

SIMULTANEOUSLY VANISHING HIGHER DERIVED LIMITS WITHOUT LARGE CARDINALS

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ABSTRACT. A question dating to Sibe Mardešić and Andrei Prasadov's 1988 work [13], and motivating a considerable amount of set theoretic work in the years since, is that of whether it is consistent with the ZFC axioms for the higher derived limits \lim^n ($n > 0$) of a certain inverse system \mathbf{A} indexed by ${}^\omega\omega$ to simultaneously vanish. An equivalent formulation of this question is that of whether it is consistent for all n -coherent families of functions indexed by ${}^\omega\omega$ to be trivial. In this paper, we prove that, in any forcing extension given by adjoining \beth_ω -many Cohen reals, $\lim^n \mathbf{A}$ vanishes for all $n > 0$. Our proof involves a detailed combinatorial analysis of the forcing extension and repeated applications of higher dimensional Δ -system lemmas. This work removes all large cardinal hypotheses from the main result of [6] and substantially reduces the least value of the continuum known to be compatible with the simultaneous vanishing of $\lim^n \mathbf{A}$ for all $n > 0$.

1. INTRODUCTION

The set theoretic study of higher derived limits traces principally to Sibe Mardešić and Andrei Prasadov's 1988 work [13]; it was in this paper that a relationship between

- (1) the continuity properties of strong homology,
- (2) the behavior of the derived limits of inverse systems indexed by functions from ω to ω , and
- (3) infinitary combinatorics and assumptions supplementary to ZFC

was first perceived. The most elementary of the systems as in (2) was denoted \mathbf{A} by Mardešić and Prasadov in [13], and the works which followed would show the behavior of its higher limits sensitive to a variety of set theoretic hypotheses; additional interest in these behaviors derived from their connection to the broader set theoretic theme of nontrivial coherence (main works in this line were [8, 22, 10, 23, 9, 16, 3, 1]; see [6, Introduction] for a brief research history). The outstanding question tracing to [13] was whether the statement “ $\lim^n \mathbf{A} = 0$ for all $n > 0$ ” is consistent with the ZFC axioms; this was affirmatively answered in [6] under the assumption of the existence of a weakly compact cardinal. Several immediately ensuing questions are listed in the conclusion of [6]. The first of these, that of the consistency strength of this statement, is answered by our main result:

Main Theorem. *It is consistent with the ZFC axioms that $\lim^n \mathbf{A} = 0$ for all $n > 0$.*

In particular, the statement “ $\lim^n \mathbf{A} = 0$ for all $n > 0$ ” carries no large cardinal strength whatsoever: as we will show, it holds in the extension of V by the forcing $\text{Add}(\omega, \beth_\omega)$ for adjoining

2010 *Mathematics Subject Classification.* 03E35, 03E75, 18E25, 55N07.

Key words and phrases. Cohen forcing, derived limit, nontrivial coherence, Delta system lemma, strong homology.

The first author was partially supported by Austrian Science Foundation (FWF) Grant Number Y1012-N35.

The second author was partially supported by a CONACyT grant A1-S-16164 and PAPIIT grant IN104220.

The work in this paper began during a visit by the third author to the Centro de Ciencias Matemáticas at UNAM Morelia while the first author was a postdoc at that institution. Both authors would like to thank the CCM for its hospitality and support.

\beth_ω -many Cohen reals. The second of the questions listed in [6] is that of the minimum value of the continuum compatible with this statement. The works [13] and [6] established for that value lower and upper bounds of \aleph_2 and a weakly inaccessible cardinal, respectively; the gap between them was substantial.

Main Corollary. *It is consistent relative to the ZFC axioms that $\lim^n \mathbf{A} = 0$ for all $n > 0$ and $2^{\aleph_0} = \aleph_{\omega+1}$.*

As noted in [6], plausible scenarios exist in which $\aleph_{\omega+1}$ is optimal, a point we return to in our conclusion below. Highly relevant in this direction is Veličković and Vignati’s recent result, in [24], that the statement “ $\lim^n \mathbf{A} \neq 0$ and $2^{\aleph_0} = \aleph_n$ ” is consistent relative to the ZFC axioms for every $n > 0$.

