Set theory and derived functors of the inverse limit

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Introduction

Overview

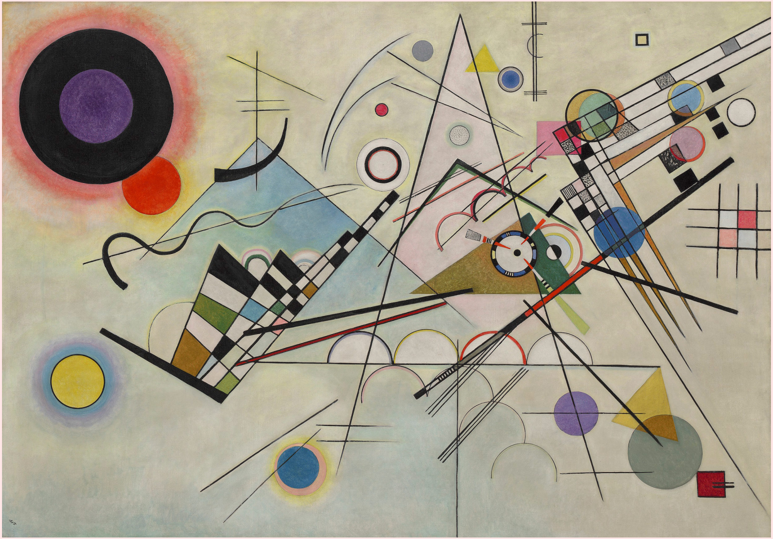
This talk will be about some interactions between set theory and homological algebra, particularly around derived functors of the inverse limit functor. This interaction has been fruitful in both directions:

- Questions coming from homological algebra have led to the development of new methods and notions in set theory.
- Set theoretic techniques have allowed for the solution of various open problems in algebra.

A brief outline of the talk:

1. An introduction to inverse systems (of abelian groups), inverse limits, and their derived functors.
2. Inverse systems indexed by ordinals, and connections to coherent Aronszajn trees.
3. Inverse systems indexed by the Baire space.
4. Applications to strong homology and condensed mathematics.

I. Inverse systems and limits



Inverse systems

Definition

Suppose that $(\Lambda, \leq)$ is a directed set. An *inverse system* (of abelian groups) indexed by Λ is a family $\mathbf{A} = \langle A_u,\ \pi_{uv} \mid u \leq v \in \Lambda \rangle$ such that:

- for all $u \in \Lambda$, A_u is an abelian group;
- for all $u \leq v \in \Lambda$, $\pi_{uv} : A_v \to A_u$ is a group homomorphism;
- for all $u \leq v \leq w \in \Lambda$, $\pi_{uw} = \pi_{uv} \circ \pi_{vw}$.

$$\begin{tikzcd}
A_w \arrow[d, "\pi_{vw}"'] \arrow[dd, bend left, "\pi_{uw}"] \\
A_v \arrow[d, "\pi_{uv}"'] \\
A_u
\end{tikzcd}$$

Level morphisms

If **A** and **B** are two inverse systems indexed by the same directed set, Λ, then a *level morphism* from **A** to **B** is a family of group homomorphisms $\mathbf{f} = \langle f_u : A_u \to B_u \mid u \in \Lambda \rangle$ such that, for all $u \leq v \in \Lambda$, $\pi^B_{uv} \circ f_v = f_u \circ \pi^A_{uv}$.

$$\begin{tikzcd}
A_v \arrow[r, "f_v"] \arrow[d, "\pi^A_{uv}"'] & B_v \arrow[d, "\pi^B_{uv}"] \\
A_u \arrow[r, "f_u"] & B_u
\end{tikzcd}$$

With this notion of morphism, the class of all inverse systems indexed by a fixed directed set Λ becomes a well-behaved category $\mathrm{Ab}^{\Lambda^{\mathrm{op}}}$ (in particular, it is an abelian category).

Inverse limits

If $\mathbf{A}$ is an inverse system indexed by Λ, then we can form the *inverse limit*, $\lim \mathbf{A}$, which is itself an abelian group. Concretely, $\lim \mathbf{A}$ can be seen as the subgroup of $\prod_{u \in \Lambda} A_u$ consisting of all sequences $\langle a_u \mid u \in \Lambda \rangle$ such that, for all $u \leq v \in \Lambda$, we have $a_u = \pi_{uv}(a_v)$.

