Higher-dimensional Δ-systems

Cornell Logic Seminar

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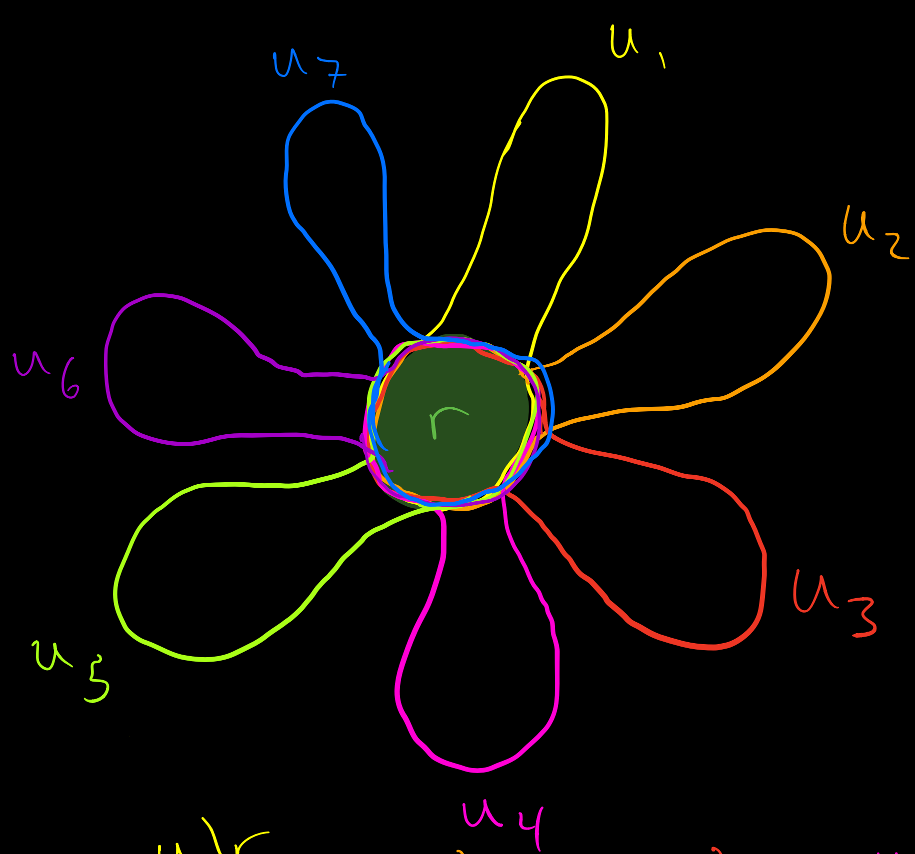
Classical Δ-systems

Def A family $\mathcal{U}$ of sets is a **Δ-system** if there is a set r, known as the **root** of the Δ-system, such that $u \cap v = r$ for all distinct $u, v \in \mathcal{U}$.



Δ-System Lemmas

Lemma (Shanin) Suppose that $\mathcal{U}$ is an uncountable family of finite sets. Then there is an uncountable subfamily $\mathcal{U}^* \subseteq \mathcal{U}$ such that $\mathcal{U}^*$ is a Δ-system.

Lemma Suppose that $\kappa < \lambda$ are infinite cardinals such that

- λ is regular
- $\nu^{<\kappa} < \lambda$ for all $\nu < \lambda$.

If $\mathcal{U}$ is a family of sets s.t. $|\mathcal{U}| \geq \lambda$ and $|u| < \kappa$ for all $u \in \mathcal{U}$, then there is $\mathcal{U}^* \subseteq \mathcal{U}$ s.t. $|\mathcal{U}^*| = \lambda$ and $\mathcal{U}^*$ is a Δ-system.

Notation around order types

Def

- If u is a set of ordinals and $\eta < \mathrm{otp}(u)$, then

  $u(\eta) =$ the unique $\alpha \in u$ s.t. $\mathrm{otp}(u \cap \alpha) = \eta$

- If $\bar{m} \subseteq \mathrm{otp}(u)$, then

  $u[\bar{m}] = \{u(\eta) \mid \eta \in \bar{m}\}$

- If $\langle u_i \mid i \in I \rangle$ is a sequence of sets of ordinals, and $w = \bigcup_{i \in I} u_i$, then

  $\mathrm{tp}(\langle u_i \mid i \in I \rangle) : \mathrm{otp}(w) \to \mathcal{P}(I)$

  $\eta \mapsto \{i \in I \mid w(\eta) \in u_i\}$

Uniform Δ-systems

Def A family $\mathcal{U}$ of sets of ordinals is a uniform Δ-system if there is a set r s.t. for all distinct $u, v \in \mathcal{U}$

- $u \cap v = r$
- $tp(\langle u, r\rangle) = tp(\langle v, r\rangle)$
  $(otp(u) = otp(v))$

Note The classical Δ-system lemmas yield uniform Δ-systems.

