Set theory and derived functors

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Prologue: Compactness

Compactness

Ubiquitous throughout mathematics are questions of *compactness*. These often take the following (vague) form:

> *To what extent are a structure's global properties determined by its local properties?*

Or, slightly more concretely:

> *Given a structure M, if all "small" substructures of M satisfy property P, must M also satisfy property P?*

Example: Given an abelian group G and a cardinal κ, we say that G is *κ-free* if every subgroup of G of cardinality $<\kappa$ is free. G is *almost free* if it is $|G|$-free. For a fixed uncountable cardinal κ, under what circumstances must all almost free abelian groups of cardinality κ be free? Does there exist a cardinal κ such that all κ-free abelian groups are free?

$\aleph_0$ and compactness

Many seminal theorems of the first half of the twentieth century can be seen as asserting that $\aleph_0$, the cardinality of the set of natural numbers, is highly compact. For example:

- Compactness Theorem for First-Order Logic
- König's Lemma: Every infinite, finitely-splitting tree has an infinite branch
- Tychonoff's Theorem
- de Bruijn – Erdős Theorem: If G is a graph and $d \in \mathbb{N}$ is such that every *finite* subgraph of G has chromatic number at most d, then the chromatic number of G is at most d.

$\aleph_1$ and incompactness

As soon as one moves to $\aleph_1$, the least uncountable cardinal, analogues of the theorems from the previous slide become false:

- The infinitary logic $L_{\omega_1\omega}$ fails to satisfy the analogue of the compactness theorem.
- There exist trees of height $\aleph_1$ with countable levels but no uncountable branches (i.e., *Aronszajn trees*).
- There exist Lindelöf spaces X and Y such that $X \times Y$ is not Lindelöf.
- There exists a graph G such that every subgraph of size $\aleph_1$ has countable chromatic number but G has uncountable chromatic number.

At $\aleph_2$ and higher, the story becomes more complicated, and independence results enter the picture.

Over the last century, set theorists have developed sophisticated tools for the study of compactness phenomena, including:

1. Tools to facilitate recursive constructions of incompact objects.
2. Tools for establishing (consistent) instances of compactness, e.g., large cardinals and forcing axioms.

Another field in which compactness phenomena, and tensions between local and global phenomena, play a central role is homological algebra, particularly with respect to the study of *derived functors*.

I. Derived functors

Abelian categories

Roughly speaking, an *abelian category* is one which sufficiently resembles the category Ab of abelian groups. For example, given objects A and B, the set $\mathrm{Hom}(A, B)$ of morphisms from A to B has a natural abelian group structure, the category has a zero object, all kernels and cokernels exist and behave as expected, etc. A pair of morphisms

$$A \xrightarrow{\pi} B \xrightarrow{\sigma} C$$

in an abelian category is *exact* at B if $\ker(\sigma) = \mathrm{im}(\pi)$. A *short exact sequence* is a sequence

$$0 \to A \xrightarrow{\pi} B \xrightarrow{\sigma} C \to 0$$

that is exact at A, B, and C.

Exact functors

Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories. We say that F is *exact* if it preserves short exact sequences, i.e., whenever

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

is a short exact sequence in $\mathcal{A}$, then

$$0 \to F(A_1) \to F(A_2) \to F(A_3) \to 0$$

is exact in $\mathcal{B}$. It is said to be *left exact* if the above sequence is always exact at $F(A_1)$ and $F(A_2)$ (equivalently, if F preserves finite limits), and *right exact* if it is always exact at $F(A_2)$ and $F(A_3)$ (equivalently, if F preserves finite colimits). Derived functors measure the failure of certain left or right functors to be exact. To illustrate this, let us consider the example of the *inverse limit* functor.

Inverse systems of abelian groups

Suppose that Λ is a directed partial order, i.e., for all $u, v \in \Lambda$, there is $w \in \Lambda$ such that $u, v \leq w$. An *inverse system of abelian groups indexed by* Λ is a structure

$$\mathbf{X} = \langle X_u, \pi_{uv} \mid u \leq v \in \Lambda \rangle$$

such that

- each X_u is an abelian group;
- each $\pi_{uv} : X_v \to X_u$ is a group homomorphism;
- for all $u \leq v \leq w$, we have $\pi_{uw} = \pi_{uv} \circ \pi_{vw}$.