If $\mathbf{A}$ and $\mathbf{B}$ are inverse systems and $\mathbf{f} : \mathbf{A} \to \mathbf{B}$, then $\mathbf{f}$ lifts to a group homomorphism $\lim \mathbf{f} : \lim \mathbf{A} \to \lim \mathbf{B}$. Concretely, this is done by letting $\lim \mathbf{f}(\langle a_u \mid u \in \Lambda \rangle) = \langle f_u(a_u) \mid u \in \Lambda \rangle$ for all $\langle a_u \mid u \in \Lambda \rangle \in \lim \mathbf{A}$.

This turns $\lim$ into a *functor* from the category $\mathrm{Ab}^{\Lambda^{\mathrm{op}}}$ of inverse systems indexed by Λ to the category Ab of abelian groups.

Question: How "nice" is this functor?

Exact sequences

In the category of inverse systems, kernels, images, and quotients can be defined pointwise in the obvious way. For example, if $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ is a level morphism, then $\ker(\mathbf{f})$ can be seen as the inverse system $\langle \ker(f_u), \pi_{uv} \mid u \leq v \in \Lambda \rangle$, where π_{uv} is simply $\pi^A_{uv} \restriction \ker(f_v)$.

We say that a pair of morphisms $\mathbf{A} \xrightarrow{\mathbf{f}} \mathbf{B} \xrightarrow{\mathbf{g}} \mathbf{C}$ is *exact at* $\mathbf{B}$ if $\mathrm{im}(\mathbf{f}) = \ker(\mathbf{g})$. A *short exact sequence* is a sequence $\mathbf{0} \to \mathbf{A} \xrightarrow{\mathbf{f}} \mathbf{B} \xrightarrow{\mathbf{g}} \mathbf{C} \to \mathbf{0}$ that is exact at $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$.

In a short exact sequence as above, we have $\ker(\mathbf{f}) = \mathbf{0}$ ($\mathbf{f}$ is *injective*) and $\mathrm{im}(\mathbf{g}) = \mathbf{C}$ ($\mathbf{g}$ is *surjective*). It can be helpful to think of $\mathbf{A}$ as a *subobject* of $\mathbf{B}$ and to think of $\mathbf{C}$ as the quotient $\mathbf{B}/\mathbf{A}$.

Exact functors

A functor F between abelian categories is said to be *exact* if it preserves short exact sequences, i.e., if, whenever $\mathbf{0} \to \mathsf{A} \xrightarrow{\mathbf{f}} \mathsf{B} \xrightarrow{\mathbf{g}} \mathsf{C} \to \mathbf{0}$ is exact in the source category of F, $\mathbf{0} \to F\mathsf{A} \xrightarrow{F\mathbf{f}} F\mathsf{B} \xrightarrow{F\mathbf{g}} F\mathsf{C} \to \mathbf{0}$ is exact in the target category of F.

The inverse limit functor is *left exact*: if $\mathbf{0} \to \mathsf{A} \xrightarrow{\mathbf{f}} \mathsf{B} \xrightarrow{\mathbf{g}} \mathsf{C}$ is exact at A and B, then $\mathbf{0} \to \lim \mathsf{A} \xrightarrow{\lim \mathbf{f}} \lim \mathsf{B} \xrightarrow{\lim \mathbf{g}} \lim \mathsf{C}$ is exact at $\lim \mathsf{A}$ and $\lim \mathsf{B}$. However, it fails to be exact, i.e., even if $\mathrm{im}(\mathbf{g}) = \mathsf{C}$, we might have $\mathrm{im}(\lim \mathbf{g}) \neq \lim \mathsf{C}$.

The failure of lim to be exact essentially amounts to the failure of lim to preserve quotients: if the quotient system B/A is defined, then it need not be the case that $\lim \mathsf{B}/\mathsf{A} \cong \lim \mathsf{B}/\lim \mathsf{A}$.

An example ($\Lambda = \omega$)

$$\mathbf{0} \longrightarrow \mathbf{A} \xrightarrow{\mathbf{f}} \mathbf{B} \xrightarrow{\mathbf{g}} \mathbf{C} \longrightarrow \mathbf{0}$$

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d, "\times 2"] & \vdots \arrow[d, "\times 2"] & \vdots \arrow[d, "\times 2"] & \vdots \arrow[d] \\
0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\times 3"] \arrow[d, "\times 2"] & \mathbb{Z} \arrow[r, "\text{mod } 3"] \arrow[d, "\times 2"] & \mathbb{Z}/3 \arrow[r] \arrow[d, "\times 2"] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\times 3"] \arrow[d, "\times 2"] & \mathbb{Z} \arrow[r, "\text{mod } 3"] \arrow[d, "\times 2"] & \mathbb{Z}/3 \arrow[r] \arrow[d, "\times 2"] & 0 \arrow[d] \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 3"] & \mathbb{Z} \arrow[r, "\text{mod } 3"] & \mathbb{Z}/3 \arrow[r] & 0
\end{tikzcd}$$

$\lim \mathbf{A} = \lim \mathbf{B} = 0$ and $\lim \mathbf{C} = \mathbb{Z}/3$, so applying lim to this short exact sequence yields $0 \to 0 \to 0 \to \mathbb{Z}/3 \to 0$, which is not exact at $\mathbb{Z}/3$.