Uniform Δ-systems and Cohen forcing

We think of $\mathrm{Add}(\omega,\theta)$ as the poset of finite partial functions $p:\theta \dashrightarrow 2$. Given $p \in \mathrm{Add}(\omega,\theta)$, define $\bar{p}:\mathrm{otp}(\mathrm{dom}(p)) \to 2$ by

$$k \mapsto p(\mathrm{dom}(p)(k))$$

Prop Suppose that $\langle p_i \mid i \in I \rangle$ is a sequence from $\mathrm{Add}(\omega,\theta)$ s.t.

- $\langle \mathrm{dom}(p_i) \mid i \in I \rangle$ is a uniform Δ-system
- $\underline{\bar{p}_i = \bar{p}_j}$ for all $i,j \in I$.

Then $p_i \parallel p_j$ for all $i,j \in I$.

Chain conditions

Cor $\mathrm{Add}(\omega, \theta)$ is $\aleph_1$-Knaster, i.e. if $\langle p_\alpha \mid \alpha < \omega_1 \rangle$ is a sequence of conditions in $\mathrm{Add}(\omega, \theta)$, then there is an unbounded $H \subseteq \omega_1$ s.t. $\langle p_\alpha \mid \alpha \in H \rangle$ consists of pairwise compatible conditions

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"Dream for a higher-dimensional version"

If $2 \leq n < \omega$, $\lambda \leq \theta$ is suft. large and $\langle p_a \mid a \in [\lambda]^n \rangle \subseteq \mathrm{Add}(\omega, \theta)$ then there is a "large" $H \subseteq \lambda$ s.t. $\langle p_a \mid a \in [H]^n \rangle$ consists of pairwise compatible conditions.

This is an impossible dream,
even for $n=2$:

Define $\langle p_{\alpha\beta} \mid \alpha<\beta<\theta \rangle$ by

- $\operatorname{dom}(p_{\alpha\beta}) = \{\alpha, \beta\}$
- $p_{\alpha\beta}(\alpha) = 0$
- $p_{\alpha\beta}(\beta) = 1$

Then $p_{\alpha\beta} \perp p_{\beta\gamma}$ for all $\alpha<\beta<\gamma$.

Double Δ-systems

$2^{\aleph_0} \xrightarrow{\text{Con}} (2^{\aleph_0}, \alpha)^2$ for all $\alpha < \omega_1$.

Def (Todorcevic) Let H be a set of ordinals. A sequence $\langle u_{\alpha\beta} \mid (\alpha,\beta) \in [H]^2 \rangle$ is a **double Δ-system** if

- For all $\alpha \in H$, $\langle u_{\alpha\beta} \mid \beta \in H \setminus (\alpha+1) \rangle$ is a Δ-system w/ root u_α^+
- For all $\beta \in H$, $\langle u_{\alpha\beta} \mid \alpha \in H \cap \beta \rangle$ is a Δ-system w/ root u_β^-
- $\langle u_\alpha^+ \mid \alpha \in H \rangle$ and $\langle u_\beta^- \mid \beta \in H \rangle$ are Δ-systems with the same root, $u_\emptyset$.

$u_\delta^- \setminus u_\emptyset$ $\quad u_{\alpha\delta} \setminus (u_\alpha^+ \cup u_\delta^-)$ $\quad u_{\beta\delta} \setminus (u_\beta^+ \cup u_\delta^-)$ $\quad u_{\gamma\delta} \setminus (u_\gamma^+ \cup u_\delta^-)$

$u_\gamma^- \setminus u_\emptyset$ $\quad u_{\alpha\gamma} \setminus (u_\alpha^+ \cup u_\gamma^-)$ $\quad u_{\beta\gamma} \setminus (u_\beta^+ \cup u_\gamma^-)$

$u_\beta^- \setminus u_\emptyset$ $\quad u_{\alpha\beta} \setminus (u_\alpha^+ \cup u_\beta^-)$

$u_\alpha^- \setminus u_\emptyset$

$u_\emptyset$ $\quad u_\alpha^+ \setminus u_\emptyset$ $\quad u_\beta^+ \setminus u_\emptyset$ $\quad u_\gamma^+ \setminus u_\emptyset$ $\quad u_\delta^+ \setminus u_\emptyset$