In order to describe the structure of our paper, we should first say a few words about our overall argument and, in particular, about how it both builds on and departs from that of [6]. In both that work and this one, the idea is to argue in a given forcing extension that an arbitrary n -coherent family Φ of functions indexed by ${}^\omega\omega$ is trivial (see Definition 2.7 below for the definitions of n -coherent and trivial). In both cases, this is achieved in two steps:

- First, a trivialization of the restricted family $\Phi \restriction A$ is found for some $A \subseteq {}^\omega\omega$.
- Second, trivializations of $\Phi \restriction A$ are shown to extend to trivializations of all of Φ .

The requirements of these two steps are in tension; what’s wanted is an A which is at once “sufficiently small” and “sufficiently large” to effect the first and second steps, respectively. This is a tension which the large cardinal assumption of [6] may be viewed as resolving: there, the weak compactness of a cardinal κ manifests as multidimensional Δ -system relationships among large families of conditions in a finite support forcing iteration of length κ . These families’ homogeneities lend them a “smallness” of the sort called for in step one; this being an iteration of Hechler forcings ensures that, nevertheless, the associated sets A are \leq^* -cofinal in ${}^\omega\omega$, from which step two follows easily.

In the present work, cardinal arithmetic and inductive hypotheses on n together take the place of the large cardinal assumption in [6]. Here again, higher-dimensional Δ -systems lie at the heart of step one, and we draw on [12] for their description and analysis. Observe, however, that without large cardinal assumptions, such systems can only appear, in general, together with some drop in cardinality. In consequence, the set A associated to such a system in step one of our argument is small in a much stronger sense than in [6]. Nevertheless, in the context of Cohen forcing, genericity arguments coupled, at each stage n , with inductive hypotheses on the triviality of k -coherent families of functions for $k < n$ allow us to propagate the triviality of $\Phi \restriction A$ to all of Φ as desired.

Our account of this argument is structured as follows. In Section 2, we record our basic conventions, some results on higher-dimensional Δ -systems, and the conversion of assertions about $\lim^n \mathbf{A}$ to assertions about the triviality of n -coherent families of functions. This section includes a brief homological interlude that those so inclined may safely ignore: from Section 2.3, only Definition 2.7, Fact 2.9, Proposition 2.10, and Remark 2.11 are needed in the remainder of the paper. In Section 3, we prove a strong form of the $n = 1$ instance of our main theorem. In Section 4, we record a framework deriving from [6] for defining trivializations of families like $\Phi \restriction A$. As the reader may have surmised, the paper from this point forward is fairly technical, and we therefore close this section with several heuristic comments that we hope will help to motivate some of the technical arguments to follow. In Section 5, we put the framework recorded in the prior section to use by completing the “step one” portion of our argument outlined above. In Section 6, we describe the “step two” portion of our argument, showing how, at each stage $n > 1$, our trivializations of $\Phi \restriction A$ will extend to all of Φ . In Section 7, we conclude with a list of some of the more prominent questions arising in

the field of set-theoretic research into higher derived limits, along with a discussion of some of their interrelations.

2. PRELIMINARIES

A useful reference for some of our motivating questions is Mardešić's *Strong Shape and Homology* [14]; readers new to the study of higher derived limits are referred to its Book III, in particular, for a valuable introduction.

2.1. Notational conventions. If X is a set and κ is a cardinal, then $[X]^\kappa = \{Y \subseteq X \mid |Y| = \kappa\}$ and $[X]^{<\kappa} = \{Y \subseteq X \mid |Y| < \kappa\}$. If κ and λ are cardinals, then we say that λ is $<\kappa$ -inaccessible if $\nu^{<\kappa} < \lambda$ for all $\nu < \lambda$. For an ordinal α , the cardinal \beth_α is defined by recursion on α by letting $\beth_0 = \aleph_0$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$, and $\beth_\beta = \sup\{\beth_\alpha \mid \alpha < \beta\}$ for limit ordinals β .

