Pseudo-Prikry sequences

(Joint and ongoing work with Spencer Unger)

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I: Historical background

Prikry forcing

Suppose κ is a measurable cardinal and U is a normal measure on κ. There is a forcing poset, which we denote $\mathbb{P}_U$, such that:

1. $\mathbb{P}_U$ is cardinal-preserving;
2. forcing with $\mathbb{P}_U$ adds an increasing sequence of ordinals, $\langle \gamma_i \mid i < \omega \rangle$, cofinal in κ;
3. $\langle \gamma_i \mid i < \omega \rangle$ diagonalizes U, i.e., for all $X \in U$, for all sufficiently large $i < \omega$, $\gamma_i \in X$.

$\mathbb{P}_U$ is known as *Prikry forcing* (with respect to U). There is now a large class of variations on Prikry forcing, known collectively as *Prikry-type forcings*, which add diagonalizing sequences to a large cardinal κ, to a set of the form $\mathcal{P}_\kappa(\lambda)$, or to a sequence of such objects.

Outside guessing of clubs

Sequences approximating Prikry sequences appear in abstract settings, as well. In these cases, we may not have a normal measure on the relevant cardinal, so we consider sub-filters of the club filter.

Theorem (Džamonja-Shelah, [3])

Suppose that:

1. *V is an inner model of W;*
2. *κ is an inaccessible cardinal in V and a singular cardinal of cofinality θ in W;*
3. $(\kappa^+)^W = (\kappa^+)^V$;
4. *$\langle C_\alpha \mid \alpha < \kappa^+ \rangle \in V$ is a sequence of clubs in κ.*

Then, in W, there is a sequence $\langle \gamma_i \mid i < \theta \rangle$ of ordinals such that, for all $\alpha < \kappa^+$ and all sufficiently large $i < \theta$, $\gamma_i \in C_\alpha$.

Generalized outside guessing of clubs

A similar theorem is proven by Gitik [4], and it is extended by Magidor and Sinapova [5], who also prove the following generalization.

Theorem (Magidor-Sinapova, [5])

Suppose that $n < \omega$ and:

1. *V is an inner model of W;*
2. *κ is a regular cardinal in V and, for all $m \leq n$, $(\kappa^{+m})^V$ has countable cofinality in W;*
3. *$(\kappa^+)^W = (\kappa^{+n+1})^V$;*
4. *$\langle D_\alpha \mid \alpha < \kappa^{+n+1} \rangle \in V$ is a sequence of clubs in $\mathcal{P}_\kappa(\kappa^{+n})$.*

Then, in W, there is a sequence $\langle x_i \mid i < \omega \rangle$ of elements of $(\mathcal{P}_\kappa(\kappa^{+n}))^V$ such that, for all $\alpha < \kappa^{+n+1}$ and all sufficiently large $i < \omega$, $x_i \in D_\alpha$.

Applications

Theorem (Cummings-Schimmerling in the context of Prikry forcing, [2])

Suppose that V is an inner model of W, κ is inaccessible in V and a singular cardinal of countable cofinality in W, and $(\kappa^+)^W = (\kappa^+)^V$.

Then $\square_{\kappa,\omega}$ holds in W.

Theorem (Brodsky-Rinot, [1])

Suppose that λ is a regular, uncountable cardinal, $2^\lambda = \lambda^+$, and $\mathbb{P}$ is a λ^+-c.c. forcing notion of size $\leq \lambda^+$. Suppose moreover that, in $V^{\mathbb{P}}$, λ is a singular ordinal and $|\lambda| > \mathrm{cf}(\lambda)$.

Then there is a λ^+-Souslin tree in $V^{\mathbb{P}}$.

II: Fat trees and pseudo-Prikry sequences

Fat trees

Definition

Suppose κ is a regular, uncountable cardinal, $n < \omega$, and, for all $m \leq n$, $\lambda_m \geq \kappa$ is a regular cardinal. Then

$$T \subseteq \bigcup_{k \leq n+1} \prod_{m<k} \kappa_m$$

is a *fat tree* of type $(\kappa, \langle \lambda_0, \ldots, \lambda_n \rangle)$ if:

1. for all $\sigma \in T$ and $\ell < \mathrm{lh}(\sigma)$, we have $\sigma \restriction \ell \in T$;
2. for all $\sigma \in T$ such that $k := \mathrm{lh}(\sigma) \leq n$, $\mathrm{succ}_T(\sigma) := \{\alpha \mid \sigma^\frown \langle \alpha \rangle \in T\}$ is $(< \kappa)$-club in κ_k.

