Generalized almost disjoint families and injective Banach spaces

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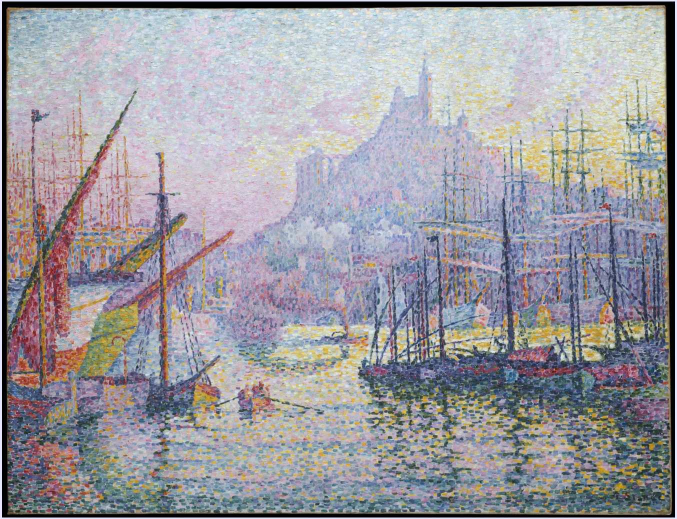
Joint work with David Schrittesser.

I. Generalized almost disjoint families



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Almost disjoint families are central objects in the study of cardinal characteristics of the continuum but have also found many applications in other fields, especially general topology and functional analysis. In search of more applications, today we will try to generalize the notion of *almost disjoint family*.

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We now introduce the notion of an *almost disjoint family* on an arbitrary topological space X. As the definition will make clear, this seems to be of most interest for compact Hausdorff spaces X that are totally disconnected but not extremally disconnected.

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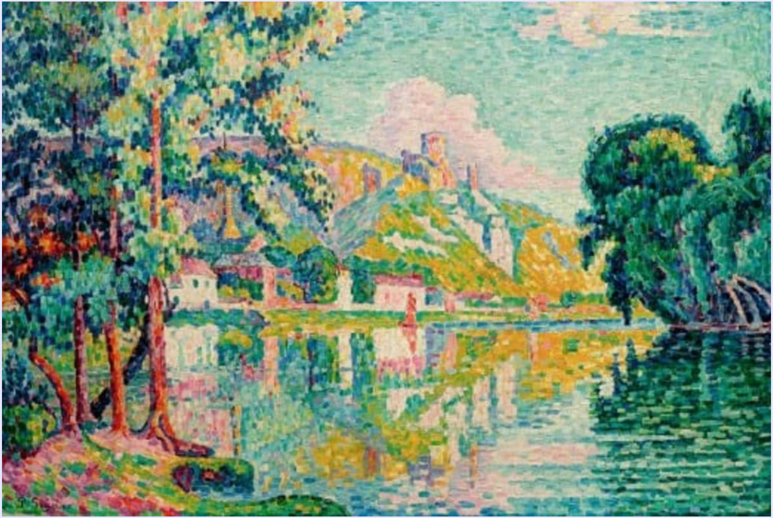
Classical a.d. families in $[\omega]^\omega$ are precisely (nontrivial) almost disjoint families on the space $\omega + 1$.

II. Injective Banach spaces



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or ∞ if no such i exists.

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$$\sigma\left(\sum_{A \in \mathcal{B}} [1_A] \right)(m) > 0.99 |\mathcal{B}|.$$

Thus, σ takes elements of the unit ball of ℓ_∞ / c_0 to elements of ℓ_∞ of arbitrarily high norm, contradicting the fact that σ is continuous (and hence bounded). $\square$

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We can do better, using the following theorem of Rosenthal.

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If E is an injective Banach space, Γ is an infinite set, and E contains a subspace isomorphic to $c_0(\Gamma)$, then it contains a subspace isomorphic to $\ell_\infty(\Gamma)$.