Given an inverse system $\mathbf{X}$, we can form its (inverse) limit $\lim \mathbf{X}$ in Ab. Concretely, this can be represented as

$$\left\{ \mathbf{x} \in \prod_{u \in \Lambda} X_u \;\middle|\; \forall u \leq v \ \mathbf{x}(u) = \pi_{uv}(\mathbf{x}(v)) \right\}.$$

Level morphisms

Given two inverse systems $\mathbf{X} = \langle X_u, \pi^X_{uv} \mid u \leq v \in \Lambda \rangle$ and $\mathbf{Y} = \langle Y_u, \pi^Y_{uv} \mid u \leq v \in \Lambda \rangle$, a morphism from $\mathbf{X}$ to $\mathbf{Y}$ is a sequence $\mathbf{f} = \langle f_u \mid u \in \Lambda \rangle$ such that

- for all $u \in \Lambda$, $f_u : X_u \to Y_u$ is a group homomorphism;
- for all $u \leq v \in \Lambda$, we have $\pi^Y_{uv} \circ f_v = f_u \circ \pi^X_{uv}$.

A morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ induces a map $\lim \mathbf{f}$ from $\lim \mathbf{X}$ to $\lim \mathbf{Y}$ defined as follows: given $\mathbf{x} \in \lim \mathbf{X}$ and $u \in \Lambda$, we set $\lim \mathbf{f}(u) = f_u(\mathbf{x}(u))$.

With this notion of morphism, the category Ab^Λ of inverse systems indexed by Λ is an abelian category, and the inverse limit becomes a functor from Ab^Λ to Ab. How well-behaved is this functor?

Exactness of the inverse limit

The functor $\lim : \mathsf{Ab}^{\Lambda} \to \mathsf{Ab}$ is left exact but not exact, i.e., if

$$0 \to \mathbf{X} \to \mathbf{Y} \to \mathbf{Z} \to 0$$

is an exact sequence in Ab^{Λ}, then the resulting sequence

$$0 \to \lim \mathbf{X} \to \lim \mathbf{Y} \to \lim \mathbf{Z} \to 0$$

is exact at $\lim \mathbf{X}$ and $\lim \mathbf{Y}$ but may fail to be exact at $\lim \mathbf{Z}$. This failure of exactness results from the fact that, even if a level morphism $\mathbf{g} : \mathbf{Y} \to \mathbf{Z}$ is such that g_u is surjective for all $u \in \Lambda$, it need not be the case that $\lim \mathbf{g} : \lim \mathbf{Y} \to \lim \mathbf{Z}$ is surjective. Note that this is manifestly an instance of *incompactness*.

An example ($\Lambda = \omega$)

$$\mathbf{0} \longrightarrow \mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Y} \xrightarrow{\mathbf{g}} \mathbf{Z} \longrightarrow \mathbf{0}$$

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle \times 2} & & \downarrow \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 3} & \mathbb{Z} & \xrightarrow{\text{mod } 3} & \mathbb{Z}/3 & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle \times 2} & & \downarrow \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 3} & \mathbb{Z} & \xrightarrow{\text{mod } 3} & \mathbb{Z}/3 & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle \times 2} & & \downarrow \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 3} & \mathbb{Z} & \xrightarrow{\text{mod } 3} & \mathbb{Z}/3 & \longrightarrow & 0
\end{array}$$

$\lim \mathbf{X} = \lim \mathbf{Y} = 0$ and $\lim \mathbf{Z} = \mathbb{Z}/3$, so applying lim to this short exact sequence yields $0 \to 0 \to 0 \to \mathbb{Z}/3 \to 0$, which is not exact at $\mathbb{Z}/3$.