Derived functors

Given any left exact functor F, there is a general procedure for producing a sequence of (right) derived functors $\langle F^n \mid n \in \omega \setminus \{0\} \rangle$ that "measure" the failure of the functor F to be exact. These derived functors then take short exact sequences

$$\mathbf{0} \longrightarrow \mathbf{A} \xrightarrow{\mathbf{f}} \mathbf{B} \xrightarrow{\mathbf{g}} \mathbf{C} \longrightarrow \mathbf{0}$$

to *long* exact sequences

$$\begin{tikzcd}
\mathbf{0} \arrow[r] & F\mathbf{A} \arrow[r, "F\mathbf{f}"] & F\mathbf{B} \arrow[r, "F\mathbf{g}"] & F\mathbf{C} \arrow[dll, "\delta"] \\
& F^1\mathbf{A} \arrow[r, "F^1\mathbf{f}"] & F^1\mathbf{B} \arrow[r, "F^1\mathbf{g}"] & F^1\mathbf{C} \arrow[dll, "\delta"] \\
& F^2\mathbf{A} \arrow[r, "F^2\mathbf{f}"] & F^2\mathbf{B} \arrow[r, "F^2\mathbf{g}"] & F^2\mathbf{C} \arrow[r] & \dots
\end{tikzcd}$$

We will be interested in the derived functors $\langle \lim^n \mid n \in \omega \setminus \{0\} \rangle$.

Derived limits and cofinality

A pair of complementary theorems from the early 1970s demonstrates a connection between the vanishing of derived inverse limits and the cofinality of the indexing poset.

Theorem (Goblot, 1970)

Suppose that Λ is a directed set, $n < \omega$, and $\mathrm{cf}(\Lambda) \leq \aleph_n$. Then, for every $\mathbf{A} \in \mathsf{Ab}^{\Lambda^{\mathrm{op}}}$, we have

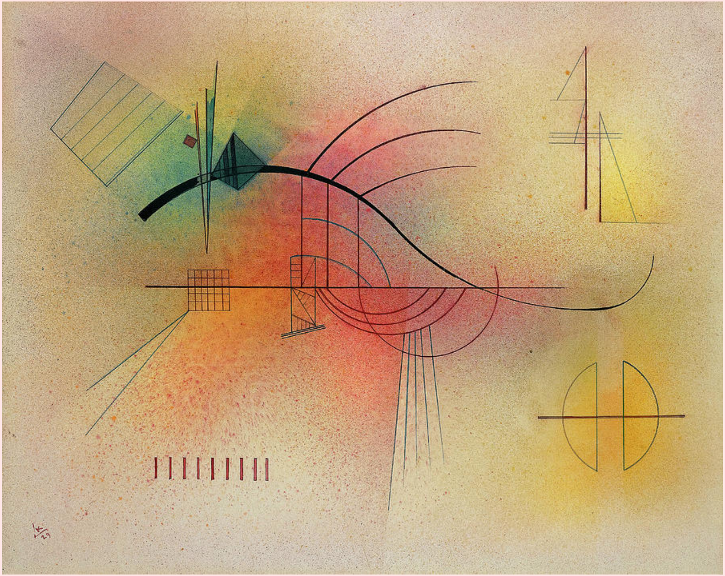
$$\lim{}^{n+2}\mathbf{A} = 0.$$

Theorem (B. Mitchell, 1973)

Suppose that Λ is a directed set, $n < \omega$, and $\mathrm{cf}(\Lambda) \geq \aleph_n$. Then there is $\mathbf{A} \in \mathsf{Ab}^{\Lambda^{\mathrm{op}}}$ such that

$$\lim{}^{n+1}\mathbf{A} \neq 0.$$

II. Ordinal-indexed systems



The systems A_δ and B_δ

Fix a limit ordinal δ and an abelian group H. For each ordinal $\alpha < \delta$, let

$$A_\alpha[H] := \bigoplus_\alpha H \qquad \text{and} \qquad B_\alpha[H] := \prod_\alpha H.$$