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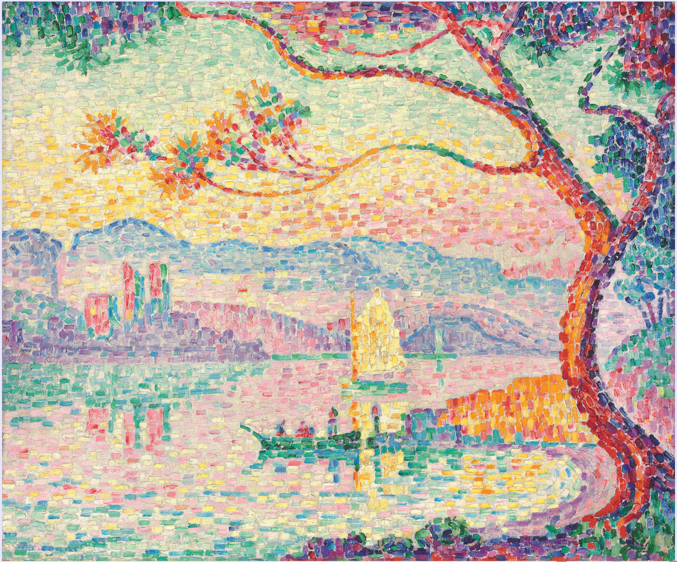
III. Function spaces and generalized a.d. familes



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If D is a dense subset of K, then $C(K)$ embeds as a closed subspace of $\ell^\infty(D)$. This leads us to be interested in quotient spaces of the form

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3. for all $b \in [\omega]^\omega$ and $f \in {}^{\omega_1}2$, either
    1. there exists $\eta < \omega_1$ such that $b \subseteq^* a_{f \restriction \eta}$; or

Recall that $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ is the Čech-Stone remainder of $\mathbb{N}$. Concretely, this is the space of nonprincipal ultrafilters on ω.

Theorem (LH-Schrittesser)

If CH *holds (or just $\mathfrak{b} = \mathfrak{c}$), then there is an almost disjoint family on $\mathbb{N}^*$ of cardinality $2^{\aleph_1}$.*

Proof sketch.

Build a sequence $\langle a_x \mid x \in {}^{<\omega_1}2\rangle$ of elements of $[\omega]^\omega$ such that

1. for all $x \subsetneq y \in {}^{<\omega_1}2$, $a_x \subsetneq^* a_y$;
2. for all incompatible $x, y \in {}^{<\omega_1}2$, $a_x \cap a_y =^* a_{x \wedge y}$;
3. for all $b \in [\omega]^\omega$ and $f \in {}^{\omega_1}2$, either
    1. there exists $\eta < \omega_1$ such that $b \subseteq^* a_{f\restriction\eta}$; or
    2. there is an infinite $b' \subseteq b$ such that, for all $\eta < \omega_1$, the set $b' \cap a_{f\restriction\eta}$ is finite.

Proof sketch (cont.)

Now every branch $f \in {}^{\omega_1}2$ through ${}^{<\omega_1}2$ determines a $\subsetneq^*$-increasing sequence $\langle a_{f \restriction \eta} \mid \eta < \omega_1 \rangle$ of elements of $[\omega]^\omega$.

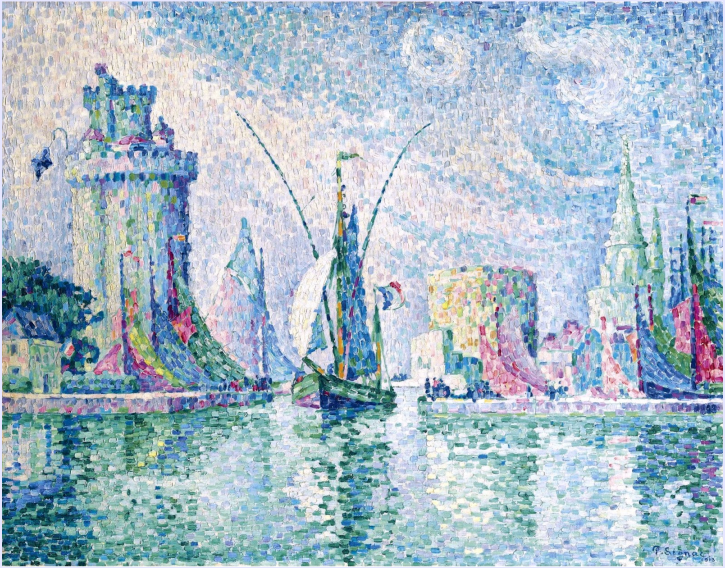
Proof sketch (cont.)