Derived functors

Given any left exact functor F betweeen (nice enough) abelian categories, there is a general procedure for producing a sequence of (right) derived functors $\langle F^n \mid n \in \omega \setminus \{0\} \rangle$ that "measure" the failure of the functor F to be exact. These derived functors then take short exact sequences

$$\mathbf{0} \longrightarrow \mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Y} \xrightarrow{\mathbf{g}} \mathbf{Z} \longrightarrow \mathbf{0}$$

to *long* exact sequences

$$\mathbf{0} \longrightarrow F\mathbf{X} \xrightarrow{F\mathbf{f}} F\mathbf{Y} \xrightarrow{F\mathbf{g}} F\mathbf{Z} \xrightarrow{\delta} F^1\mathbf{X} \xrightarrow{F^1\mathbf{f}} F^1\mathbf{Y} \xrightarrow{F^1\mathbf{g}} F^1\mathbf{Z} \xrightarrow{\delta} F^2\mathbf{X} \xrightarrow{F^2\mathbf{f}} F^2\mathbf{Y} \xrightarrow{F^2\mathbf{g}} F^2\mathbf{Z} \longrightarrow \ldots$$

Hom and Ext

Given an abelian category $\mathcal{A}$,

$$\mathrm{Hom}(\cdot,\cdot) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathsf{Ab}$$

is a (bi)functor that is contravariant in the first coordinate and covariant in the second. It is left exact in both coordinates, i.e., if $0 \to A \to B \to C \to 0$ is exact in $\mathcal{A}$ and $X \in \mathcal{A}$, then

$$0 \to \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(X, C)$$

is exact at $\mathrm{Hom}(X, A)$ and $\mathrm{Hom}(X, B)$ and

$$0 \to \mathrm{Hom}(C, X) \to \mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X)$$

is exact at $\mathrm{Hom}(C, X)$ and $\mathrm{Hom}(B, X)$. The derived functors of Hom are Ext^n for $n > 0$.

Some facts about Ext

Given two abelian groups C and A:

- $\mathrm{Ext}^1(C, A)$ is the group of equivalence classes of short exact sequences of the form

$$0 \to A \to B \to C \to 0.$$

- $\mathrm{Ext}^n(C, A) = 0$ for all $n \geq 2$.
- C is a *free* abelian group if and only if $\mathrm{Ext}^1(C, G) = 0$ for *every* abelian group G.

Whitehead's problem

An abelian group C is called a *Whitehead* group if $\mathrm{Ext}^1(C, \mathbb{Z}) = 0$. Every free abelian group is Whitehead.

Question (Whitehead, 1940s)

Is every Whitehead group free?

Theorem (Stein '51)

All countable Whitehead groups are free.

Theorem (Shelah '74)

Whitehead's problem is independent of the ZFC *axioms.*

An idea of the proof

Suppose that C is an abelian group. How might we determine whether or not C is Whitehead? First note that there is a (canonical) short exact sequence of abelian groups of the form

$$0 \to K \to F \to C \to 0,$$

where $K \lhd F$ are both free. Now apply the functor $\mathrm{Hom}(\cdot, \mathbb{Z})$ to this sequence; the beginning of the resulting long exact sequence is:

$$0 \to \mathrm{Hom}(C, \mathbb{Z}) \to \mathrm{Hom}(F, \mathbb{Z}) \to \mathrm{Hom}(K, \mathbb{Z}) \to \mathrm{Ext}^1(C, \mathbb{Z}) \to 0$$

Thus, $\mathrm{Ext}^1(C, \mathbb{Z}) = 0$ iff the map from $\mathrm{Hom}(F, \mathbb{Z})$ to $\mathrm{Hom}(K, \mathbb{Z})$ is surjective. But this map simply takes a morphism from F to $\mathbb{Z}$ and restricts it to K. Thus, C is Whitehead iff every homomorphism from K to $\mathbb{Z}$ lifts to a homomorphism from F to $\mathbb{Z}$.

$\diamondsuit$

Recall that Jensen's $\diamondsuit$ principle asserts the existence of a sequence $\langle f_\alpha \mid \alpha < \omega_1 \rangle$ such that

- each f_α is a map from α to $\mathbb{Z}$;
- for all $f : \omega_1 \to \mathbb{Z}$, there are many $\alpha < \omega_1$ for which $f \restriction \alpha = f_\alpha$.