Let $A_\alpha := A_\alpha[\mathbb{Z}]$ and $B_\alpha := B_\alpha[\mathbb{Z}]$. For $\alpha \leq \beta < \delta$, let $\pi_{\alpha\beta} : B_\beta \to B_\alpha$ be the projection maps. These restrict to maps from A_β to A_α. Let $\mathbf{A}_\delta = \langle A_\alpha, \pi_{\alpha\beta} \mid \alpha \leq \beta < \delta \rangle$ and $\mathbf{B}_\delta = \langle B_\alpha, \pi_{\alpha\beta} \mid \alpha \leq \beta < \delta \rangle$. Note that $\lim \mathbf{B}_\delta \cong \prod_\delta \mathbb{Z}$, while $\lim \mathbf{A}_\delta$ consists of all $\varphi \in \prod_\delta \mathbb{Z}$ such that $\mathrm{sppt}(\varphi) \cap \alpha$ is finite for all $\alpha < \delta$. If $\mathrm{cf}(\delta) > \omega$, this is $\bigoplus_\delta \mathbb{Z}$. If $\mathrm{cf}(\delta) = \omega$, it also includes φ such that $\mathrm{sppt}(\varphi)$ has order type ω and is cofinal in δ.

The system $\mathbf{B}_\delta$ has the property that $\lim^n \mathbf{B}_\delta = 0$ for all $n \geq 1$. What about $\lim^n \mathbf{A}_\delta$?

(Side remark: $\lim^n \mathbf{A}_\delta$ is equal to the Čech cohomology group $\check{H}^n(\delta, \mathbb{Z})$, where δ is given the order topology induced by the ordinal ordering.)

Computing derived limits

Consider the short exact sequence

$$0 \to \mathbf{A}_\delta \xrightarrow{\iota} \mathbf{B}_\delta \xrightarrow{p} \mathbf{B}_\delta/\mathbf{A}_\delta \to 0.$$

Applying lim yields the exact sequence

$$0 \to \lim \mathbf{A}_\delta \xrightarrow{\lim \iota} \lim \mathbf{B}_\delta \xrightarrow{\lim p} \lim \mathbf{B}_\delta/\mathbf{A}_\delta \to \lim^1 \mathbf{A}_\delta \to \lim^1 \mathbf{B}_\delta = 0.$$

Thus, we have

$$\lim^1 \mathbf{A}_\delta \cong \frac{\lim \mathbf{B}_\delta/\mathbf{A}_\delta}{\mathrm{im}(\lim(p))}.$$

Elements of $\lim \mathbf{B}_\delta/\mathbf{A}_\delta$ are (equivalence classes of) sequences $\langle \varphi_\alpha \in \prod_\alpha \mathbb{Z} \mid \alpha < \delta \rangle$ such that, for all $\alpha < \beta < \kappa$, we have $\varphi_\alpha =^* \varphi_\beta \restriction \alpha$ (where $=^*$ denotes equality mod finite). Recall that $\lim \mathbf{B}_\delta = \prod_\delta \mathbb{Z}$. Elements of $\mathrm{im}(\lim(p))$ are thus all (eq. classes of) sequences $\langle \varphi_\alpha \mid \alpha < \delta \rangle$ for which there exists a single function $\psi \in \prod_\delta \mathbb{Z}$ such that $\varphi_\alpha =^* \psi \restriction \alpha$ for all $\alpha < \delta$.

Coherent Aronszajn trees

Therefore, $\lim^1 \mathbf{A}_\delta \neq 0$ if and only if there is a sequence $\langle \varphi_\alpha : \alpha \to \mathbb{Z} \mid \alpha < \delta \rangle$ that is

1. (coherent) $\varphi_\alpha =^* \varphi_\beta \restriction \alpha$ for all $\alpha \leq \beta < \delta$;
2. (nontrivial) there is no $\psi : \delta \to \mathbb{Z}$ such that $\psi \restriction \alpha =^* \varphi_\alpha$ for all $\alpha < \delta$.

When δ is a regular cardinal, those familiar with combinatorial set theory may recognize that this is equivalent to the existence of a *coherent δ-Aronszajn tree*. These are well-studied set-theoretic objects, facts about which immediately transfer to yield the following:

- $\lim^1 \mathbf{A}_\omega = 0$;
- $\lim^1 \mathbf{A}_{\omega_1} \neq 0$;
- if $V = L$, then $\lim^1 \mathbf{A}_\kappa \neq 0$ for all uncountable, regular cardinals κ that are not weakly compact;
- the Proper Forcing Axiom implies that $\lim^1 \mathbf{A}_\kappa = 0$ for all regular $\kappa \geq \omega_2$.