If X is a set of ordinals, then $\text{otp}(X)$ denotes the order-type of X . If X and Y are nonempty sets of ordinals, then we write $X < Y$ to denote the assertion that $\alpha < \beta$ for all $\alpha \in X$ and all $\beta \in Y$. We will often view finite sets of ordinals as finite increasing sequences of ordinals, and vice versa. For example, if $a \in [\text{Ord}]^{<\omega}$ and $\ell < \text{otp}(a)$, then $a(\ell)$ is the unique $\alpha \in a$ such that $|a \cap \alpha| = \ell$. If $\mathbf{m} \subseteq \text{otp}(a)$, then $a[\mathbf{m}] = \{a(\ell) \mid \ell \in \mathbf{m}\}$. For any set X of ordinals and natural number n , the notation $(\alpha_0, \dots, \alpha_{n-1}) \in [X]^n$ will denote the conjunction of the statements $\{\alpha_0, \dots, \alpha_{n-1}\} \in [X]^n$ and $\alpha_0 < \dots < \alpha_{n-1}$. Frequently in what follows we index objects by finite sets, either of ordinals or of functions or of other finite sets. Our use of commas or curly brackets in the associated subscripts or superscripts is, in general, according to no other principle than readability. We will often, for example, write expressions like $q_{\alpha\beta}$ for expressions like $q_{\{\alpha,\beta\}}$; we handle these matters with some greater care, however, in the context of the more technical Section 5.

The forcings appearing herein will all be of the form $\mathbb{P} = \text{Add}(\omega, \chi)$, i.e., the forcing to add χ -many Cohen reals, where χ is an uncountable cardinal. For more on the forcing technique, readers are referred to [11], though several of our notations diverge from that text's. We think of the conditions of \mathbb{P} as finite partial functions from $\chi \times \omega$ to ω , ordered by reverse inclusion. Recall that a poset \mathbb{Q} is κ -Knaster if for every $A \in [\mathbb{Q}]^\kappa$ there exists a $B \in [A]^\kappa$ consisting of pairwise compatible elements of \mathbb{Q} , and that $\mathbb{P} = \text{Add}(\omega, \chi)$ is κ -Knaster for all regular uncountable κ . Forcing with \mathbb{P} produces a generic function $F : \chi \times \omega \rightarrow \omega$. For a fixed $\alpha < \chi$ we call the function $F(\alpha, \cdot) : \omega \rightarrow \omega$ the α^{th} Cohen real added by \mathbb{P} , and we will typically denote this function by f_α ; we denote the canonical \mathbb{P} -name in V for f_α by \dot{f}_α . If G is \mathbb{P} -generic over V and $W \subseteq \chi$ then G_W denotes $\{p \in G \mid \text{dom}(p) \subseteq W \times \omega\}$. For any condition p in \mathbb{P} let $u(p)$ denote the set $\{\alpha < \chi \mid \text{dom}(p) \cap (\{\alpha\} \times \omega) \neq \emptyset\}$ and let \bar{p} denote the finite partial function from $\text{otp}(u(p)) \times \omega$ to ω defined as follows: for all $i < \text{otp}(u(p))$ and all $j < \omega$, define (i, j) to be in the domain of \bar{p} if and only if $(u(p)(i), j) \in \text{dom}(p)$; if so, let $\bar{p}(i, j) = p(u(p)(i), j)$. Intuitively, \bar{p} is a “collapsed” version of p . Notice that the set $\{\bar{p} \mid p \in \mathbb{P}\}$ is a subset of the set of finite partial functions from $\omega \times \omega$ to ω and is therefore countable.

For notational conventions pertaining more directly to coherent families of functions, see Section 2.3 below.

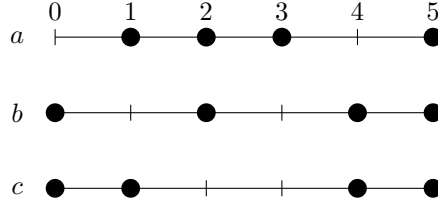
2.2. Higher-dimensional Δ -systems. Our proofs will make repeated use of multidimensional Δ -system lemmas. Similar lemmas appear as far back as the 1980s in works of Todorćević [21] and Shelah [19], [18], and they have been employed with increasing frequency in recent years, notably in work of Zhang [25] in which they are also used to analyze the forcing extension by $\text{Add}(\omega, \beth_\omega)$. We use here a version of the multidimensional Δ -system lemma isolated by the third author in [12]; our reasons for choosing this formulation include the facts that it holds from provably optimal

cardinality assumptions and that its purely combinatorial formulation seems particularly well-suited to the problem at hand. With this in mind, we recall some relevant definitions and results from [12].