If $\diamondsuit$ holds and C is a nonfree abelian group of cardinality $\aleph_1$, one can recursively build a homomorphism $g : K \to \mathbb{Z}$ that does not lift to a homomorphism $g' : F \to \mathbb{Z}$. The $\diamondsuit$ sequence is used to anticipate potential liftings of g; one can then continue the construction in a way that obstructs this lifting. Thus, $\diamondsuit$ implies that every Whitehead group of size $\aleph_1$ is free. More generally, if $V = L$, then every Whitehead group is free.

Martin's Axiom

In ZFC, one can construct a nonfree abelian group of size $\aleph_1$ satisfying a technical condition called *Chase's condition*. Given such a group C, the canonical short exact sequence

$$0 \to K \to F \to C \to 0,$$

and a homomorphism $g : K \to \mathbb{Z}$, there is a forcing notion $\mathbb{P}$ with the countable chain condition that adds a homomorphism $g' : F \to \mathbb{Z}$ lifting g. If Martin's Axiom ($\mathrm{MA}_{\aleph_1}$), which is a consistent strengthening of the Baire Category Theorem, holds, then such a g' already exists in the ground model. Thus, $\mathrm{MA}_{\aleph_1}$ implies that there exists a nonfree Whitehead group of cardinality $\aleph_1$.

Cofinality and derived limits

Definition

Given a directed partial order Λ, the *cofinality* of Λ, denoted $\mathrm{cf}(\Lambda)$, is the least cardinality of a subset $\Gamma \subseteq \Lambda$ such that, for all $u \in \Lambda$, there is $v \in \Gamma$ such that $u \leq v$.

Theorem (Goblot, '70)

Suppose that Λ is a directed partial order, $n \in \mathbb{N}$, $\mathrm{cf}(\Lambda) \leq \aleph_n$, and $\mathbf{X}$ is an inverse system of abelian groups indexed by Λ. Then $\lim^m \mathbf{X} = 0$ for all $m > n + 1$.

Theorem (Mitchell, '72)

Suppose that Λ is a directed partial order, $n \in \mathbb{N}$, and $\mathrm{cf}(\Lambda) \geq \aleph_n$. Then there is an inverse system of abelian groups $\mathbf{X}$ indexed by Λ such that $\lim^{n+1} \mathbf{X} \neq 0$.

Global dimension and the continuum

Definition

Let R be a ring. The *homological dimension* of R is

$$\sup\{n \in \mathbb{N} \mid \exists C, A \in \mathrm{Mod}-R\ [\mathrm{Ext}^n(C, A) \neq 0]\}.$$

- The homological dimension of $\mathbb{Z}$ is 1.
- The homological dimension of a field is 0.

Theorem (Osofsky, '70)

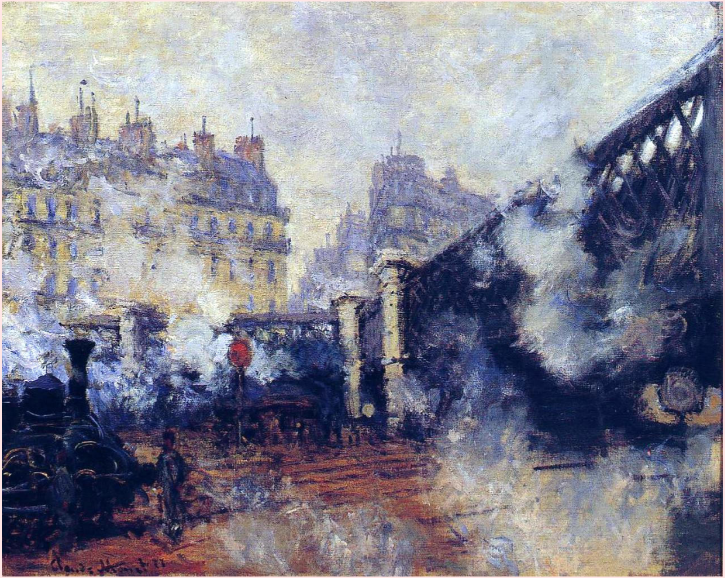
Suppose that R is a countably infinite product of fields. If $n \in \mathbb{N}$ and $|\mathbb{R}| = \aleph_n$, then the homological dimension of R is $n+1$. If $|\mathbb{R}| > \aleph_\omega$, then the homological dimension of R is infinite.