Higher dimensions

Similar, but higher dimensional, characterizations exist for the nonvanishing of higher derived limits. For example, $\lim^2 \mathbf{A}_\delta \neq 0$ if and only if there is a family $\langle \varphi_{\alpha\beta} : \alpha \to \mathbb{Z} \mid \alpha \leq \beta < \delta \rangle$ that is

- (2-coherent) for all $\alpha \leq \beta \leq \gamma < \delta$, we have

$$\varphi_{\beta\gamma} \restriction \alpha - \varphi_{\alpha\gamma} + \varphi_{\alpha\beta} =^* 0, \text{ i.e., } \varphi_{\alpha\beta} + \varphi_{\beta\gamma} =^* \varphi_{\alpha\gamma};$$

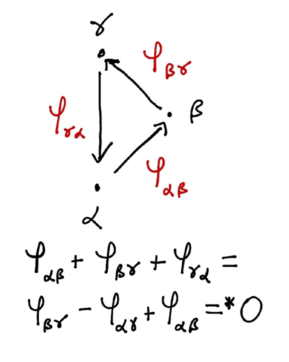
- (nontrivial) there is no sequence $\langle \psi_\alpha : \alpha \to \mathbb{Z} \mid \alpha < \delta \rangle$ such that, for all $\alpha \leq \beta < \delta$, we have

$$\varphi_{\alpha\beta} =^* \psi_\beta \restriction \alpha - \psi_\alpha.$$

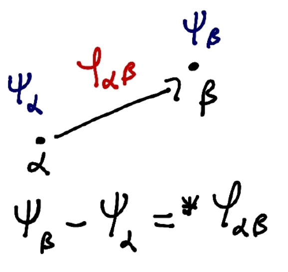
Unlike coherent Aronszajn trees, these are genuinely new combinatorial objects. Note that a 2-coherent family is *locally trivial*, i.e., given $\gamma < \delta$, then the sequence $\langle -\varphi_{\alpha\gamma} \mid \alpha < \gamma \rangle$ witnesses that the initial segment $\langle \varphi_{\alpha\beta} \mid \alpha \leq \beta < \gamma \rangle$ is trivial. Therefore, nontrivial 2-coherent families can be interpreted as instances of set-theoretic incompactness.

A reframing

Coherence and triviality can be reframed in terms of oriented sums of functions indexed by maximal faces of simplices whose vertices are labeled by ordinals. For example, a 2-dimensional family $\langle \varphi_{\alpha\beta} \mid \alpha \leq \beta < \kappa \rangle$ is 2-coherent if the oriented sum on the boundary of every 2-simplex vanishes mod finite:



A 2-d family is trivial if its 2-d information reduces (mod finite) to a 1-d family $\langle \psi_\alpha \mid \alpha < \kappa \rangle$.



Nonvanishing results

- (Mitchell) For all $1 \leq n < \omega$, $\lim^n \mathbf{A}_{\aleph_n}[\bigoplus_{\aleph_n} \mathbb{Z}] \neq 0$.

It is a major open question whether this can be improved to $\lim^n \mathbf{A}_{\aleph_n} \neq 0$ in ZFC. This is true for $n = 1$ due to the existence of coherent Aronszajn trees.

- (Bergfalk–LH) If $V = L$, and $1 \leq n < \omega$, then $\lim^n \mathbf{A}_\kappa \neq 0$ for all regular cardinals $\kappa \geq \aleph_n$ that are not weakly compact.

The proof of this goes through the following general stepping-up lemma:

Lemma (Bergfalk–LH)

Suppose that $\lambda < \kappa$ are regular uncountable cardinals, $\lim^n \mathbf{A}_\lambda \neq 0$, and there is a stationary $S \subseteq \kappa \cap \mathrm{cof}(\lambda)$ such that $\square(\kappa, S) + \diamondsuit_\kappa(S)$ holds. Then $\lim^{n+1} \mathbf{A}_\kappa \neq 0$.

