Guessing models, cardinal arithmetic, and ultrafilters

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This talk contains joint work with Radek Honzik, Assaf Rinot, Šárka Stejskalová, and Jing Zhang.

I. Guessing models

Set-theoretic compactness

This is a talk about set-theoretic compactness, and the extent to which a structure's *global* behavior is determined by its *local* behavior. Many questions in this area take the following general form:

> *Suppose a structure M is such that all "small" substructures of M satisfy property P. Must M also satisfy property P?*

Examples of questions of compactness about a fixed cardinal κ:

- Suppose that A is an abelian group such that every subgroup of A of cardinality $<\kappa$ is free. Must A be free?
- Suppose that G is a graph such that every subgraph of G with fewer than κ-many vertices has countable chromatic number. Must G have countable chromatic number?

Large cardinals

Instances of set-theoretic compactness are often furnished by large cardinals. For example, if κ is a strongly compact cardinal, then both of the questions on the previous slide have positive answers.

(Recall that an uncountable cardinal κ is strongly compact iff every κ-satisfiable theory in the infinitary logic $L_{\kappa\kappa}$ is satisfiable iff every κ-complete filter can be extended to a κ-complete ultrafilter.)

On the other hand, if $V = L$, then examples of incompactness abound. For example, in L, for every infinite cardinal κ, there is a graph G of cardinality κ^+ such that every subgraph of G with at most κ-many vertices has countable chromatic number, but the chromatic number of G is κ^+.

Small cardinals

Much work in combinatorial set theory investigates the extent to which certain compactness properties of large cardinals can consistently hold at smaller uncountable cardinals. For example:

- An uncountable cardinal κ is weakly compact if and only if there are no κ-Aronszajn trees. By work of Mitchell and Silver, the assertion that there are no $\aleph_2$-Aronszajn trees is equiconsistent with the existence of a weakly compact cardinal.
- Let μ_0 denote the least fixed point of the function $\alpha \mapsto \aleph_\alpha$. By work of Magidor and Shelah, it is consistent (relative to the consistency of certain large cardinals) that whenever A is an abelian group such that every subgroup of A of cardinality $\leq \mu_0$ is free, A itself must be free (and this is best possible).

Guessing models

Guessing models, introduced by Viale and Weiss, provide a useful tool for capturing much of the power of large cardinals in a way that can consistently hold at smaller cardinals.

Definition

Let $x \in M \prec H(\theta)$, and fix a subset $d \subseteq x$.

1. We say that d is *M-approximated* if, for every countable $z \in M$, we have $d \cap z \in M$.
2. We say that d is *M-guessed* if there is $b \in M$ such that $b \cap M = d \cap M$.

M is a *guessing model* if every set that is M-approximated is M-guessed.

The guessing model property

Definition

Let $\kappa > \omega_1$ be a regular cardinal. Then GMP_κ is the assertion that, for every regular $\theta \geq \kappa$, the set of guessing models is stationary in $\mathscr{P}_\kappa H(\theta)$. Equivalently, for every regular $\theta \geq \kappa$ and every $x \subseteq H(\theta)$ with $|x| < \kappa$, there is a guessing model $M \prec H(\theta)$ with $x \subseteq M$ and $|M| < \kappa$.

Theorem (Magidor)

A cardinal κ is supercompact iff it is inaccessible and GMP_κ holds.

Theorem (Viale–Weiss)

The Proper Forcing Axiom (PFA) implies $\mathsf{GMP}_{\aleph_2}$.

II. Cardinal arithmetic

Singular Cardinals Hypothesis

GMP_κ can be seen as asserting that κ behaves in many ways like a strongly compact or supercompact cardinal. One place this can be seen is in its effect on cardinal arithmetic.

Definition

The Singular Cardinals Hypothesis (SCH) is the assertion that, for every singular strong limit cardinal μ, we have $2^\mu = \mu^+$.

Theorem (Solovay)

If κ is a strongly compact cardinal, then SCH *holds above κ.*

Theorem (Viale, Krueger)

If GMP_κ *holds, then* SCH *holds above κ. In particular,* $\mathsf{GMP}_{\aleph_2}$ *implies* SCH.

Pseudopowers

SCH is a somewhat unsatisfactory statement, as it only applies to *strong limit* singular cardinals, i.e., those singular cardinals μ such that $2^\lambda < \mu$ for all $\lambda < \mu$.

To rectify this situation, Shelah introduced the *pseudopower* function, pp, a PCF-theoretic function that attempts to provide a more refined measure of the size of the power set of singular cardinals by "washing away" the influence of smaller cardinals.

For example, if one starts in a model satisfying GCH and forces to add $>\aleph_{\omega+1}$-many Cohen reals, then one obtains a model in which one trivially has $2^{\aleph_\omega} > \aleph_{\omega+1}$. Nonetheless, in this forcing extension we still have $\mathrm{pp}(\aleph_\omega) = \aleph_{\omega+1}$.

For all singular cardinals μ, we always have $\mu^+ \leq \mathrm{pp}(\mu) \leq 2^\mu$.

Shelah's Strong Hypothesis

Definition

Shelah's Strong Hypothesis (SSH) is the assertion that $\mathrm{pp}(\mu) = \mu^+$ for all singular cardinals μ.

SSH is a strengthening of SCH, imposing requirements on *all* singular cardinals regardless of whether or not they are strong limit cardinals.

