Generalized almost disjoint families and injective Banach spaces

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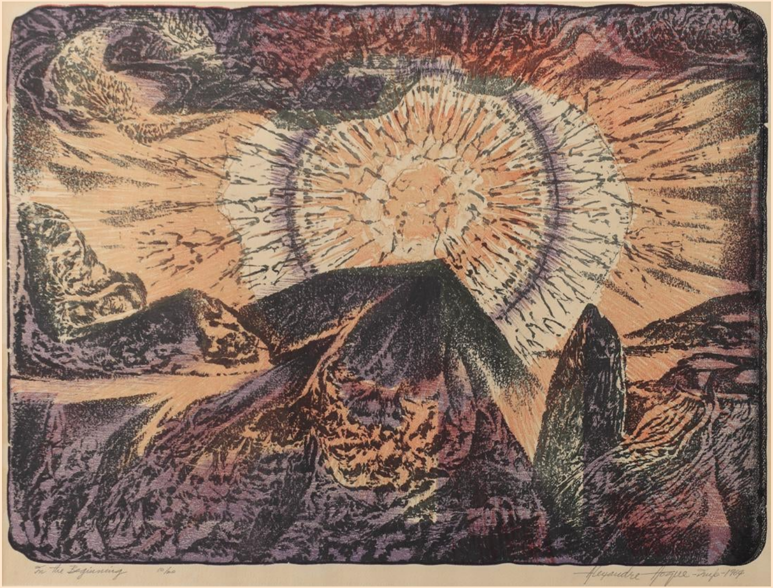
I. Injective Banach spaces



Injective Banach spaces

Definition

A Banach space E is *injective* if, for every Banach space X and every subspace $Y \subset X$, every continuous linear map $T : Y \to E$ extends to a continuous linear map $T' : X \to E$.

Recall that, given a set Γ, we can form the following spaces.

- $\ell^\infty(\Gamma)$ is the space of all bounded functions $f : \Gamma \to \mathbb{R}$.
- $c_0(\Gamma)$ is the space of all functions $f : \Gamma \to \mathbb{R}$ such that, for all $\varepsilon > 0$, the set
$$\{\gamma \in \Gamma \mid |f(\gamma)| > \varepsilon\}$$
is finite.

$$\ell^\infty = \ell^\infty(\mathbb{N}), \quad c_0 = c_0(\mathbb{N})$$

Fact

A Banach space E is injective iff it is a complemented subspace of $\ell^\infty(\Gamma)$ for some set Γ, i.e., $\ell^\infty(\Gamma) \cong E \bigoplus X$ for some space X.

Injective dimension

The category of Banach spaces has *enough* injective objects: every Banach space embeds as a closed subspace of an injective Banach space. Given a Banach space X, we can then form an *injective resolution* of X, i.e., an exact sequence of maps

$$0 \to X \to E_0 \to E_1 \to E_2 \to \cdots$$

such that each E_i is injective. (Recall that *exactness* is the requirement that the image of each map equals the kernel of the following map.) The *injective dimension* of X is the least $i \in \mathbb{N}$ such that there exists an injective resolution of the form

$$0 \to X \to E_0 \to \cdots \to E_i \to 0 \to \cdots$$

or ∞ if no such i exists.

Question

What is the injective dimension of c_0?

Proposition (Phillips)

c_0 is not injective.

Proof sketch.

Consider the short exact sequence

$$0 \to c_0 \xrightarrow{\iota} \ell^\infty \xrightarrow{\pi} \ell^\infty / c_0 \to 0.$$

If c_0 were injective, this sequence would *split*, i.e., there would be a continuous linear map $\sigma : \ell^\infty / c_0 \to \ell^\infty$ such that $\pi \circ \sigma = \mathrm{id}$. Such a map σ would select an element from each equivalence class in ℓ^∞ / c_0.

Proof sketch (cont.)

Let $\mathcal{A} \subseteq [\omega]^\omega$ be an uncountable almost disjoint family. For each $A \in \mathcal{A}$, let 1_A be its characteristic function. Then, for all finite nonempty $\mathcal{B} \subseteq \mathcal{A}$,

$$\left\| \sum_{A \in \mathcal{B}} [1_A] \right\|_{\ell^\infty / c_0} = 1.$$

For each $A \in \mathcal{A}$, fix $m_A \in A$ such that $\sigma([1_A])(m_A) > 0.99$. Find an $m \in \omega$ and an uncountable $\mathcal{A}' \subseteq \mathcal{A}$ such that $m_A = m$ for all $A \in \mathcal{A}'$. Now, for all finite nonempty $\mathcal{B} \subseteq \mathcal{A}'$, we have

$$\sigma\left(\sum_{A \in \mathcal{B}} [1_A] \right)(m) > 0.99|\mathcal{B}|.$$

Thus, σ takes elements of the unit ball of ℓ^∞ / c_0 to elements of ℓ^∞ of arbitrarily high norm, contradicting the fact that σ is continuous (and hence bounded). $\square$

One more step

We can do better, using the following theorem of Rosenthal.