Vanishing results

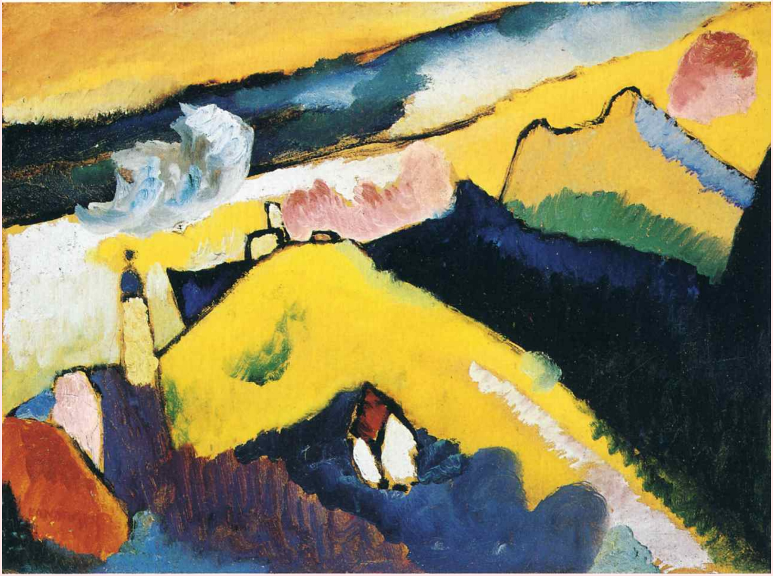
1. (Todorcevic) The P-Ideal Dichotomy (and hence the Proper Forcing Axiom) implies that $\lim^1 \mathbf{A}_\kappa = 0$ for all regular $\kappa \geq \aleph_2$.
2. (Bergfalk–LH–J. Zhang) If λ is strongly compact, then $\lim^n \mathbf{A}_\kappa[H] = 0$ for all $1 \leq n < \omega$, all regular $\kappa \geq \lambda$, and all abelian groups H.
3. (Bergfalk–LH–J. Zhang) Relative to the consistency of a supercompact cardinal, it is consistent that $\lim^n \mathbf{A}_{\aleph_{\omega+1}}[H] = 0$ for all $1 \leq n < \omega$ and all abelian groups H.

Question

What else can we say about the situation below $\aleph_\omega$? Is it consistent, that, e.g., $\lim^2 \mathbf{A}_{\aleph_3} = 0$*?*

This would be a higher-dimensional analogue of the result that, consistently, there are no coherent $\aleph_2$-Aronszajn trees. We expect a positive answer, but it seems to require genuinely new ideas.

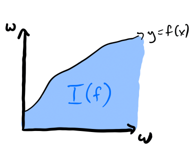
III. ${}^{\omega}\omega$-indexed systems



The system $\mathbf{A}[H]$

Fix an abelian group H. Given a function $f : \omega \to \omega$, let

$$I(f) := \{(k, m) \in \omega \times \omega \mid m \leq f(k)\}$$



and let $A_f[H] := \bigoplus_{I(f)} H$. Given $f, g \in {}^{\omega}\omega$, let $f \leq g$ iff $f(k) \leq g(k)$ for all $k < \omega$; in this case, let $\pi_{fg} : A_g[H] \to A_f[H]$ be the projection map. This defines an inverse system

$$\mathbf{A}[H] = \langle A_f, \pi_{fg} \mid f, g \in {}^{\omega}\omega, \ f \leq g \rangle.$$

If $H = \mathbb{Z}$, we omit it in the notation. The system $\mathbf{A}[H]$ and its (derived) limits naturally arise in a variety of mathematical contexts, and the vanishing of its derived limits is of considerable interest.

$\lim^1 \mathbf{A}$

The first derived limit $\lim^1 \mathbf{A}$ was extensively investigated by set theorists in the late 1980s and early 1990s. We have a similar characterization of its vanishing as with $\mathbf{A}_\kappa$: $\lim^1 \mathbf{A}[H] \neq 0$ iff there is a family of functions

$$\langle \varphi_f : I(f) \to H \mid f \in {}^\omega\omega \rangle$$

that is

- (coherent) $\varphi_f =^* \varphi_g \restriction I(f)$ for all $f \leq g$ in ${}^\omega\omega$; and
- (nontrivial) there is no function $\psi : \omega \times \omega \to H$ such that $\varphi_f =^* \psi \restriction I(f)$ for all $f \in {}^\omega\omega$.

Early results

- (Mardešić–Prasolov, Simon, 1988) If the Continuum Hypothesis holds, then $\lim^1 \mathbf{A} \neq 0$.
- (Dow–Simon–Vaughan, 1989) If $\mathfrak{d} = \aleph_1$, then $\lim^1 \mathbf{A} \neq 0$.
- (Dow–Simon–Vaughan, 1989) If the Proper Forcing Axiom holds, then $\lim^1 \mathbf{A} = 0$.
- (Todorcevic, 1989) If the Open Coloring Axiom holds, then $\lim^1 \mathbf{A} = 0$.
- (Kamo, 1993) After adding $\aleph_2$-many Cohen reals to any model of ZFC, we have $\lim^1 \mathbf{A} = 0$.