In this paper, as well as in [–], statements on homological dimension were found to be equivalent to the continuum hypothesis. In these works, if $2^{\aleph_0} \neq \aleph_1$, then $\aleph_1$ appears in the role of a stumbling block in getting from $\aleph_0$ to $2^{\aleph_0}$ . . . There is no way in these papers to get one's hands on $\aleph_1$. Such a situation is aesthetically (or intuitively, if you prefer) repugnant to me . . . For those reasons, the hypothesis $2^{\aleph_0} = \aleph_1$ appears to me to be the natural one applying to the axiom system in which homological algebra is done, and $2^{\aleph_0} > \aleph_\omega$ has somewhat upsetting consequences.

Barbara Osofsky, "Homological dimension and cardinality", 1970

Intermission

II. A family of inverse systems



The systems $\mathbf{A}[H]$

We now introduce a specific family of inverse systems that occurs naturally in various mathematical contexts. We denote the set of all functions from ω to ω by ${}^{\omega}\omega$. Given a function $f \in {}^{\omega}\omega$ and $H \in \mathsf{Ab}$, let

$$I(f) := \{(k, m) \in \omega \times \omega \mid m < f(k)\}$$

and $A_f[H] = \bigoplus_{I(f)} H$. Given $f \leq g$ in ${}^{\omega}\omega$, there is a projection map $\pi_{fg} : A_g[H] \to A_f[H]$. We thus obtain an inverse system

$$\mathbf{A}[H] = \langle A_f[H], \pi_{fg} \mid f \leq g \in {}^{\omega}\omega \rangle.$$

Note that $\lim \mathbf{A}[H] = \bigoplus_{\omega} \prod_{\omega} H$. We omit "$H$" from the notation if $H = \mathbb{Z}$.

Describing $\lim^1 \mathbf{A}$

Define an inverse system $\mathbf{B} = \langle B_f, \pi_{fg} \mid f, g \in {}^\omega\omega,\ f \leq g \rangle$ by letting $B_f = \prod_{I(f)} \mathbb{Z}$. Note that $\lim \mathbf{B} = \prod_{\omega \times \omega} \mathbb{Z}$. This gives rise to a short exact sequence

$$0 \to \mathbf{A} \to \mathbf{B} \to \mathbf{B}/\mathbf{A} \to 0,$$

which then induces a long exact sequence

$$0 \to \lim \mathbf{A} \to \lim \mathbf{B} \to \lim \mathbf{B}/\mathbf{A} \to \lim^1 \mathbf{A} \to \lim^1 \mathbf{B} \to \dots$$

$\mathbf{B}$ has the property that $\lim^n \mathbf{B} = 0$ for all $n > 0$. Therefore, we get

$$\lim^1 \mathbf{A} \cong \frac{\lim \mathbf{B}/\mathbf{A}}{\mathrm{im}(\lim \mathbf{B})}.$$

Describing $\lim^1 \mathbf{A}$

Let $=^*$ denote equality mod finite. For $f \in {}^\omega\omega$, elements of B_f/A_f are the $=^*$-equivalence classes of functions from $I(f)$ to $\mathbb{Z}$. Therefore, elements of $\lim \mathbf{B}/\mathbf{A}$ are (equivalence classes of) families of functions

$$\langle \varphi_f : I(f) \to \mathbb{Z} \mid f \in {}^\omega\omega \rangle$$

that are *coherent*, i.e., $\varphi_f =^* \varphi_g$ (on their common domain $I(f) \cap I(g)$) for all $f, g \in {}^\omega\omega$.

Elements of $\mathrm{im}(\lim \mathbf{B})$ are precisely those coherent families of functions for which there is a single function $\psi : \omega \times \omega \to \mathbb{Z}$ such that $\psi \restriction I(f) =^* \varphi_f$ for all $f \in {}^\omega\omega$. Such families are called *trivial*.

We thus see that $\lim^1 \mathbf{A} = 0$ iff every coherent family of functions is trivial.