Definition 2.1. Suppose that a and b are sets of ordinals.

- (1) We say that a and b are *aligned* if $\text{otp}(a) = \text{otp}(b)$ and $\text{otp}(a \cap \gamma) = \text{otp}(b \cap \gamma)$ for all $\gamma \in a \cap b$. In other words, if γ is a common element of two aligned sets a and b , then it occupies the same relative position in both a and b .
- (2) If a and b are aligned then we let $\mathbf{r}(a, b) := \{i < \text{otp}(a) \mid a(i) = b(i)\}$. Notice that, in this case, $a \cap b = a[\mathbf{r}(a, b)] = b[\mathbf{r}(a, b)]$.

To help digest this definition, we provide some simple examples of aligned and non-aligned sets. In the following picture, a , b , and c are all 4-elements subsets of $\{0, \dots, 5\}$. The sets a and b are aligned, with $\mathbf{r}(a, b) = \{1, 3\}$, and the sets b and c are aligned, with $\mathbf{r}(b, c) = \{0, 2, 3\}$. However, the sets a and c are not aligned, since $1 \in a \cap c$ but $|a \cap 1| = 0 \neq 1 = |c \cap 1|$.



Definition 2.2. Suppose that H is a set of ordinals, n is a positive integer, and u_b is a set of ordinals for all $b \in [H]^n$. We call $\langle u_b \mid b \in [H]^n \rangle$ a *uniform n -dimensional Δ -system* if there are an ordinal ρ and, for each $\mathbf{m} \subseteq n$, a set $\mathbf{r}_{\mathbf{m}} \subseteq \rho$ satisfying the following statements.

- (1) $\text{otp}(u_b) = \rho$ for all $b \in [H]^n$.
- (2) For all $a, b \in [H]^n$ and $\mathbf{m} \subseteq n$, if a and b are aligned with $\mathbf{r}(a, b) = \mathbf{m}$, then u_a and u_b are aligned with $\mathbf{r}(u_a, u_b) = \mathbf{r}_{\mathbf{m}}$.
- (3) For all $\mathbf{m}_0, \mathbf{m}_1 \subseteq n$, we have $\mathbf{r}_{\mathbf{m}_0 \cap \mathbf{m}_1} = \mathbf{r}_{\mathbf{m}_0} \cap \mathbf{r}_{\mathbf{m}_1}$.

The following lemma records a crucial feature of these Δ -systems. Here and in similar places later in the paper, m denotes the set of natural numbers less than m , so, for instance, $b[m] = \{b(\ell) \mid \ell < m\}$, $\mathbf{r}_m = \mathbf{r}_{\{\ell \mid \ell < m\}}$, and the statement that $b[m] = a$ amounts to asserting that b end-extends a .

Lemma 2.3. Suppose that $1 \leq n < \omega$ and $\langle u_b \mid b \in [H]^n \rangle$ is a uniform n -dimensional Δ -system, as witnessed by ρ and $\langle \mathbf{r}_{\mathbf{m}} \mid \mathbf{m} \subseteq n \rangle$, and suppose that H has no largest element. For each $m < n$ and each $a \in [H]^m$, define a set u_a by choosing $b \in [H]^n$ such that $b[m] = a$ and setting $u_a = u_b[\mathbf{r}_m]$. Then the following hold.

- (1) These definitions are independent of our choice of b .
- (2) For each $a \in [H]^{<n}$, the collection

$$\{u_{a \cup \{\beta\}} \mid \beta \in H \setminus (\max(a) + 1)\}$$

is a (1-dimensional) Δ -system with root u_a .