Theorem (Rosenthal)

If E is an injective Banach space, Γ is a set, and E contains a subspace isomorphic to $c_0(\Gamma)$, then it contains a subspace isomorphic to $\ell^\infty(\Gamma)$.

Corollary (Amir)

ℓ^∞/c_0 is not injective.

Proof.

In the notation of the previous proof, $\{[1_A] \mid A \in \mathcal{A}\}$ generates a copy of $c_0(\mathcal{A})$ in ℓ^∞/c_0. We can take $|\mathcal{A}| = 2^{\aleph_0}$. If ℓ^∞/c_0 were injective, it would then contain a copy of $\ell^\infty(2^{\aleph_0})$, but it is too small for this. $\square$

Injective dimension can be reformulated as follows. Given *any* injective resolution

$$0 \xrightarrow{\iota_{-1}} X = E_{-1} \xrightarrow{\iota_0} E_0 \xrightarrow{\iota_1} E_1 \xrightarrow{\iota_2} \cdots,$$

the injective dimension of X is the least i such that $E_{i-1}/\mathrm{im}(\iota_{i-1})$ is injective.

Corollary

The injective dimension of c_0 is at least 2.

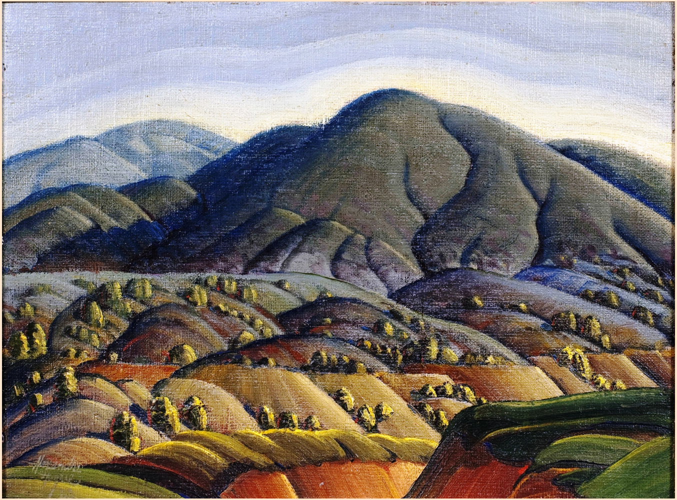
Proof.

c_0 has an injective resolution beginning

$$0 \to c_0 \to \ell^\infty \to \cdots.$$

□

II. Generalized almost disjoint families



Generalized AD families

Definition

Let K be a topological space (usually compact, Hausdorff, totally disconnected, not extremally disconnected). A *generalized almost disjoint family in* K is a collection $\mathcal{A}$ of subsets of K such that:

1. For all finite, nonempty $\mathcal{B} \subseteq \mathcal{A}$, the set
$$\bigcup \mathcal{B}$$
is a regular open subset of K that is not closed.
2. For all distinct $A, B \in \mathcal{A}$, the set $A \cap B$ is closed.

If $K = \omega + 1$, then a generalized almost disjoint family in K is simply a classical (nontrivial) almost disjoint family $\mathcal{A} \subseteq [\omega]^\omega$.

Function spaces

If K is a compact Hausdorff space, then $C(K)$ is the Banach space of continuous functions from K to $\mathbb{R}$.

Fact

If K is extremally disconnected, then $C(K)$ is injective.

If D is a dense subset of K, then $C(K)$ embeds as a closed subspace of $\ell^\infty(D)$. This leads us to be interested in quotient spaces of the form

$$\ell^\infty(D)/C(K).$$

Let K be a compact Hausdorff space and D a dense subset of K.

Theorem (LH-Schrittesser)

If there is a generalized AD family in K of cardinality κ, then $\ell^\infty(D)/C(K)$ contains a subspace isomorphic to $c_0(\kappa)$.

Proof sketch.

Let $\mathcal{A}$ be a generalized AD family of cardinality κ. Then

$$\{[1_A \upharpoonright D] \mid A \in \mathcal{A}\}$$

generates a copy of $c_0(\kappa)$ inside $\ell^\infty(D)/C(K)$. □

Corollary

If K contains a generalized AD family $\mathcal{A}$ such that $2^{|\mathcal{A}|} > 2^{|D|}$, then $\ell^\infty(D)/C(K)$ is not injective, so the injective dimension of $C(K)$ is at least 2.

