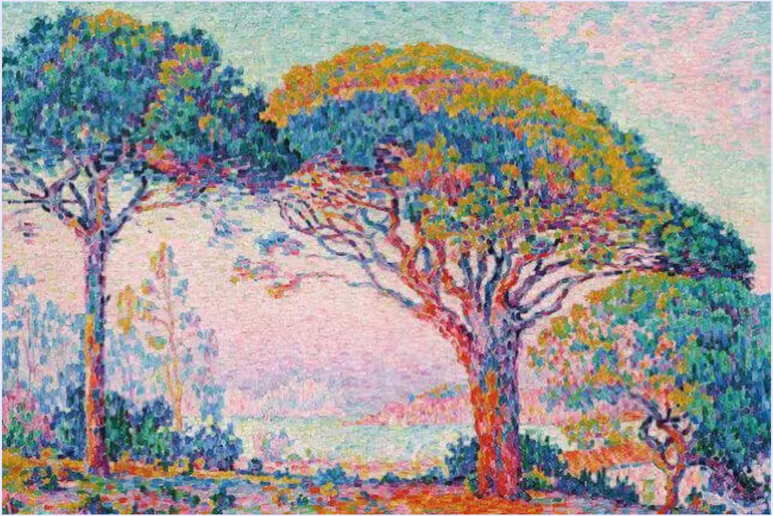
I. Introduction



Motivating question: Suppose that a mathematical object X is represented as a limit of simpler, well-behaved objects $\{X_i \in i \in I\}$. Under what circumstances does X inherit the nice properties of the X_i's?

Motivating thesis: For each $n < \omega$, the cardinal $\aleph_n$ is characterized by its n-dimensional combinatorial behavior.

Inverse systems

If Λ is a directed set, an *inverse system* indexed by Λ is a system

$$\mathbf{X} = \langle X_u, \pi^X_{uv} \mid u \leq v \in \Lambda \rangle$$

such that

- for all $u \leq v \in \Lambda$, $\pi^X_{uv} : X_v \to X_u$; and
- for all $u \leq v \leq w \in \Lambda$, $\pi^X_{uw} = \pi^X_{uv} \circ \pi^X_{vw}$.

Given an inverse system $\mathbf{X}$, we can form the inverse limit, $\lim \mathbf{X}$, together with maps $\pi^X_u : \lim \mathbf{X} \to X_u$ for all $u \in \Lambda$.

Level morphisms

Given two inverse systems $\mathbf{X}$ and $\mathbf{Y}$ indexed by Λ, a *level morphism* $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is a system $\langle f_u : X_u \to Y_u \mid u \in \Lambda \rangle$ such that, for all $u \leq v \in \Lambda$,

$$\pi^Y_{uv} \circ f_v = v_u \circ \pi^X_{uv}.$$

A level morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ induces a map $\lim \mathbf{f} : \lim \mathbf{X} \to \lim \mathbf{Y}$.

Mittag-Leffler's Theorem

Theorem (Mittag-Leffler, '76)

Suppose that $U \subseteq \mathbb{C}$ is open and $A \subseteq U$ is a set with no limit points in U. Then there exists a meromorphic function $f : U \to \mathbb{C}$ whose poles are precisely the points in A.

Proof sketch.

Write $U = \bigcup_{n<\omega} U_n$ as an increasing union of open sets such that $U_n \cap A$ is finite for all n. For each $n < \omega$, fix a set X_n of functions from $\bar{U}_n$ to $\mathbb{C}$ such that, for all $m < n$ and $f \in X_n$:

- f is meromorphic on U_n with poles precisely $U_n \cap A$;
- $f \restriction \bar{U}_m \in X_m$.

Do this in a way such that each X_m is a complete metric space and, for all $m < n < \omega$, $\{f \restriction \bar{U}_m \mid f \in X_n\}$ is dense in X_m.