$u_{\alpha\beta}$

Uniform double Δ-systems

"**Def**" A **uniform double Δ-system** is a double Δ-system $\langle u_{\alpha\beta} \mid (\alpha,\beta) \in [H]^2 \rangle$ such that

- each $u_{\alpha\beta}$ is a set of ordinals
- for all $(\alpha,\beta), (\gamma,\delta) \in [H]^2$,

$$tp(u_{\alpha\beta}, u^+_\alpha, u^-_\beta, u_\emptyset) = tp(u_{\gamma\delta}, u^+_\gamma, u^-_\delta, u_\emptyset)$$

- if $\{\alpha,\beta\} \cap \{\gamma,\delta\} = \emptyset$, then $u_{\alpha\beta} \cap u_{\gamma\delta} = u_\emptyset$,

Aligned sets

Def Suppose that a and b are sets of ordinals.

(1) We say that a and b are ***aligned*** if $\operatorname{otp}(a) = \operatorname{otp}(b)$ and, for all $\gamma \in a \cap b$, $\operatorname{otp}(a \cap \gamma) = \operatorname{otp}(b \cap \gamma)$

Ex
- $\{1, 3, \omega, \omega_2\}, \{2, 3, \omega, \omega_1\}$ ✓
- $\{1, 3, \omega, \omega_2\}, \{1, 2, 3, \omega_2\}$ ✗

(2) If a and b are aligned, then $\bar{r}(a,b) := \{ i < \operatorname{otp}(a) \mid a(i) = b(i) \}$

Note $a \cap b = a[\bar{r}(a,b)] = b[\bar{r}(a,b)]$

Uniform double Δ-systems + Cohen forcing

Prop Suppose that $\langle p_{\alpha\beta} \mid (\alpha,\beta) \in [H]^2 \rangle$ is a sequence of conditions in $\mathrm{Add}(\omega, \theta)$ such that

- $\langle \mathrm{dom}(p_{\alpha\beta}) \mid (\alpha,\beta) \in [H]^2 \rangle$ is a uniform double Δ-system
- $\bar{p}_{\alpha\beta} = \bar{p}_{\gamma\delta}$ for all $(\alpha,\beta),(\gamma,\delta) \in [H]^2$

Then $p_{\alpha\beta} \parallel p_{\gamma\delta}$ for all $(\alpha,\beta),(\gamma,\delta) \in [H]^2$ s.t. $\{\alpha,\beta\}$ and $\{\gamma,\delta\}$ are aligned.

Uniform n-dimensional Δ-systems

Def Suppose that H is a set of ordinals, $0<n<\omega$, and, for each $b \in [H]^n$, u_b is a set of ordinals. Then $\langle u_b \mid b \in [H]^n \rangle$ is a **uniform n-dimensional Δ-system** if there are an ordinal ρ and, for each $\overline{m} \subseteq n$, a set $\overline{r}_{\overline{m}} \subseteq \rho$ s.t.

(1) $\mathrm{otp}(u_b) = \rho$ for all $b \in [H]^n$

(2) for all $a, b \in [H]^n$, if a and b are aligned and $\overline{r}(a,b) = \overline{m}$, then u_a and u_b are aligned and $\overline{r}(u_a, u_b) = \overline{r}_{\overline{m}}$.

(3) for all $\overline{m}_0, \overline{m}_1 \subseteq n$ $\overline{r}_{\overline{m}_0 \cap \overline{m}_1} = \overline{r}_{\overline{m}_0} \cap \overline{r}_{\overline{m}_1}$

Uniform n-dimensional Δ-systems + Cohen forcing

Prop Suppose that $\langle p_b \mid b \in [H]^n \rangle$ is a sequence of conditions in $\mathrm{Add}(\omega, \theta)$ such that

- $\langle \mathrm{dom}(p_b) \mid b \in [H]^n \rangle$ is a uniform n-dimensional Δ-system
- $\bar{p}_a = \bar{p}_b$ for all $a, b \in [H]^n$.

Then $p_a \parallel p_b$ for all aligned $a, b \in [H]^n$

Def If λ is an infinite regular cardinal, recursively define $\sigma(\lambda, n)$ for $1 \leq n < \omega$ by letting

- $\sigma(\lambda, 1) = \lambda$
- $\sigma(\lambda, n+1) = \left(2^{<\sigma(\lambda,n)}\right)^+$

Higher-dimensional Δ-system lemma

Lemma Suppose that

- $1 \le n < \omega$
- $\kappa < \lambda$ are infinite cardinals, λ is regular, $\nu^{<\kappa} < \lambda$ for all $\nu < \lambda$, and $\mu = \sigma(\lambda, n)$
- for all $b \in [\mu]^n$, $u_b \in [On]^{<\kappa}$
- $g : [\mu]^n \to 2^{<\kappa}$

Then there is $H \in [\mu]^\lambda$ s.t.