Higher coherence

An analogous characterization of the nonvanishing of $\lim^n \mathbf{A}$ for $n > 1$ exists in terms of higher-dimensional families of functions. For example, $\lim^2 \mathbf{A} \neq 0$ if and only if there is a family

$$\Phi = \langle \varphi_{fg} : I(f) \cap I(g) \to \mathbb{Z} \mid f, g \in {}^\omega\omega \rangle$$

that is

- alternating: $\varphi_{fg} = -\varphi_{gf}$ for all $f, g \in {}^\omega\omega$;
- 2-coherent: $\varphi_{gh} - \varphi_{fh} + \varphi_{fg} =^* 0$ for all $f, g, h \in {}^\omega\omega$;
- nontrivial: there is no family $\langle \psi_f : I(f) \to \mathbb{Z} \mid f \in {}^\omega\omega \rangle$ such that $\varphi_{fg} =^* \psi_g - \psi_f$ for all $f, g \in {}^\omega\omega$.

Some history

The system **A** arose in 1980s work of Mardešić and Prasolov on strong homology. In particular, they showed that if strong homology is additive, even on the class of closed subsets of Euclidean space, then $\lim^n \mathbf{A} = 0$ for all $n > 0$.

- (Mardešić–Prasolov, '88) The Continuum Hypothesis implies that $\lim^1 \mathbf{A} \neq 0$.
- (Dow–Simon–Vaughan, '89) The Proper Forcing Axiom, a strengthening of Martin's Axiom, implies that $\lim^1 \mathbf{A} = 0$.
- (Kamo, '94) After adding ω_2-many Cohen reals to any model of ZFC, we have $\lim^1 \mathbf{A} = 0$.

Compactly Borel sets

Recall that a *Polish space* is a separable completely metrizable topological space (e.g., $\mathbb{R}$). A subset X of a Polish space is called *analytic* if it is a continuous image of a Borel subset of a Polish space. Jayne and Rogers, and later Fremlin, asked questions in the '70s and '80s of the following form:

> *Suppose that A is a subset of a Polish space X and $A \cap K$ is Borel/analytic/...for every compact $K \subseteq X$ (i.e., A is compactly Borel/analytic/...). What can be said about A?*

- (Kunen–Miller, '82) It is consistent with ZFC that every compactly analytic set is analytic.
- (Todorcevic, '98) If $\lim^1 \mathbf{A} \neq 0$, then there exists a subset A of the irrational numbers $\mathbf{I}$ such that A is not analytic but $A \cap K$ is F_σ for every compact $K \subseteq \mathbf{I}$.

Recent results

The ‘80s and ‘90s saw much work on $\lim^1 \mathbf{A}$. Recent years have seen the development of higher-dimensional combinatorial set-theoretic tools that have allowed us to begin to understand its higher derived limits.

- (Bergfalk, ‘17) The Proper Forcing Axiom implies that $\lim^2 \mathbf{A} \neq 0$.
- (Bergfalk–LH, ‘21) If there exists a weakly compact cardinal, then there is a forcing extension in which $\lim^n \mathbf{A} = 0$ for all $n > 0$.
- (Bergfalk–Hrušák–LH, ‘23) If one adds $\aleph_\omega$-many Cohen reals to any model of $\mathsf{ZFC} + \mathsf{GCH}$, then $\lim^n \mathbf{A} = 0$ for all $n > 0$ in the resulting extension.
- (Bannister, ‘24) In either of the above two models, we in fact have $\lim^n \mathbf{A}[H] = 0$ for all $n > 0$ and all abelian groups H.

The value of the continuum

In the models from the previous slide in which $\lim^n \mathbf{A}[H] = 0$ for all n and H, we have $|\mathbb{R}| > \aleph_\omega$. By recent joint work with Casarosa, building on earlier work of Veličković and Vignati, this is necessary.

Theorem (Casarosa–LH, '24)

Suppose that $\lim^n \mathbf{A}[H] = 0$ for all $n > 0$ and all abelian groups H. Then $|\mathbb{R}| > \aleph_\omega$.