Lemma (Mittag-Leffler, Bourbaki)

Suppose that $\mathbf{Y} = \langle Y_m, \pi_{mn} : Y_n \to Y_m \mid m \leq n < \omega \rangle$ *is an inverse sequence of nonempty complete metric spaces such that, for all* $m < n < \omega$, $\pi_{mn}[Y_n]$ *is dense in* Y_m. **Then**, *for all* $m < \omega$, $\pi_m[\lim \mathbf{Y}]$ *is dense in* Y_m. *In particular,* $\lim \mathbf{Y}$ *is nonempty.*

Back to proof of Theorem.

Recall that $\mathbf{X} = \langle X_m, \pi_{mn} \mid m \leq n < \omega \rangle$ is an inverse system of complete metric spaces such that each X_m consists of functions on $\bar{U}_m$, meromorphic on U_m, with poles precisely $U_m \cap A$. The maps π_{mn} are restriction maps. Thus, any element of $\lim \mathbf{X}$ would be a meromorphic function on U with poles precisely A. The above lemma implies that $\lim \mathbf{X} \neq \emptyset$. $\square$

Derived limits

Inverse systems of abelian groups

From now on, we consider inverse systems of abelian groups. Given a directed set Λ, let Ab^{Λ} denote the category of inverse systems of abelian groups indexed by Λ. The inverse limit is then a functor from Ab^{Λ} to Ab.

Exactness of lim

The functor $\lim : \mathrm{Ab}^\Lambda \to \mathrm{Ab}$ is *left exact* but not exact, i.e., if

$$\mathbf{0} \to \mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Y} \xrightarrow{\mathbf{g}} \mathbf{Z} \to \mathbf{0}$$

is exact in Ab^Λ, then the induced sequence

$$0 \to \lim \mathbf{X} \xrightarrow{\lim \mathbf{f}} \lim \mathbf{Y} \xrightarrow{\lim \mathbf{g}} \lim \mathbf{Z} \to 0,$$

is exact at $\lim \mathbf{X}$ and $\lim \mathbf{Y}$, but might not be exact at $\lim \mathbf{Z}$. Concretely, this failure of exactness comes from the fact that even if a morphism $\mathbf{g} : \mathbf{Y} \to \mathbf{Z}$ consists of surjective maps, the limit map $\lim \mathbf{g} : \lim \mathbf{Y} \to \lim \mathbf{Z}$ need not be surjective.

Derived limits

Derived limits measure the failure of the inverse limit functor to be exact. For each $0 < n < \omega$, there is a derived functor $\lim^n : \mathsf{Ab}^\Lambda \to \mathsf{Ab}$ such that every short exact sequence

$$\mathbf{0} \to \mathbf{X} \to \mathbf{Y} \to \mathbf{Z} \to \mathbf{0}$$

induces a *long* exact sequence

$$\begin{array}{l} \mathbf{0} \longrightarrow \lim \mathbf{X} \longrightarrow \lim \mathbf{Y} \longrightarrow \lim \mathbf{Z} \longrightarrow \\ \longrightarrow \lim^1 \mathbf{X} \longrightarrow \lim^1 \mathbf{Y} \longrightarrow \lim^1 \mathbf{Z} \longrightarrow \\ \longrightarrow \lim^2 \mathbf{X} \longrightarrow \lim^2 \mathbf{Y} \longrightarrow \lim^2 \mathbf{Z} \longrightarrow \dots \end{array}$$

Vanishing derived limits

For each $0 < n < \omega$, $\lim^n$ can be thought of as measuring n-dimensional obstructions to the exactness of lim.

Instances of $\lim^n \mathbf{X} = 0$ yield positive instances of our motivating question. For example, for $\mathbf{X} \in \mathrm{Ab}^\Lambda$, if $\lim^1 \mathbf{X} = 0$, **then**, whenever $\mathbf{g} : \mathbf{Y} \to \mathbf{Z}$ is a surjective level morphism in Ab^Λ with $\ker(\mathbf{g}) = \mathbf{X}$, the induced map $\lim \mathbf{g} : \lim \mathbf{Y} \to \lim \mathbf{Z}$ is surjective.