(1) $\langle u_b \mid b \in [H]^n \rangle$ is a uniform n-dim. Δ-system

(2) $g \restriction [H]^n$ is constant.

Moreover ...

Pf Sketch Induction on n

$n=1$: Classical Δ-system lemma
+ pigeonhole principle

Fix $n>1$, and assume all instances of the lemma for $n-1$.

Let θ be a suff. large regular cardinal and fix

$$M \prec (H(\theta), \in, \lhd, g, \langle u_b \mid b \in [\mu]^n \rangle)$$

s.t.
- ${}^{<\sigma(\lambda,n-1)}M \subseteq M$
- $\mu_M := M \cap \mu \in \mu$

(possible since $\sigma(\lambda, n-1)$ is regular and $\mu = \sigma(\lambda, n) = (2^{<\sigma(\lambda,n-1)})^+$).

Recursively build an increasing sequence $\langle \alpha_\eta \mid \eta < \sigma(\lambda, n-1) \rangle$ of ordinals in M such that, for every $\xi < \sigma(\lambda, n-1)$, letting

$A_\xi := \{\alpha_\eta \mid \eta < \xi\}$,

$\langle u_{a^\frown \langle \alpha_\xi \rangle} \mid a \in [A_\xi]^{n-1} \rangle$ "looks like"

$\langle u_{a^\frown \langle \mu_M \rangle} \mid a \in [A_\xi]^{n-1} \rangle$, i.e.,

- $\mathrm{tp}(\langle u_{a^\frown \langle \alpha_\xi \rangle} \mid a \in [A_\xi]^{n-1} \rangle) = \mathrm{tp}(\langle u_{a^\frown \langle \mu_M \rangle} \mid a \in [A_\xi]^{n-1} \rangle)$
- $u_{a^\frown \langle \alpha_\xi \rangle} \cap M = u_{a^\frown \langle \mu_M \rangle} \cap M$ for all a
- $g(a^\frown \langle \alpha_\xi \rangle) = g(a^\frown \langle \mu_M \rangle)$ for all a

$\vdots$

Apply the inductive hypothesis to

$\langle u_{a^\frown\langle \mu_M \rangle} \mid a \in [A]^{n-1} \rangle$ to find

$H_0 \in [A]^\lambda$ s.t.

- $\langle u_{a^\frown\langle \mu_M \rangle} \mid a \in [H_0]^{n-1} \rangle$ is a uniform $(n-1)$-dimensional Δ-system, as witnessed by p and $\langle \bar{s}_{\bar{m}} \subseteq p \mid \bar{m} \subseteq n-1 \rangle$

- g is constant on $\langle u_{a^\frown\langle \mu_M \rangle} \mid a \in [H_0]^{n-1} \rangle$ with value k

- $a \mapsto \{i \mid u_{a^\frown\langle \mu_M \rangle}(i) \in M\}$ is constant on $[H_0]^{n-1}$, with value $\bar{i}$

Now g is constant on $[H_0]^n$, since, by construction,

$g(b) = g(b \upharpoonright (n-1) ^\frown \langle \mu_M \rangle) = k$

for all $b \in [H_0]^n$.

Now argue that we can thin out H_0 to an unbounded $H \subseteq H_0$ s.t. $\langle u_b \mid b \in [H]^n \rangle$ is a uniform n-dimensional Δ-system, as witnessed by p and $\langle \bar{r}_{\bar{m}} \mid \bar{m} \subseteq n \rangle$, where

$$\bar{r}_{\bar{m}} = \begin{cases} \bar{s}_{\bar{m} \cap (n-1)} & \text{if } n-1 \in \bar{m} \\ \bar{s}_{\bar{m}} \cap \bar{\iota} & \text{if } n-1 \notin \bar{m} \end{cases}$$

$\boxed{\text{M}}$

Polarized Partition Relations

Def Let $2 \leq n < \omega$. Then Θ_n is the least cardinal θ s.t. for every function $f : \theta^n \to \omega$ there is a sequence $\langle A_i \mid i < n \rangle$ of infinite subsets of θ s.t. $f \upharpoonright \prod_{i<n} A_i$ is constant.