Recall that $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ is the Čech-Stone remainder of $\mathbb{N}$. Concretely, this is the space of nonprincipal ultrafilters on ω.

Theorem (LH-Schrittesser)

If CH *holds (or just* $\mathfrak{b} = \mathfrak{c}$*), then* $\mathbb{N}^*$ *contains a generalized AD family of cardinality* $2^{\aleph_1}$*.*

Proof sketch.

Build a sequence $\langle a_x \mid x \in {}^{<\omega_1}2\rangle$ of elements of $[\omega]^\omega$ such that

1. for all $x \sqsubseteq y \in {}^{<\omega_1}2$, $a_x \subsetneq^* a_y$;
2. for all incompatible $x, y \in {}^{<\omega_1}2$, $a_x \cap a_y =^* a_{x \wedge y}$;
3. for all $b \in [\omega]^\omega$, either
    1. $b \subseteq^* a_{x_0} \cup \ldots \cup a_{x_n}$ for some $x_0, \ldots, x_n \in {}^{<\omega_1}2$; or
    2. for all $x_0, \ldots, x_n \in {}^{<\omega_1}2$, there is $y \in {}^{<\omega_1}2$ incompatible with each x_i such that $|a_y \cap (b \setminus (a_{x_0} \cup \ldots \cup a_{x_n})| = \aleph_0$.

Proof sketch (cont.)

Now every branch $f \in {}^{\omega_1}2$ through ${}^{<\omega_1}2$ determines a $\subsetneq^*$-increasing sequence $\langle a_{f\restriction\alpha} \mid \alpha < \omega_1 \rangle$ of elements of $[\omega]^\omega$. Let A_f be the collection of all $\mathcal{U} \in \mathbb{N}^*$ for which there exists $\alpha < \omega_1$ such that $a_{f\restriction\alpha} \in \mathcal{U}$. Then $\{A_f \mid f \in {}^{\omega_1}2\}$ is a generalized almost disjoint family. $\square$

Back to c_0

Fact

ℓ^∞/c_0 is isomorphic to $C(\mathbb{N}^*)$.

Also, $\mathbb{N}^*$ has a dense subset D of cardinality $2^{\aleph_0}$. Therefore, c_0 has an injective resolution beginning

$$0 \to c_0 \xrightarrow{\iota_0} \ell^\infty \xrightarrow{\iota_1} \ell^\infty(2^{\aleph_0}) \to \cdots$$

such that

$$\ell^\infty(2^{\aleph_0})/\mathrm{im}(\iota_1) \cong \ell^\infty(D)/C(\mathbb{N}^*).$$

Thus, if $\ell^\infty(D)/C(\mathbb{N}^*)$ is not injective, then the injective dimension of c_0 is at least 3.

Theorem

If CH *holds, then the injective dimension of c_0 is at least 3.*

Proof.

If CH holds, then $\mathbb{N}^*$ contains a generalized AD family of size $2^{\aleph_1}$. Thus, $\ell^\infty(D)/C(\mathbb{N}^*)$ contains a copy of $c_0(2^{\aleph_1})$. If $\ell^\infty(D)/C(\mathbb{N}^*)$ were injective, then it would contain a copy of $\ell^\infty(2^{\aleph_1})$, but it is too small for this, since

$$|\ell^\infty(2^{\aleph_1})| = 2^{2^{\aleph_1}}$$

but

$$|\ell^\infty(D)/C(\mathbb{N}^*)| = 2^{2^{\aleph_0}}.$$

□

Questions

Question

Can the cardinal arithmetic assumptions be removed from these results?

Question

Is the injective dimension of c_0 infinite?

Questions

Question

Is $\mathrm{Ext}^2(c_0(\aleph_1), c_0) \neq 0$ *in* ZFC*?*

This question reduces to the following: Is there a continuous linear map from $c_0(\aleph_1)$ to $\ell^\infty(\mathbb{N}^*)/C(\mathbb{N}^*)$ that does not lift to a continuous linear map from $c_0(\aleph_1)$ to $\ell^\infty(\mathbb{N}^*)$?

The earlier construction using an uncountable AD family shows that $\mathrm{Ext}^1(c_0(\aleph_1), c_0) \neq 0$. On the other hand, $\mathrm{Ext}^n(c_0, c_0) = 0$ for all $n \geq 1$.

Aviles et al. proved that, under CH, $\mathrm{Ext}^2(c_0(\aleph_1), c_0) \neq 0$. It is conceivable that $\mathrm{MA}_{\aleph_1}$ or PFA would yield the opposite conclusion.

Thank you!