Goblot's Theorem

It turns out that higher derived limits provably vanish on inverse systems indexed by directed sets of small cofinality.

Theorem (Goblot)

Suppose that $n < \omega$ and Λ is a directed set with $\mathrm{cf}(\Lambda) \leq \aleph_n$. Then $\lim^{n+2} \mathbf{X} = 0$ for all $\mathbf{X} \in \mathrm{Ab}^{\Lambda}$.

This can be slightly improved if we put some additional requirement on the inverse systems under consideration.

The Mittag-Leffler condition

Definition (Bourbaki)

Suppose that $\mathbf{X} = \langle X_u, \pi^X_{uv} \mid u \leq v \in \Lambda \rangle$ is an inverse system of abelian groups. We say that $\mathbf{X}$ satisfies the *Mittag-Leffler condition* if, for all $u \in \Lambda$, there exists $v_u \geq u$ such that, for all $v \geq v$, $\pi_{uv}[X_v] = \pi_{uv_u}[X_{v_u}]$.

Fact (Grothendieck)

Suppose that $\mathbf{X} \in \mathrm{Ab}^\omega$ *is an inverse sequence of abelian groups satisfying the Mittag-Leffler condition. Then* $\lim^1 \mathbf{X} = 0$ *(and hence, by Goblot's theorem,* $\lim^n \mathbf{X} = 0$ *for all* $0 < n < \omega$*).*

Characterizing Mittag-Leffler

The Mittag-Leffler condition does not *characterize* vanishing of the first derived limit: there are inverse sequences of abelian groups that fail to have the ML-condition but nonetheless have a vanishing first derived limit. However, we do have the following:

Theorem (Emmanouil, '96)

Suppose that $\mathbf{X} \in \mathsf{Ab}^\omega$ *is an inverse sequence of abelian groups. The following are equivalent:*

1. $\mathbf{X}$ *has the Mittag-Leffler condition.*
2. $\lim^1 \bigoplus_\omega \mathbf{X} = 0$.
3. $\lim^1 \bigoplus_I \mathbf{X} = 0$ *for every index set* I.

Higher cofinalities

For $n > 0$, if $\mathrm{cf}(\Lambda) = \aleph_n$ and $\mathbf{X} \in \mathsf{Ab}^\Lambda$ has the Mittag-Leffler condition, then $\lim^{n+1} \mathbf{X} = 0$. However, we no longer have a characterization *à la* Emmanouil unless Λ has a linearly ordered cofinal subset.

Theorem (LH – Momblona Rodríguez)

Suppose that $0 < n < \omega$ and $\mathbf{X} \in \mathsf{Ab}^{\omega_n}$ is an inverse system of abelian groups on ω_n. The following are equivalent:

1. *$\mathbf{X}$ has the Mittag-Leffler condition.*
2. *$\lim^{n+1} \bigoplus_{\omega_n} \mathbf{X} = 0$.*
3. *$\lim^{n+1} \bigoplus_I \mathbf{X} = 0$ for every index set I.*

Higher generalizations

Let $n > 0$ and let Λ be a directed set with $\mathrm{cf}(\Lambda) = \aleph_n$. Fix a smooth filtration $\langle \Lambda_\alpha \mid \alpha < \omega_n \rangle$ of Λ such that each Λ_α is a directed set of cofinality $\aleph_{n-1}$. Fix $\mathbf{X} \in \mathsf{Ab}^\Lambda$. This induces systems $(\mathbf{X} \restriction \Lambda_\alpha)$ for $\alpha < \omega_n$. Moreover, for $\alpha < \beta < \omega_n$, we get projection maps

$$\lim^k(\mathbf{X} \restriction \Lambda_\beta) \to \lim^k(\mathbf{X} \restriction \Lambda_\alpha)$$

for all $k < \omega$.