Facts

- $\Theta_n \leq \beth_{n-1}^{+}$ (Erdős-Rado)
- $\Theta_n \geq \aleph_n$
- Consistently $\Theta_2 = \aleph_3$

Thm Fix $2 \leq n < \omega$ and an infinite cardinal χ, and let $\mathbb{P} = \mathrm{Add}(\omega, \chi)$. Then, in $V^{\mathbb{P}}$, $\Theta_n \leq (\beth_{n-1}^{+})^{V}$

Pf Sketch Let $\mu = (\beth_{n-1}^{+})^{V}$.

Fix $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{c} : [\mu]^n \to \omega$. We will find $q \leq p$ and names $\langle \dot{A}_i \mid i < n \rangle$ s.t.

$q \Vdash$ " $\dot{A}_0 < \dot{A}_1 < \dots < \dot{A}_{n-1}$, $\dot{A}_i \subseteq \mu$ is infinite, and $\dot{c} \restriction \prod \dot{A}_i$ is constant"

For each $b \in [\mu]^n$, fix $q_b \le p$ deciding the value of $\dot{c}(b)$ as some $k_b < \omega$. Note $\beth^+_{n-1} = \sigma(\lambda_1, n)$, so we can apply the Lemma to obtain $H \in [\mu]^{\lambda_1}$ ($\mathrm{otp}(H) = \omega_1$) s.t.

- $\langle \mathrm{dom}(q_b) \mid b \in [H]^n \rangle$ is a uniform n-dim. Δ-system
  ($u_b = \mathrm{dom}(q_b)$)

- the function $b \mapsto (k_b, \bar{q}_b)$ is constant on $[H]^n$, taking value $(k, \bar{q})$.

Let $p = |\bar{q}|$ and $\langle \bar{r}_{\bar{m}} \mid \bar{m} \leq n \rangle$ witness that $\langle u_b \mid b \in [H]^n \rangle$ is a uniform n-dim Δ-system.

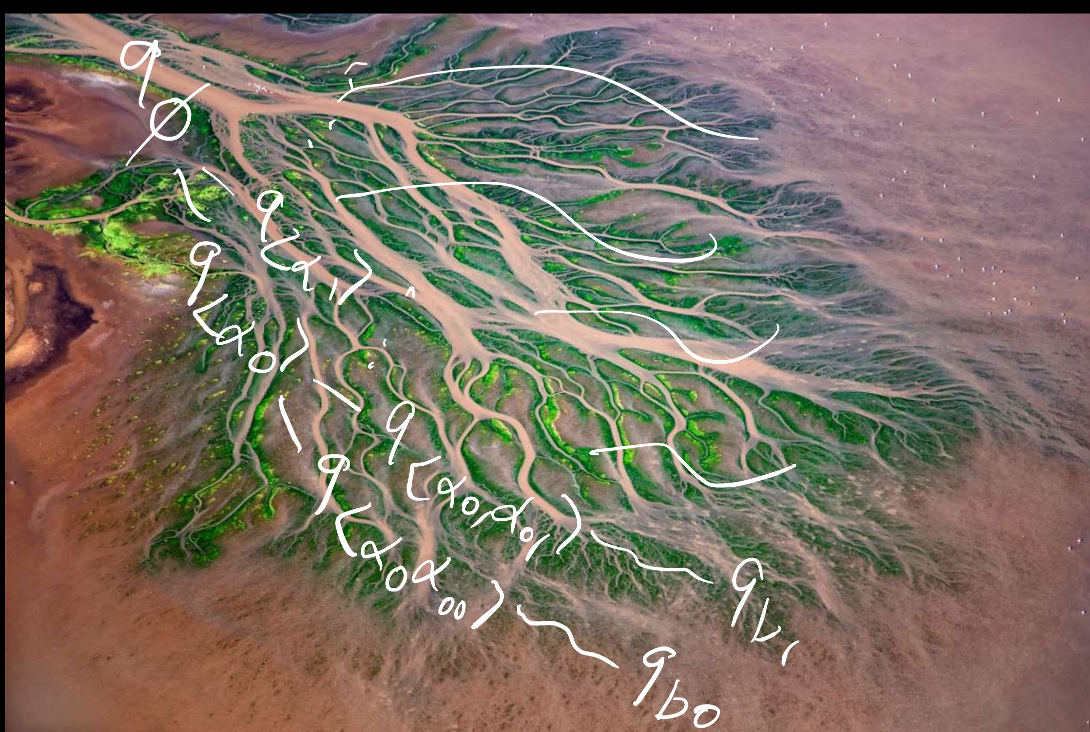
For each $m < n$ and $a \in [H]^m$, define u_a by fixing $b \in [H]^n$ s.t. $b[m] = a$ and letting $u_a = u_b[\bar{r}_m]$ and then let $q_a = q_b \restriction u_a$. Since $\langle u_b \mid b \in [H]^n \rangle$ is a n-Δ-system and $\overline{q_b} = \bar{q}$ for all $b \in [H]^n$, these assignments are independent of choice of b.