Theorem (LH–Stejskalová [3])

If GMP_κ *holds, then* SSH *holds above* κ*.*

$2^{\aleph_0}$ **vs.** $2^{\aleph_1}$

Recall that PFA $\Rightarrow$ $\mathrm{GMP}_{\aleph_2}$. PFA implies that $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$. By work of Cox and Krueger, $\mathrm{GMP}_{\aleph_2}$ is compatible with any value of the continuum greater than $\aleph_1$. In fact:

Theorem (Honzik–LH–Stejskalová [1])

Suppose that GMP_κ holds. Then GMP_κ continues to hold in the forcing extension after adding any number of Cohen reals.

Thus, $\mathrm{GMP}_{\aleph_2}$ cannot imply $2^{\aleph_0} = 2^{\aleph_1}$, since it is compatible with $\mathrm{cf}(2^{\aleph_0}) = \aleph_1$. But it *does* imply the next best thing:

Theorem (LH–Stejskalová [3])

Suppose that $\mathrm{GMP}_{\aleph_2}$ holds. Then

$$2^{\aleph_1} = \begin{cases} 2^{\aleph_0} & \textit{if } \mathrm{cf}(2^{\aleph_0}) > \aleph_1 \\ (2^{\aleph_0})^+ & \textit{if } \mathrm{cf}(2^{\aleph_0}) = \aleph_1. \end{cases}$$

Weak almost guessing property

The audience may have noticed an asymmetry in some of the earlier results. For instance, GMP_κ implies SCH above κ, and GMP_κ characterizes *supercompact* cardinals among inaccessible cardinals.

On the other hand, Solovay proved that SCH holds above *strongly compact* cardinals.

In work with Stejskalová [4], we isolated a weakening of GMP_κ, the *weak almost guessing property* (wAGP_κ) that characterizes *strongly compact cardinals* among inaccessible cardinals and can be forced to hold at $\aleph_2$ starting from a strongly compact cardinal.

wAGP_κ suffices for many of the applications of GMP_κ, including all of the applications presented in this section. Its statement is rather technical, though, so we will not say more about it in this talk.

III. Indecomposable ultrafilters

Indecomposable ultrafilters

Definition

Let U be a nonprincipal ultrafilter over an uncountable cardinal κ. Then U is said to be *indecomposable* if, for all $\lambda < \kappa$ and all functions $f : \kappa \to \lambda$, there is a set $A \in U$ such that $f[A]$ is countable.

Indecomposability is a natural weakening of κ-*completeness*. It has a number of implications for the behavior of ultraproducts. For example:

Theorem (Prikry)

Suppose that U is an indecomposable ultrafilter over a strong limit cardinal κ. Then, for all infinite cardinals $\lambda < \kappa$, we have

$$|\prod_{\kappa} \lambda / U| \leq 2^{\lambda}.$$

Silver's question

There are two scenarios in which a cardinal κ trivially carries an indecomposable ultrafilter:

1. κ is measurable;
2. κ is a limit of countably many measurable cardinals.

If κ is measurable and one does Prikry forcing or adds κ-many Cohen reals, then this preserves the fact that κ carries an indecomposable ultrafilter. Silver asked whether these are essentially the only scenarios in which indecomposable ultrafilters exist. More specifically, he asked whether an inaccessible cardinal carrying an indecomposable ultrafilter must be measurable. This was answered negatively by Sheard:

Theorem (Sheard)

It is consistent, relative to the consistency of a measurable cardinal, that there is an inaccessible cardinal that is not weakly compact but carries an indecomposable ultrafilter.

Goldberg's theorem

However, Goldberg recently proved that Silver's question has a positive answer above a strongly compact cardinal.

Theorem (Goldberg)

Suppose that κ is strongly compact and $\mu \geq \kappa$ carries an indecomposable ultrafilter. Then either

1. *μ is measurable; or*
2. *μ is a limit of countably many measurable cardinals.*

This naturally raises the question of whether GMP_κ yields the same consequences.

The influence of guessing models

Theorem (LH–Rinot–Zhang [2])

Suppose that $\mathrm{GMP}_{\aleph_2}$ *holds and* $\mu > 2^{\aleph_0}$ *carries an indecomposable ultrafilter. Then either*

1. μ *is measurable; or*
2. μ *is a limit of countably many measurable cardinals.*

We also showed that this theorem is sharp in the following two ways:

- $\mathrm{GMP}_{\aleph_2}$ is compatible with $2^{\aleph_0}$ carrying an indecomposable ultrafilter.
- PFA (or even MM) is compatible with the existence of an inaccessible cardinal μ that is not weakly compact but carries a nonprincipal ultrafilter U such that, for all $\lambda < \mu$ and $f : \mu \to \lambda$, there is $A \in U$ such that $|f[A]| \leq \aleph_1$.

References

Radek Honzik, Chris Lambie-Hanson, and Šárka Stejskalová, *Indestructibility of some compactness principles over models of* PFA, Ann. Pure Appl. Logic **175** (2024), no. 1, part A, Paper No. 103359, 17. MR 4640983

Chris Lambie-Hanson, Assaf Rinot, and Jing Zhang, *Squares, ultrafilters, and forcing axioms*, (2024), submitted.

Chris Lambie-Hanson and Šárka Stejskalová, *Guessing models, trees, and cardinal arithmetic*, Isr. J. Math. (2024), to appear.

_______, *Strong tree properties, Kurepa trees, and guessing models*, Monatsh. Math. **203** (2024), no. 1, 111–148. MR 4688349

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Thank you!