Definition

We say that $\mathbf{X}$ satisfies ML_n if there is a club $C \subseteq \omega_n$ such that, for all $\alpha \in C$ and all $\beta \in (\alpha, \omega_n)$, the projection map

$$\lim^n(\bigoplus_{\omega_{n-1}} \mathbf{X} \restriction \Lambda_\beta) \to \lim^n(\bigoplus_{\omega_{n-1}} \mathbf{X} \restriction \Lambda_\alpha)$$

is surjective.

A characterization

Theorem (LH – Momblona Rodríguez)

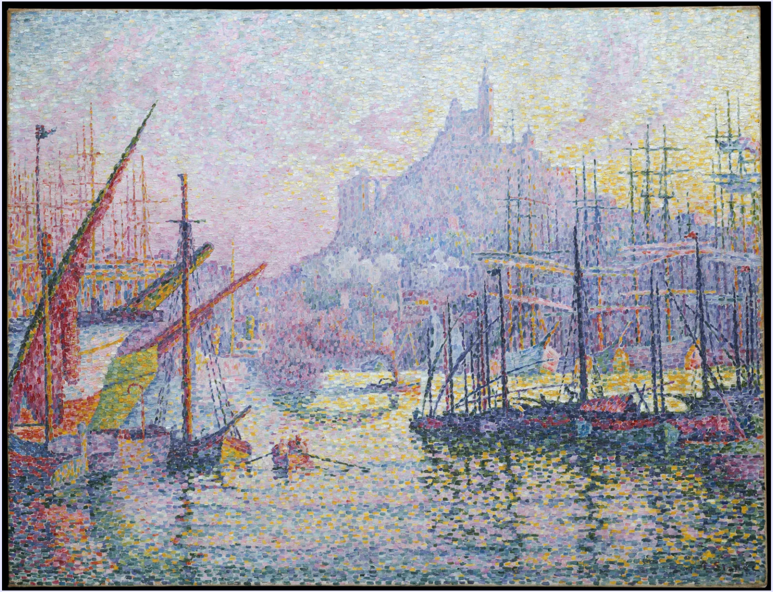
Suppose that $0 < n < \omega$, Λ *is a directed set with* $\mathrm{cf}(\Lambda) = \aleph_n$, *and* $\mathbf{X} \in \mathsf{Ab}^\Lambda$. *Then the following are equivalent:*

1. $\mathbf{X}$ *satisfies* ML_n;
2. $\lim^{n+1} \bigoplus_{\omega_n} \mathbf{X} = \emptyset$;
3. $\lim^{n+1} \bigoplus_I \mathbf{X} = \emptyset$ *for every* I.

Question

Are the above conditions equivalent to $\lim^{n+1} \bigoplus_\omega \mathbf{X} = \emptyset$?

An application



An inverse system

Fix a field K and a cardinal κ, and consider the polynomial ring $R = K[\{x_\alpha \mid \alpha < \kappa\}]$ over K with κ-many variables. Let Λ be the set of all monic monomials in R, ordered by divisibility, e.g.,

$$x_\alpha x_\beta < x_\alpha^2 x_\beta < x_\alpha^2 x_\beta^3 x_\gamma.$$

For $u \in \Lambda$, let $\langle u \rangle$ denote the ideal in R generated by u. Note that $u \leq v \Rightarrow \langle u \rangle \supseteq \langle v \rangle$. We can therefore define an inverse system $\mathbf{Z}$ over Λ by letting

$$Z_u = \frac{K[\{x_\alpha \mid \alpha < \kappa\}]}{\langle u \rangle}$$

and, if $u < v$, letting $\pi^Z_{uv} : Z_v \to Z_u$ be the quotient map.