<u>Note</u> • For all $a \in [H]^{<n}$

$\langle u_{a^\frown\langle\beta\rangle} \mid \beta \in H \setminus (\max(a)+1)\rangle$

is a Δ-system with root u_a.

• As a result, if G is $\mathbb{P}$-generic and $q_a \in G$, then, by genericity, there are unbdd'ly many $\beta \in H$ s.t. $q_{a^\frown\langle\beta\rangle} \in G$.



Now let G be $\mathbb{P}$-generic with $q_\emptyset \in G$. Work in $V[G]$.

Apply the observation above n times to find $\delta_0 < \delta_1 < \dots < \delta_{n-1}$ from H s.t., letting $d = \{\delta_0, \dots, \delta_{n-1}\}$, then

- $q_d \in G$
- $H \cap \delta_0$ is infinite
- $H \cap (\delta_{m+1} \setminus \delta_m)$ is infinite for all $m < n-1$.

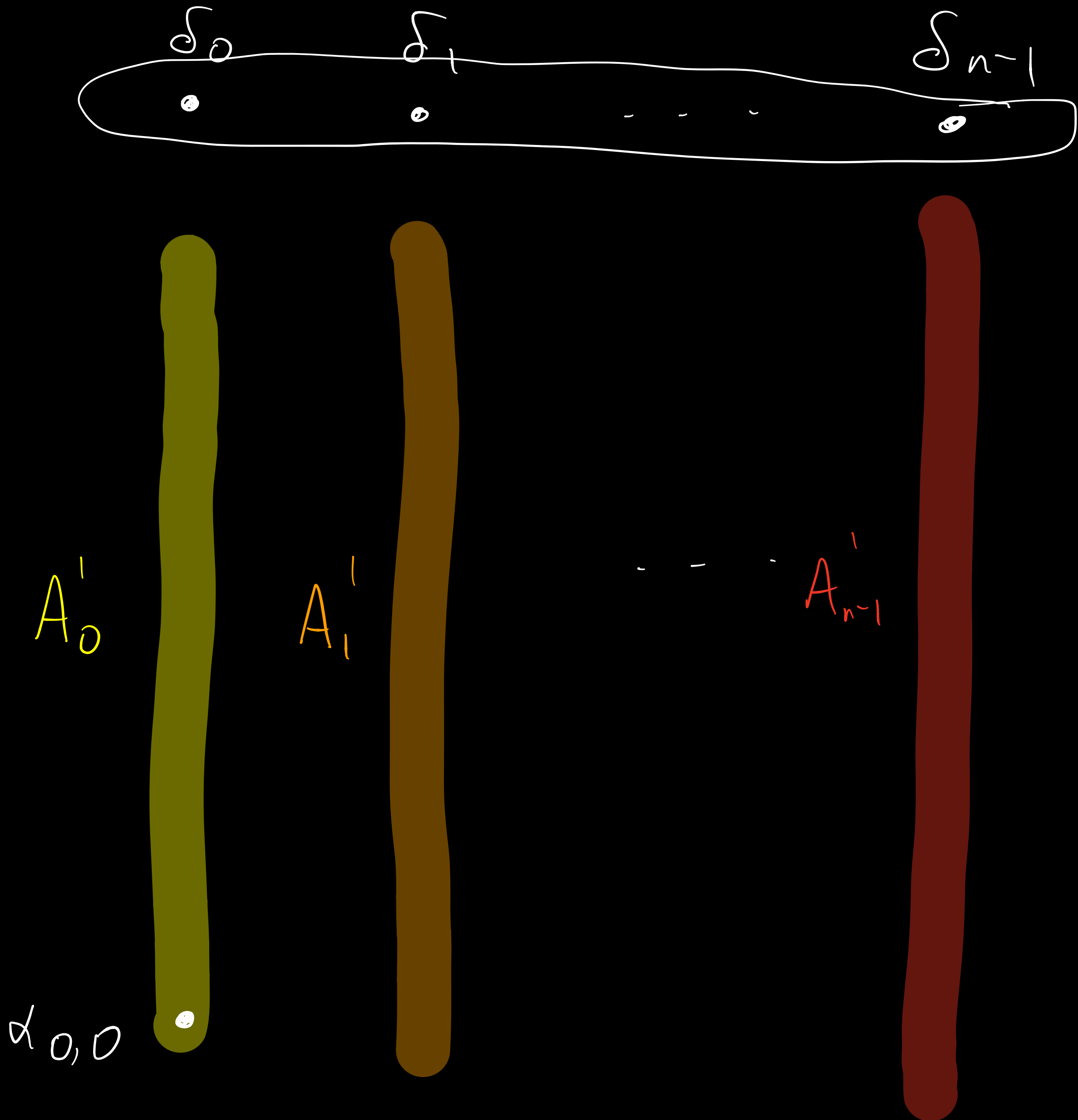
Let A_0^1 = first ω-many elements of $H \cap \delta_0$