Proof. We first show (1). Indeed, fix $a \in [H]^{<n}$ and suppose that $b, b' \in [H]^n$ are such that $b[m] = a = b'[m]$. Since H has no largest element, we can find $b'' \in [H]^n$ such that $b''[m] = a$ and $b''(m) > \max(b \cup b')$. In particular, b and b'' are aligned and, likewise, b' and b'' are aligned. Moreover, we have $m \subseteq \mathbf{r}(b, b'')$ and $m \subseteq \mathbf{r}(b', b'')$. It follows that u_b and $u_{b''}$ are aligned and $u_b[\mathbf{r}_m] = u_{b''}[\mathbf{r}_m]$. Similarly, $u_{b'}[\mathbf{r}_m] = u_{b''}[\mathbf{r}_m]$, so $u_b[\mathbf{r}_m] = u_{b'}[\mathbf{r}_m]$.

We now show (2). Fix $a \in [H]^{<n}$, let $m = |a|$, and suppose that $\beta < \beta'$ are elements of $H \setminus (\max(a) + 1)$. Fix ordinals $\gamma_{m+1} < \gamma_{m+2} < \dots < \gamma_{n-1}$ in $H \setminus (\beta' + 1)$. Let $b = a \cup \{\beta\} \cup \{\gamma_{m+1}, \dots, \gamma_{n-1}\}$ and $b' = a \cup \{\beta'\} \cup \{\gamma_{m+1}, \dots, \gamma_{n-1}\}$. Then b and b' are aligned, with $\mathbf{r}(b, b') = n \setminus \{m\}$, and hence $u_b \cap u_{b'} = u_b[\mathbf{r}_{n \setminus \{m\}}] = u_{b'}[\mathbf{r}_{n \setminus \{m\}}]$. Moreover, we have $u_{a \cup \{\beta\}} = u_b[\mathbf{r}_{m+1}]$ and $u_{a \cup \{\beta'\}} = u_{b'}[\mathbf{r}_{m+1}]$. Putting this all together, we obtain

$$\begin{aligned} u_{a \cup \{\beta\}} \cap u_{a \cup \{\beta'\}} &= u_b[\mathbf{r}_{m+1}] \cap u_{b'}[\mathbf{r}_{m+1}] \\ &= u_b[\mathbf{r}_{m+1}] \cap u_{b'}[\mathbf{r}_{m+1}] \cap u_b[\mathbf{r}_{n \setminus \{m\}}] \cap u_{b'}[\mathbf{r}_{n \setminus \{m\}}] \\ &= u_b[\mathbf{r}_m] \cap u_{b'}[\mathbf{r}_m] \\ &= u_a \cap u_a = u_a, \end{aligned}$$

where the second equality holds by the previous observation that $u_b \cap u_{b'} = u_b[\mathbf{r}_{n \setminus \{m\}}] = u_{b'}[\mathbf{r}_{n \setminus \{m\}}]$ and the third equality holds by Clause (3) of Definition 2.2. Therefore, $\{u_{a \cup \{\beta\}} \mid \beta \in H \setminus (\max(a) + 1)\}$ is indeed a Δ -system with root u_a . \square

We now recall a fact which follows from the main result of [12]. In its statement, we use the following notation.

Definition 2.4. Suppose that λ is an infinite regular cardinal. Recursively define cardinals $\sigma(\lambda, n)$ for $1 \leq n < \omega$ by letting $\sigma(\lambda, 1) = \lambda$ and, given $1 \leq n < \omega$, letting $\sigma(\lambda, n+1) = (2^{<\sigma(\lambda, n)})^+$.

Fact 2.5 ([12, Theorem 3.8]). *Suppose that*

- $1 \leq n < \omega$;
- $\kappa < \lambda$ are infinite cardinals, λ is regular and $<\kappa$ -inaccessible, and $\mu = \sigma(\lambda, n)$;
- $c : [\mu]^n \rightarrow 2^{<\kappa}$;
- for all $b \in [\mu]^n$, we are given a set $u_b \in [\text{On}]^{<\kappa}$.

Then there are $H \in [\mu]^\lambda$ and $k < 2^{<\kappa}$ such that

- (1) $c(b) = k$ for all $b \in [H]^n$;
- (2) $\langle u_b \mid b \in [H]^n \rangle$ is a uniform n -dimensional Δ -system.

In order to motivate these definitions, let us highlight a way in which Fact 2.5 will be employed.