Lemma

If $\mathcal{C}$ is a club in $\mathcal{P}_\kappa(\kappa^{+n})$, then there is a fat tree of type $(\kappa, \langle \kappa^{+n}, \kappa^{+n-1}, \ldots, \kappa \rangle)$ such that, for every maximal $\sigma \in T$, there is $x \in \mathcal{C}$ such that, for all $m \leq n$, $\sup(x \cap \kappa^{+m}) = \sigma(n-m)$.

Outside guessing of fat trees

Theorem

Suppose that:

1. *V is an inner model of W;*
2. *in V, $\kappa < \lambda$ are cardinals, with κ regular;*
3. *in W, $\theta < \theta^{+2} < |\kappa|$, θ is a regular cardinal, and there is a $\subseteq$-increasing sequence $\langle x_i \mid i < \theta \rangle$ from $(\mathcal{P}_\kappa(\lambda))^V$ such that $\bigcup_{i<\theta} x_i = \lambda$;*
4. *$(\lambda^+)^V$ remains a cardinal in W;*
5. *$n < \omega$ and, in V, $\langle \lambda_i \mid i \leq n \rangle$ is a sequence of regular cardinals from $[\kappa, \lambda]$ and $\langle T(\alpha) \mid \alpha < \lambda^+ \rangle$ is a sequence of fat trees of type $(\kappa, \langle \lambda_0, \ldots, \lambda_n \rangle)$.*

Then, in W, there is a sequence $\langle \sigma_i \mid i < \theta \rangle$ such that, for all $\alpha < \lambda^+$ and all sufficiently large $i < \theta$, σ_i is a maximal element of $T(\alpha)$.

Proof sketch ($n = 0$)

Our sequence of fat trees is just a sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ of clubs in λ_0. Let $X = (\mathcal{P}_\kappa(\lambda))^V$. If $f : X \to \lambda_0$ and $C \subseteq \lambda_0$ is unbounded, define $f^C : X \to \lambda_0$ by $f^C(x) = \min(C \setminus f(x))$.
Work first in V. Fix a sequence $\langle e_\beta \mid \beta < \lambda^+ \rangle$ such that $e_\beta : \beta \to \lambda$ is an injection.
Define a sequence $\vec{f} = \langle f_\beta \mid \beta < \lambda^+ \rangle$ of functions from X to λ_0 satisfying:

1. for all $\beta < \gamma < \lambda^+$ and all $x \in X$, if $e_\gamma(\beta) \in x$, then $f_\beta(x) < f_\gamma(x)$;
2. for all $\gamma \in S^{\lambda^+}_{<\kappa}$, there is a club D_γ in γ such that, for all $\beta \in D_\gamma$, $f_\beta < f_\gamma$;
3. for all $\alpha, \beta < \lambda^+$, there is $\gamma < \lambda^+$ such that $f_\beta^{C_\alpha} < f_\gamma$.

Proof sketch (cont.)

Move now to W, where we have $\langle x_i \mid i < \theta \rangle$. Define a sequence $\vec{g} = \langle g_\beta \mid \beta < \lambda^+ \rangle$ from θ to λ_0 by letting $g_\beta(i) = f_\beta(x_i)$. Note that:

1. $\vec{g}$ is $<^*$-increasing;
2. for all $\gamma \in S^{\lambda^+}_{>\theta}$, there is a club D_γ in γ such that, for all $\beta \in D_\gamma$, $g_\beta < g_\gamma$;
3. $\theta^{+3} < \lambda^+$.

Therefore, $\vec{g}$ has an *exact upper bound*, i.e. a $<^*$-upper bound h such that, for every $h' <^* h$, there is $\beta < \lambda^+$ such that $h' <^* g_\beta$. Moreover, we may assume $\mathrm{cf}(h)(i) > \theta$ for all $i < \theta$, so $h : \theta \to \lambda_0$. For $i < \theta$, let $\gamma_i = h(i)$. We claim that this works.

Proof sketch (cont.)