A_{m+1}^1 = first ω-many elements of $H \cap (\delta_{m+1} \setminus \delta_m)$

By recursion on the anti-lex order on $n \times \omega$, construct a matrix

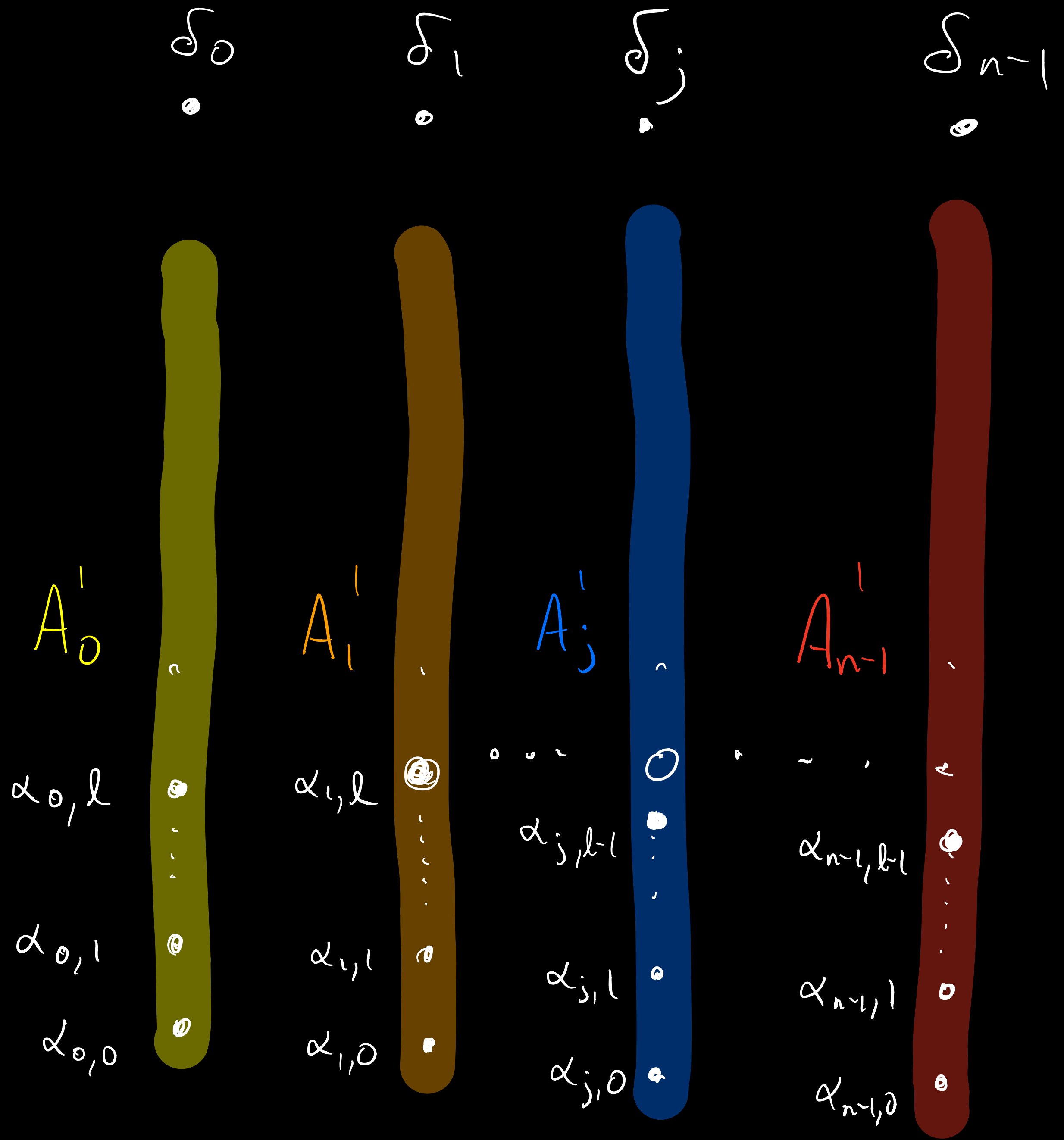
$\langle \alpha_{m,l} \mid m<n,\ l<\omega \rangle$ s.t.