Now every branch $f \in {}^{\omega_1}2$ through ${}^{<\omega_1}2$ determines a $\subsetneq^*$-increasing sequence $\langle a_{f\restriction\eta} \mid \eta < \omega_1\rangle$ of elements of $[\omega]^\omega$. Let A_f be the collection of all $\mathcal{U} \in \mathbb{N}^*$ for which there exists $\eta < \omega_1$ such that $a_{f\restriction\eta} \in \mathcal{U}$.

Proof sketch (cont.)

Now every branch $f \in {}^{\omega_1}2$ through ${}^{<\omega_1}2$ determines a $\subsetneq^*$-increasing sequence $\langle a_{f\restriction\eta} \mid \eta < \omega_1 \rangle$ of elements of $[\omega]^\omega$. Let A_f be the collection of all $\mathcal{U} \in \mathbb{N}^*$ for which there exists $\eta < \omega_1$ such that $a_{f\restriction\eta} \in \mathcal{U}$. Then $\{A_f \mid f \in {}^{\omega_1}2\}$ is an almost disjoint family on $\mathbb{N}^*$. $\square$

IV. Back to c_0



Back to c_0

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ℓ_∞ / c_0 is isomorphic to $C(\mathbb{N}^*)$.

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Also, $\mathbb{N}^*$ has a dense subset D of cardinality $2^{\aleph_0}$. Therefore, c_0 has an injective resolution beginning

$$0 \to c_0 \xrightarrow{\iota_0} \ell_\infty \xrightarrow{\iota_1} \ell_\infty(2^{\aleph_0}) \to \cdots$$

such that

$$\ell_\infty(2^{\aleph_0})/\mathrm{im}(\iota_1) \cong \ell_\infty(D)/C(\mathbb{N}^*).$$

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such that

$$\ell_\infty(2^{\aleph_0})/\mathrm{im}(\iota_1) \cong \ell_\infty(D)/C(\mathbb{N}^*).$$

Thus, if $\ell_\infty(D)/C(\mathbb{N}^*)$ is not injective, then the injective dimension of c_0 is at least 3.

Theorem

If CH *holds, then the injective dimension of* c_0 *is at least 3.*

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If CH holds, then $\mathbb{N}^*$ contains a generalized AD family of size $2^{\aleph_1}$. Thus, $\ell_\infty(D)/C(\mathbb{N}^*)$ contains a copy of $c_0(2^{\aleph_1})$.

Theorem

If CH *holds, then the injective dimension of c_0 is at least 3.*

Proof.

If CH holds, then $\mathbb{N}^*$ contains a generalized AD family of size $2^{\aleph_1}$. Thus, $\ell_\infty(D)/C(\mathbb{N}^*)$ contains a copy of $c_0(2^{\aleph_1})$. If $\ell_\infty(D)/C(\mathbb{N}^*)$ were injective, then it would contain a copy of $\ell_\infty(2^{\aleph_1})$, but it is too small for this, since

$$|\ell_\infty(2^{\aleph_1})| = 2^{2^{\aleph_1}}$$

and

$$|\ell_\infty(D)/C(\mathbb{N}^*)| = 2^{2^{\aleph_0}} = 2^{\aleph_1}.$$

$\square$

Questions

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Question

Is the injective dimension of c_0 infinite?

Thank you!