If not, then there is $\alpha < \lambda^+$ and an unbounded $A \subseteq \theta$ such that, for all $i \in A$, $\gamma_i \notin C_\alpha$. Define $h' : \theta \to \lambda_0$ by

$$h'(i) = \begin{cases} 0 & \text{if } i \notin A \\ \max(C_\alpha \cap \gamma_i) & \text{if } i \in A \end{cases}$$

$h' < h$, so there is $\beta < \lambda^+$ such that $h' <^* g_\beta$. But then there is $\gamma < \lambda^+$ such that $f_\beta^{C_\alpha} < f_\gamma$. Now, for all sufficiently large $i \in A$, we have

$$\max(C_\alpha \cap h(i)) < g_\beta(i) < h(i) < \min(C_\alpha \setminus g_\beta(i)) < g_\gamma(i).$$

In particular, h is *not* a $<^*$-upper bound for $\vec{g}$. Contradiction!

III: Diagonal sequences

Diagonal clubs

Definition

Suppose that θ is a regular cardinal and $\vec{\mu} = \langle \mu_i \mid i < \theta \rangle$ is an increasing sequence of regular cardinals.

1. A *diagonal club in* $\vec{\mu}$ is a sequence $\langle C_i \mid i < \theta \rangle$ such that, for all $i < \theta$, C_i is club in μ_i.
2. If $\kappa \leq \mu_0$ is a regular cardinal, then a *diagonal club in* $\mathcal{P}_\kappa(\vec{\mu})$ is a sequence $\langle D_i \mid i < \theta \rangle$ such that, for all $i < \theta$, D_i is club in $\mathcal{P}_\kappa(\mu_i)$.

Diagonal ordinal sequences

Theorem

Suppose that:

1. V *is an inner model of* W*;*
2. *in* V*,* μ *is a singular cardinal of cofinality* θ*;*
3. *there is* $\kappa < \mu$ *such that every* V*-regular cardinal in* $[\kappa, \mu)$ *has cofinality* θ *in* W*;*
4. *in* W*,* $(\mu^+)^V$ *remains a cardinal and* $\theta^{+2} < |\mu|$*.*

Then there are:

- *an increasing sequence of regular cardinals* $\vec{\mu} = \langle \mu_i \mid i < \theta \rangle \in V$*, cofinal in* μ*;*
- *a function* $g \in \prod_{i<\theta} \mu_i$ *in* W

such that, for every $\langle C_i \mid i < \theta \rangle \in V$ *that is a diagonal club in* $\vec{\mu}$*, for all sufficiently large* $i < \theta$*,* $g(i) \in C_i$*.*

Generalized diagonal sequences

Theorem

Suppose that:

1. V *is an inner model of* W*;*
2. *in* V, $\mathrm{cf}(\mu) = \theta < \kappa = \mathrm{cf}(\kappa) < \mu$ *are cardinals, with* μ *strong limit;*
3. *in* V, $\vec{\mu} = \langle \mu_i \mid i < \theta \rangle$ *is an increasing sequence of regular cardinals, cofinal in* μ, *with* $\kappa \leq \mu_0$*;*
4. *in* W, *there is a* $\subseteq$*-increasing sequence* $\langle x_i \mid i < \theta \rangle$ *from* $(\mathcal{P}_\kappa(\mu))^V$ *such that* $\bigcup_{i<\theta} x_i = \mu$*;*
5. *in* W, $(\mu^+)^V$ *remains a cardinal and* $\mu \geq 2^\theta$*;*
6. *in* V, $\langle \vec{D}(\alpha) \mid \alpha < \mu^+ \rangle$ *is a sequence of diagonal clubs in* $\mathcal{P}_\kappa(\vec{\mu})$.

Then, in W, *there is* $\langle y_i \mid i < \theta \rangle$ *such that, for all* $\alpha < \mu^+$ *and all sufficiently large* $i < \theta$, $y_i \in D(\alpha)_i$.

References

- Ari Meir Brodsky and Assaf Rinot, *More notions of forcing add a Souslin tree*, Preprint.
- James Cummings and Ernest Schimmerling, *Indexed squares*, Israel Journal of Mathematics **131** (2002), 61–99. MR 1942302
- Mirna Džamonja and Saharon Shelah, *On squares, outside guessing of clubs and $I_{<f}[\lambda]$*, Fundamenta Mathematicae **148** (1995), no. 2, 165–198. MR 1360144
- Moti Gitik, *Some results on the nonstationary ideal. II*, Israel Journal of Mathematics **99** (1997), 175–188. MR 1469092
- Menachem Magidor and Dima Sinapova, *Singular cardinals and square properties*, Proceedings of the American Mathematical Society, To appear.

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