Higher-dimensional coherence

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I. Coherent Aronszajn trees

Coherent Aronszajn trees

Let κ be a regular uncountable cardinal. Given two functions f and g, let $f =^* g$ denote the assertion that

$$\{\alpha \in \mathrm{dom}(f) \cap \mathrm{dom}(g) \mid f(\alpha) \neq g(\alpha)\}$$

is finite. A *coherent κ-Aronszajn tree* (with maps into ω) is a subtree $T \subseteq {}^{<\kappa}\omega$ of height κ such that

1. for all $s, t \in T$, we have $s =^* t$;
2. there is no $f \in {}^{\kappa}\omega$ such that $\forall \alpha < \kappa \; f \restriction \alpha \in T$.

Some facts

- (Aronszajn) coherent $\aleph_1$-Aronszajn trees exist;
- (B. König) if $\square(\kappa)$ holds, then there exists a coherent κ-Aronszajn tree;
- (Todorčević) If the P-ideal dichotomy (PID) holds and $\kappa > \aleph_1$ is regular, then there are no coherent κ-Aronszajn trees.

Nontrivial coherence and cohomology

The existence of a coherent κ-Aronszajn tree is equivalent to the existence of a family of functions $\Phi = \langle \varphi_\alpha : \alpha \to \mathbb{Z} \mid \alpha < \kappa \rangle$ that is

- *(coherent)* for all $\alpha < \beta < \kappa$, $\varphi_\alpha =^* \varphi_\beta$; and
- *(nontrivial)* there is no $\psi : \kappa \to \mathbb{Z}$ such that $\psi =^* \varphi_\alpha$ for all $\alpha < \kappa$.

They both end up being equivalent to the assertion that the first cohomology group of κ (with the order topology) with coeffients in $\mathbb{Z}$ is nontrivial, i.e., $H^1(\kappa, \mathbb{Z}) \neq 0$.

II. Higher coherence

2-**coherence**

Higher analogues of coherent Aronszajn trees naturally arise through cohomological considerations.

Definition

A family of functions $\Phi = \langle \varphi_{\alpha\beta} : \alpha \to \mathbb{Z} \mid \alpha < \beta < \kappa \rangle$ is

- *2-coherent* if, for all $\alpha < \beta < \gamma < \kappa$, we have

$$\varphi_{\beta\gamma} - \varphi_{\alpha\gamma} + \varphi_{\alpha\beta} =^* 0$$

  i.e., $\varphi_{\alpha\beta} + \varphi_{\beta\gamma} =^* \varphi_{\alpha\gamma}$.

- *2-trivial* if there is a family $\Psi = \langle \psi_\alpha : \alpha \to \mathbb{Z} \mid \alpha < \kappa \rangle$ such that, for all $\alpha < \beta < \kappa$, we have

$$\psi_\beta - \psi_\alpha =^* \varphi_{\alpha\beta}.$$

2-trivial families are 2-coherent. $H^2(\kappa, \mathbb{Z}) = 0$ iff every 2-coherent family as above is 2-trivial.

Some facts

We can similarly define n-coherent and n-trivial families for $2 < n < \omega$; $H^n(\kappa, \mathbb{Z}) = 0$ iff every n-coherent family on κ is n-trivial.

- (Goblot) If $n < \omega$ and $\kappa < \aleph_n$, then $H^n(\kappa, \mathbb{Z}) = 0$.
- If κ is weakly compact, then $H^n(\kappa, \mathbb{Z}) = 0$ for all $1 \leq n < \omega$.
- If κ is regular and above the least ω_1-strongly compact cardinal, then $H^n(\kappa, \mathbb{Z}) = 0$ for all $1 \leq n < \omega$.
- (Bergfalk–LH) If $V = L$, then $H^n(\kappa, \mathbb{Z}) \neq 0$ for all $1 \leq n < \omega$ and all regular $\kappa \geq \aleph_n$ such that κ is not weakly compact. (Some combinations of squares and diamonds are all that are needed for this.)

Recent vanishing results

There have recently been some nontrivial vanishing results about $H^n(\kappa, \mathbb{Z})$ for "small" values of κ.

Theorem (Bergfalk–LH–Zhang)

If κ is regular and carries and $\aleph_1$-indecomposable ultrafilter, then $H^n(\kappa, \mathbb{Z}) = 0$ for all $1 \leq n < \omega$. In particular, it is consistent that $H^n(\omega_{\omega+1}, \mathbb{Z}) = 0$ for all $1 \leq n < \omega$.

Recently, Eskew and Hayut constructed a model of ZFC in which the $\aleph_n$'s carry many dense ideals. By arguments of Bergfalk–LH–Zhang, in this model we have

$$H^n(\omega_m, \mathbb{Z}) = 0$$

for all $1 \leq n < m < \omega$.

This raises the natural question: must $H^n(\omega_n, \mathbb{Z}) \neq 0$?

- (Aronszajn) $H^1(\omega_1, \mathbb{Z}) \neq 0$.
- (B. Mitchell) $H^n(\omega_n, \bigoplus_{\omega_n} \mathbb{Z}) \neq 0$.
- (Bergfalk–LH–Zhang) $H^n(\omega_n \cdot \omega_n, \mathbb{Z}) \neq 0$.

Theorem (Bergfalk–LH–Zhang)

If $H^1(\omega_2, \mathbb{Z}) = 0$, then $H^2(\omega_2, \mathbb{Z}) \neq 0$.

We believe the arguments of the above theorem should generalize to yield that for all $1 \leq n < \omega$, there is $1 \leq m \leq n$ such that $H^m(\omega_n, \mathbb{Z}) \neq 0$, but details remain to be checked.

Trivializing families

However, recent work indicates that, in contrast with the 1-dimensional case, nontrivial 2-coherent families on ω_2 are typically *fragile*.

Theorem (Bergfalk–LH–Zhang)

Suppose that CH *holds and* Φ *is a 2-coherent family on* ω_2. *Then*

1. *there is a* $(<\omega_2)$*-distributive forcing* $\mathbb{P}$ *such that* $\Vdash_{\mathbb{P}}$ *"*Φ *is 2-trivial";*
2. *there is a countably closed,* $\aleph_2$*-cc forcing* $\mathbb{Q}$ *such that* $\Vdash_{\mathbb{Q}}$ *"*Φ *is 2-trivial".*

In contrast, there are always coherent ω_1-Aronszajn trees that cannot be trivialized without collapsing ω_1. It is currently unclear whether either of the posets identified above can be iterated to yield the consistency of $H^2(\omega_2, \mathbb{Z}) = 0$.

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