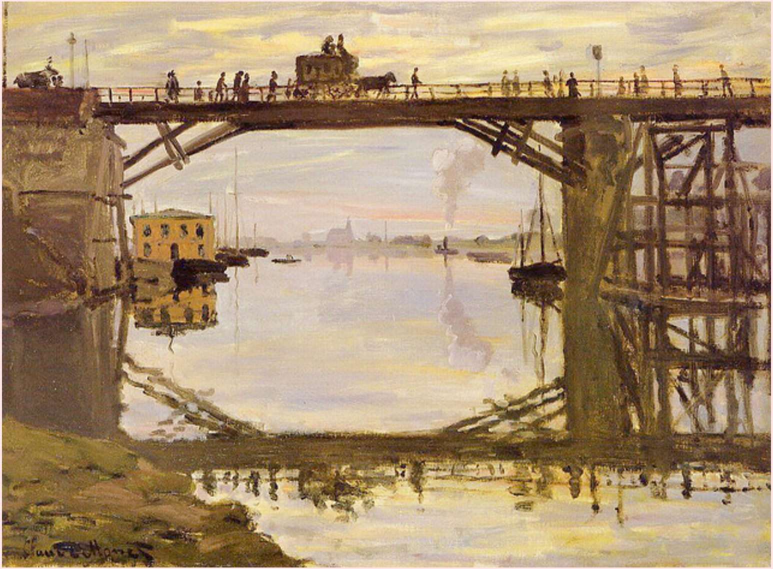
The vanishing of these derived limits joins a growing list of "nice" higher-dimensional combinatorial statements about the reals that are consistent with ZFC but imply that the continuum is greater than $\aleph_\omega$.

The value of the continuum

Other examples of such statements:

- (Hindman–Leader–Strauss, Komjáth+, Zhang) For every finite set d and every function $f : \mathbb{R} \to d$, there is an infinite $X \subseteq \mathbb{R}$ such that $f \restriction (X + X)$ is constant.
- (LH–Zucker) For every finite sequence $\langle Y_i \mid i < d \rangle$ of perfect Polish spaces and every function $f : \prod_{i<d} Y_i \to \omega$, there are somewhere dense subsets $Z_i \subseteq Y_i$ such that $f \restriction \prod_{i<d} Z_i$ is constant.
- (Shelah, Raghavan–Todorcevic) $\mathbb{Q}$ has finite topological Ramsey degree in $\mathbb{R}$ in all finite dimensions, i.e., there is $h : \omega \to \omega$ such that, for all $0 < \ell, n < \omega$ and all functions $c : [\mathbb{R}]^n \to \ell$, there is $X \subseteq \mathbb{R}$ homeomorphic to $\mathbb{Q}$ such that c takes at most $h(n)$-many values on $[X]^n$.

Epilogue: Condensed mathematics



Condensed mathematics

Condensed mathematics is a framework, introduced recently by Clausen and Scholze, to allow for the application of algebraic tools in contexts in which algebraic objects carry topologies.

Problem: Classical categories of algebraic objects carrying topologies, such as the category TopAb of topological abelian groups, fail to be abelian categories.

Solution: Embed these classical categories into richer, "condensed" categories. E.g., TopAb embeds into the category Cond(Ab) of condensed abelian groups.

Condensed abelian groups

Let ED denote the category of extremally disconnected compact Hausdorff spaces. A *condensed abelian group* is a contravariant functor $T : \mathrm{ED} \to \mathrm{Ab}$ such that

1. $T(\emptyset) = 0$ (i.e., the one-element group);
2. for all $S_0, S_1 \in \mathrm{ED}$, $T(S_0 \sqcup S_1) = T(S_0) \times T(S_1)$.

Given $X \in \mathrm{TopAb}$, define $\underline{X} \in \mathrm{Cond(Ab)}$ by setting $\underline{X}(S) = \mathrm{Cont}(S, X)$ for all $S \in \mathrm{ED}$. This describes an embedding of TopAb into Cond(Ab); it is fully faithful on the class of compactly generated topological abelian groups.

Cond(Ab) is a (very nice) abelian category; e.g., all limits and colimits exist; arbitrary products, direct sums, and filtered colimits are exact; and the category is generated by compact projective objects.