Higher limits

There has recently been a resurgence of research into the derived limits of $\mathbf{A}$ and related inverse systems, spurred especially by some breakthroughs in the study of the higher derived limits. We first note that the nonvanishing of such limits can be characterized in a similar way to the systems $\mathbf{A}_\kappa$. For example, $\lim^2 \mathbf{A}[H] \neq 0$ iff there is a family

$$\langle \varphi_{fg} : I(f) \to H \mid f, g \in {}^\omega\omega,\ f \leq g \rangle$$

that is

- (2-coherent) $\varphi_{gh} \restriction I(f) - \varphi_{fh} + \varphi_{fg} =^* 0$ for all $f \leq g \leq h$ in ${}^\omega\omega$;
- (nontrivial) there is no family $\langle \psi_f : I(f) \to H \mid f \in {}^\omega\omega \rangle$ such that $\varphi_{fg} =^* \psi_g \restriction I(f) - \psi_f$ for all $f \leq g$ in ${}^\omega\omega$.

Some recent results

- (Bergfalk, 2017) PFA implies $\lim^2 \mathbf{A} \neq 0$.
- (Veličković–Vignati, 2023) For all $n \geq 1$, it is consistent that $\lim^n \mathbf{A} \neq 0$.
- (Bergfalk–LH, 2021) After adding a weakly compact number of Hechler reals to any model of set theory, $\lim^n \mathbf{A} = 0$ for all $n \geq 1$.
- (Bergfalk–Hrušák–LH, 2023) After adding $\beth_\omega$-many Cohen reals to any model of set theory, $\lim^n \mathbf{A} = 0$ for all $n \geq 1$.
- (Bannister, 2023) In both of the two preceding results, we in fact obtain $\lim^n \mathbf{A}[H] = 0$ for all $n \geq 1$ and all abelian groups H. This is optimal by the following result:
- (LH, 2023) If $2^{\aleph_0} < \aleph_\omega$, then there is $n \geq 1$ and an abelian group H such that $\lim^n \mathbf{A}[H] \neq 0$.

IV. Some applications

Strong homology

Strong homology is a homology theory for topological spaces that is strong shape invariant. It was developed by Lisica and Mardešić, and was designed to reflect the properties of spaces with pathological local behavior more reliably than, e.g., singular homology.

Given a space X and a $p < \omega$, let $\bar{H}_p(X)$ denote the p^{th} strong homology group of X.

Additivity

A desirable property for a homology theory to have is *additivity*:

Definition

A homology theory is *additive* on a class of topological spaces $\mathcal{C}$ if, for every natural number p and every family $\{X_i \mid i \in J\}$ such that each X_i and $\coprod_J X_i$ are in $\mathcal{C}$, we have

$$\bigoplus_J \mathrm{H}_p(X_i) \cong \mathrm{H}_p(\coprod_J X_i)$$

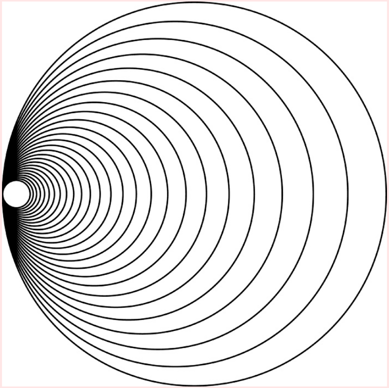
via the map induced by the inclusions

$$X_i \hookrightarrow \coprod_J X_i.$$

Question: Is strong homology additive?

Infinite earring spaces

Let X^n denote the n-dimensional infinite earring space, i.e., the one-point compactification of the disjoint union of countably infinitely many copies of the n-dimensional open unit ball.



X^1

Theorem (Mardešić-Prasolov, '88)

Suppose that $0 < p < n$ are natural numbers. Then

$$\bigoplus_{\mathbb{N}} \bar{\mathrm{H}}_p(X^n) = \bar{\mathrm{H}}_p(\coprod_{\mathbb{N}} X^n)$$

if and only if $\lim^{n-p} \mathbf{A} = 0$.

Consequently, if strong homology is additive, even on closed subsets of Euclidean space, then $\lim^n \mathbf{A} = 0$ for all $n \geq 1$. Getting these derived limits to vanish ended up removing all obstacles to additivity of strong homology on a robust class of spaces.