Question (Alvite Pazó)

For $0 < n < \omega$, *what is* $\lim^n \mathbf{Z}$*?*

A short exact sequence

For every $u \in \Lambda$,

$$0 \to \langle u \rangle \to K[\{x_\alpha \mid \alpha < \kappa\}] \to \frac{K[\{x_\alpha \mid \alpha < \kappa\}]}{\langle u \rangle} \to 0$$

is exact. Define a system $\mathbf{X}$ on Λ by letting $X_u = \langle u \rangle$ and letting π^X_{uv} be the inclusion map for all $u \leq v$ in Λ. Let $\mathbf{Y}$ be the constant system on Λ with $Y_u = K[\{x_\alpha \mid \alpha < \kappa\}]$ for all $u \in \Lambda$. Then

$$\mathbf{0} \to \mathbf{X} \to \mathbf{Y} \to \mathbf{Z} \to \mathbf{0}$$

is exact in Ab^Λ. Moreover, we have $\lim^n \mathbf{Y} = 0$ for all $0 < n < \omega$.

A long exact sequence

Applying lim to the short exact sequence

$$\mathbf{0} \to \mathbf{X} \to \mathbf{Y} \to \mathbf{Z} \to \mathbf{0}$$

yields

$$\mathbf{0} \longrightarrow \lim \mathbf{X} \longrightarrow \lim \mathbf{Y} \longrightarrow \lim \mathbf{Z} \longrightarrow$$

$$\longrightarrow \lim^1 \mathbf{X} \longrightarrow 0 \longrightarrow \lim^1 \mathbf{Z} \longrightarrow$$

$$\longrightarrow \lim^2 \mathbf{X} \longrightarrow 0 \longrightarrow \lim^2 \mathbf{Z} \longrightarrow \dots$$

It follows that $\lim^n \mathbf{Z} = \lim^{n+1} \mathbf{X}$ for all $0 < n < \omega$.

An answer

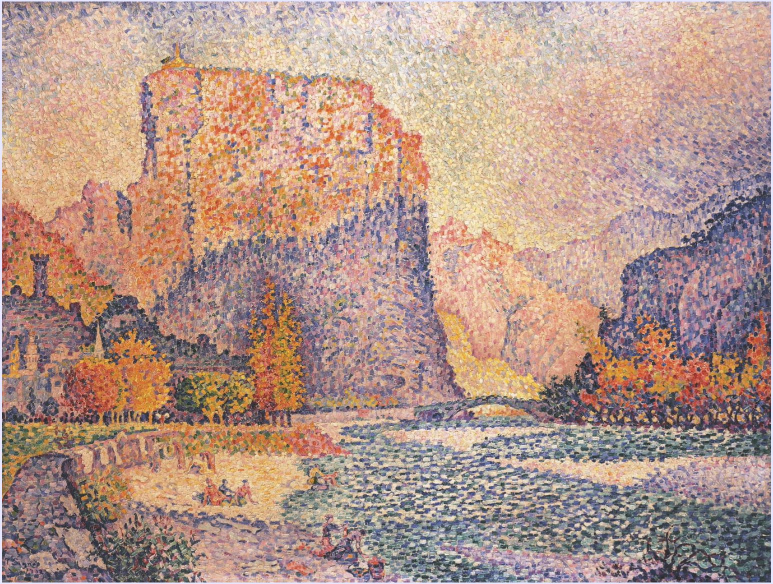
If $\kappa = \aleph_n$ for some $0 < n < \omega$, then $\mathrm{cf}(\Lambda) = \aleph_n$ and $\mathbf{X}$ fails to satisfy ML_n. This can then be used to argue that $\lim^{n+1} \mathbf{X} \neq 0$, and hence $\lim^n \mathbf{Z} \neq 0$. Somewhat surprisingly, this is an equivalence.

Theorem (LH – Momblona Rodríguez)

For each $0 < n < \omega$, we have $\lim^n \mathbf{Z} \neq 0$ if and only if $\kappa = \aleph_n$.

This provides a particularly pure instance of this talk's motivating thesis: nontrivial n-dimensional combinatorics shows up in the derived limits of this system precisely when there are exactly $\aleph_n$-many variables.

Thank you!



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