Internal Hom

For $T_0, T_1 \in \mathsf{Cond(Ab)}$, $\mathrm{Hom}(T_0, T_1)$ is an abelian group. Cond(Ab) also has a tensor product and an *internal Hom functor*, $\underline{\mathrm{Hom}}(\cdot, \cdot)$, which takes values in Cond(Ab). It satisfies the adjunction

$$\mathrm{Hom}(T_0, \underline{\mathrm{Hom}}(T_1, T_2)) \cong \mathrm{Hom}(T_0 \otimes T_1, T_2).$$

Internal Ext

$\underline{\mathrm{Hom}}(\cdot,\cdot)$ has derived functors, $\underline{\mathrm{Ext}}^n(\cdot,\cdot)$. When Whitehead's problem is formulated in terms of $\underline{\mathrm{Ext}}^1$ (applied to abelian groups with the discrete topology), it turns out that it is *not* independent of ZFC.

Theorem (Clausen–Scholze)

Suppose that C *is an abelian group and* $\underline{\mathrm{Ext}}^1(\underline{C},\underline{\mathbb{Z}}) = 0$*. Then* C *is free.*

Say that an abelian group C is *condensed Whitehead* if $\underline{\mathrm{Ext}}^1(\underline{C},\underline{\mathbb{Z}}) = 0$. Then the above result amounts to saying that every condensed Whitehead group is free.

Continuous families of morphisms

Clausen and Scholze's proof is very slick and relies on deep analysis of the category of condensed abelian groups. In joint work with Bergfalk and Šaroch, we produced a more set-theoretic proof of the result.

Suppose that C is an abelian group and $0 \to K \to F \to C \to 0$ is a short exact sequence with K and F free. Recall that C is Whitehead iff every element of $\mathrm{Hom}(K, \mathbb{Z})$ lifts to an element of $\mathrm{Hom}(F, \mathbb{Z})$. It turns out that C is *condensed Whitehead* iff a continuously indexed version of this holds:

> *C is condensed Whitehead iff for every compact Hausdorff space S and every continuous function $\varphi : S \to \mathrm{Hom}(K, \mathbb{Z})$, there is a continuous $\varphi' : S \to \mathrm{Hom}(F, \mathbb{Z})$ such that $\varphi'(s) \restriction K = \varphi(s)$ for all $s \in S$.*

If C is nonfree of size κ, then there is always a continuous $\varphi : \prod_\kappa \{0, 1\} \to \mathrm{Hom}(K, \mathbb{Z})$ that does not lift continuously as above.

Pro-abelian groups

A *pro-abelian group* is a topological abelian group that can be expressed as the inverse limit of an inverse system of (discrete) abelian groups.

Question (Clausen–Scholze)

Does the category of pro-abelian groups embed fully faithfully into Cond(Ab) *(at the level of derived categories)?*

This reduces to the following question: is it the case that, for all index sets I, J, and K, and all $0 < n < \omega$, we have

$$\mathrm{Ext}^n_{\mathsf{Cond(Ab)}}\left(\prod_I \bigoplus_J \underline{\mathbb{Z}}, \bigoplus_K \underline{\mathbb{Z}}\right) = 0?$$

An equivalence

Clausen and Scholze observed that the following conditions are equivalent:

1. For all $0 < n < \omega$ and every cardinal μ, we have

$$\mathrm{Ext}^n_{\mathsf{Cond(Ab)}}\left(\prod_{\omega}\bigoplus_{\omega}\underline{\mathbb{Z}}, \bigoplus_{\mu}\underline{\mathbb{Z}}\right) = 0.$$

2. Whenever $M_0 \twoheadleftarrow M_1 \twoheadleftarrow M_2 \twoheadleftarrow \cdots$ is a sequential system of countable abelian groups with surjective transition maps and N is *any* abelian group, we have, for all $n \geq 0$,

$$\mathrm{Ext}^n_{\mathsf{Cond(Ab)}}(\lim \underline{M_i}, \underline{N}) \cong \mathrm{colim}\ \mathrm{Ext}^n_{\mathsf{Cond(Ab)}}(\underline{M_i}, \underline{N}).$$

3. $\lim^n \mathbf{A}[H] = 0$ for all $n \geq 1$ and all abelian groups H.

Thank you!