Theorem (Bannister–Bergfalk–Moore, 2023, Bannister, 2023)

After adding weakly compact-many Hechler reals or $\beth_\omega$-many Cohen reals to any model of ZFC*, strong homology is additive on the class of locally compact separable metric spaces.*

A ZFC counterexample

Let $\mathbf{A}_{\omega,\omega_1}$ be the inverse system defined analogously to $\mathbf{A}$, but indexed by functions $f : \omega \to [\omega_1]^{<\omega}$. Let $X^n_{\omega_1}$ denote the one-point compactification of the disjoint union of ω_1-many copies of the n-dimensional open unit ball. This is an analogue of the classical earring space, and its strong homology is related to $\mathbf{A}_{\omega,\omega_1}$ in the same way that the strong homology of X^n is related to $\mathbf{A}$.

Theorem (Bergfalk–LH, 2023)

- $\lim^1 \mathbf{A}_{\omega,\omega_1} \neq 0$;
- *for all* $n \geq 2$,

$$\bigoplus_{\mathbb{N}} \bar{\mathrm{H}}_{n-1}(X^n_{\omega_1}) \neq \bar{\mathrm{H}}_{n-1}\Big(\coprod_{\mathbb{N}} X^n_{\omega_1}\Big).$$

A more complicated ZFC counterexample to the additivity of strong homology was found by Prasolov in 2005.

Condensed mathematics

Condensed mathematics is a framework, developed recently by Dustin Clausen and Peter Scholze, for applying algebraic tools to the study of algebraic structures carrying topologies. The basic idea is to embed classical categories (e.g., topological abelian groups, topological vector spaces, etc.) into richer categories with nicer algebraic structures. Loosely speaking, objects in these richer categories are contravariant functors from the category of compact Hausdorff spaces to the classical category of interest satisfying certain properties (sheaf conditions).

For example, given a topological abelian group A, one obtains an associated condensed abelian group $\underline{A} : \mathsf{CHaus} \to \mathsf{Ab}$ defined by $\underline{A}(S) = \mathrm{Cont}(S, A)$ for all $S \in \mathsf{CHaus}$.

Fully faithful embeddings

This describes an embedding of various classical categories into their condensed analogues. When restricted to nice subcategories, these embeddings are fully faithful, even at the level of derived categories.

For example, the embedding of the category of locally compact abelian groups into condensed abelian groups is fully faithful.

Question: To what extent is this embedding fully faithful on larger subcategories?

Our results above yield some negative answers here. For example, the fact that $\lim^1 \mathbf{A}_{\omega,\omega_1} \neq 0$ can be used to show that the embedding of the category of pro-abelian groups into the category of condensed abelian groups is not fully faithful (in particular, it is not full).

Banach–Smith Duality

Definition

A *Smith space* is a complete, compactly generated locally convex topological vector space X having a universal compact set K, i.e., for every compact $T \subseteq X$, there is $r > 0$ such that $r \cdot K \supseteq T$.

Smith spaces are dual to Banach spaces:

- If Y is a Banach space, then $C(Y, \mathbb{R})$ *with the compact-open topology* is a Smith space.
- If X is a Smith space, then $C(X, \mathbb{R})$ *with the compact-open topology* is a Banach space.

This remains true for *p*-adic Banach and Smith spaces, replacing $\mathbb{R}$ with $\mathbb{Q}_p$.

Condensed Banach–Smith Duality

Question: To what extent does this Banach–Smith duality persist in the derived condensed setting?

The internal dual of a condensed Smith space (in the derived category) is always a condensed Banach space. The reverse need not be true.

However, if $\lim^n \mathbf{A}[H] = 0$ for all $n > 0$ and all abelian groups H, then this duality holds between condensed separable p-adic Banach spaces and condensed coseparable p-adic Smith spaces (in the derived setting).

Interesting questions remain regarding the situation in the archimedean setting.

The continuum

Thus, the assumption that $\lim^n \mathbf{A}[H] = 0$ for all $n > 0$ and all abelian groups H is a natural and attractive one to make in the context of condensed mathematics. Recall that this assumption implies $2^{\aleph_0} > \aleph_\omega$, and is compatible with $2^{\aleph_0} = \aleph_{\omega+1}$.

This provides another example in a growing list of what can be seen as combinatorial questions about the real numbers in which

- there is a "nice" answer to the question;
- the "nice" answer is known to be consistent;
- the "nice" answer entails a large continuum (i.e., $2^{\aleph_0} > \aleph_\omega$).

I would argue that this suggests that it would be worthwhile to further investigate "canonical" models of ZFC or additional natural axioms which would imply the "nice" answers to these questions (and hence would also imply a large continuum).

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Thank you for your attention!