- for all $m<n$ $\langle \alpha_{m,l} \mid l<\omega \rangle$ is an increasing sequence from A_m'
- letting $A_m = \{\alpha_{m,l} \mid l<\omega\} \cup \{\delta_m\}$, for all $b \in \prod_{m<n} A_m$, we have $q_b \in G$ and hence $c(b) = k$



First step: $\langle u_{\langle \alpha, \delta_1, \dots, \delta_{n-1} \rangle} \mid \alpha \in A_0' \rangle$ is a Δ-system with root r and $q_{\langle \alpha, \delta_1, \dots, \delta_{n-1} \rangle} \restriction r \in G$

So, by genericity, we can find $\alpha_{0,0} \in A_0'$ s.t. $q_{\langle \alpha_0, \delta_1, \dots, \delta_{n-1} \rangle} \in G$.



Step (j, l): For all $b \in \prod_{m<n} A_m$ (as so far defined), $\langle u_{b[j\mapsto\alpha]} \mid \alpha \in A_j \rangle$ is a Δ-system, so by genericity, there is $\alpha_{j,l}$ as desired. ■ Thm

Recent Applications

Coherent, nontrivial n-dimensional families of functions

Def
- Given $f \in {}^\omega\omega$, $I(f) = \{(i,j) \in \omega^2 \mid j \leq f(i)\}$
- Given $n \geq 2$, a family of functions

$$\Phi = \langle \varphi_{\vec{f}} : I(\wedge \vec{f}) \to \mathbb{Z} \mid \vec{f} \in ({}^\omega\omega)^n \rangle \text{ is}$$

- **n-coherent** if it is alternating and $\sum_{i=0}^{n} (-1)^i \varphi_{\vec{f}^i} =^* 0$ for all $\vec{f} \in ({}^\omega\omega)^{n+1}$
- **n-trivial** if there is an alternating $\Psi = \langle \psi_{\vec{f}} : I(\wedge \vec{f}) \to \mathbb{Z} \mid \vec{f} \in ({}^\omega\omega)^{n-1} \rangle$ s.t. $\varphi_{\vec{f}} =^* \sum_{i=0}^{n-1} (-1)^i \psi_{\vec{f}^i}$ for all $\vec{f} \in ({}^\omega\omega)^n$.

The existence of n-coherent, non-n-trivial families of functions entails the non-additivity of strong homology, even on closed subsets of Euclidean space.

<u>Thm</u>* (Bergfalk-Hrušák-LH)

Let $\mathbb{P} = \mathrm{Add}(\omega, \beth_\omega)$. Then, in $V^{\mathbb{P}}$, for all $n \geq 1$, every n-coherent family of functions is n-trivial.

Additive partition relations

Def For $r<\omega$, $\mathbb{R} \to^+ (\aleph_0)_r$ is the assertion that for every $c : \mathbb{R} \to r$, there is an infinite $X \subseteq \mathbb{R}$ s.t. $c \restriction (X+X)$ is constant.

Thm (Hindman, Leader, Strauss '17)

If $2^{\aleph_0} < \aleph_\omega$, then there is $r<\omega$ s.t. $\mathbb{R} \not\to^+ (\aleph_0)_r$

Thm (Zhang '20)

- $\mathbb{R} \to^+ (\aleph_0)_2$ holds in ZFC
- In $V^{\mathrm{Add}(\omega, \beth_\omega)}$, then $\mathbb{R} \to^+ (\aleph_0)_r$ for all r.

The proof proceeds by identifying $\mathbb{R}$ with $\bigoplus_{\alpha<2^\omega} \mathbb{Q}$ and actually proves that, in $V^{\mathrm{Add}(\omega, \beth_\omega)}$,

$$\bigoplus_{\alpha<(\beth_\omega)^V} \mathbb{N} \to^+ (\aleph_0)_r \quad \text{for all } r.$$

Moreover, for a fixed r, it suffices to consider elements of $\bigoplus_{\alpha<(\beth_\omega)^V} \mathbb{N}$ with support of size $\leq 2r$. In this way, $2r$-dimensional Δ-systems come into play.

Thank You!