Lemma 2.6. *Suppose that n is a positive integer, H is a set of ordinals, and $\langle p_b \mid b \in [H]^n \rangle$ is a sequence of conditions in some $\mathbb{P} = \text{Add}(\omega, \chi)$ such that*

- *there is a fixed \bar{p} such that $\bar{p} = \bar{p}_b$ for all $b \in [H]^n$, and*
- *$\langle u(p_b) \mid b \in [H]^n \rangle$ is a uniform n -dimensional Δ -system.*

Then for all $a, a' \in [H]^n$, if a and a' are aligned, then p_a and $p_{a'}$ are compatible in \mathbb{P} .

Proof. Fix $a, a' \in [H]^n$ such that a and a' are aligned. To show that p_a and $p_{a'}$ are compatible, it suffices to show that, for every $(\alpha, m) \in \text{dom}(p_a) \cap \text{dom}(p_{a'})$, we have $p_a(\alpha, m) = p_{a'}(\alpha, m)$. To this end, fix $(\alpha, m) \in \text{dom}(p_a) \cap \text{dom}(p_{a'})$, so $\alpha \in u(p_a) \cap u(p_{a'})$. Since $\langle p_b \mid b \in [H]^n \rangle$ is a uniform n -dimensional Δ -system and $a, a' \in [H]^n$ are aligned, it follows that $u(p_a)$ and $u(p_{a'})$ are aligned. There is therefore a single $i < \omega$ such that $\alpha = u(p_a)(i) = u(p_{a'})(i)$. But then, since $\bar{p} = \bar{p}_\alpha = \bar{p}_{\alpha'}$, we have $p_a(\alpha, m) = \bar{p}(i, m) = p_{a'}(\alpha, m)$, as desired. \square

2.3. Higher-dimensional coherence and triviality. Our main theorem is an assertion about the derived limits of an inverse system \mathbf{A} ; just as in [13] and [6] (and, indeed, as in all the intervening works cited in the latter), our analysis of these limits will be via their reformulation in terms of multidimensionally coherent indexed families of functions from subsets of ω^2 to \mathbb{Z} . Readers are

referred to [6] for fuller details of this reformulation. We turn now to the relevant conventions and definitions.

Given functions $f, g : \omega \rightarrow \omega$, let $f \leq g$ if and only if $f(j) \leq g(j)$ for all $j \in \omega$. We say that a subset $X \subseteq {}^\omega\omega$ is *downwards closed* if, for all $f \in X$ and $g \in {}^\omega\omega$, if $g \leq f$, then $g \in X$. Let $I(f)$ denote the set $\{(j, k) \in \omega^2 \mid k \leq f(j)\}$; visually, this is the region below the graph of f . Given a sequence $\vec{f} = (f_0, \dots, f_n)$ of elements of ${}^\omega\omega$, let $\wedge \vec{f}$ denote the greatest lower \leq -bound of the functions f_0, \dots, f_n . For any such sequence and $i \leq n$, let \vec{f}^i denote the sequence of length n obtained by removing the i^{th} entry of \vec{f} ; in symbols, $\vec{f}^i = (f_0, \dots, f_{i-1}, f_{i+1}, \dots, f_n)$, sometimes written as $(f_0, \dots, \hat{f}_i, \dots, f_n)$. If π is a permutation of $(0, \dots, n)$, then $\text{sgn}(\pi)$ denotes the *sign* or *parity* of π , recorded as a 1 or -1 . The notation $\pi(\vec{f})$ denotes the sequence $(f_{\pi(0)}, \dots, f_{\pi(n)})$.

If φ and ψ are partial functions from ω^2 to \mathbb{Z} , then the expression $\varphi =^* \psi$ will mean that the set $\{(j, k) \in \text{dom}(\varphi) \cap \text{dom}(\psi) \mid \varphi(j, k) \neq \psi(j, k)\}$ is finite. Implicit in this expression, in other words, are restrictions of φ and ψ to their shared domain; a similar convention will apply to sums of such functions below.

Definition 2.7. Fix an $X \subseteq {}^\omega\omega$ and a positive integer n and suppose that

$$\Phi = \left\langle \varphi_{\vec{f}} : I(\wedge \vec{f}) \rightarrow \mathbb{Z} \mid \vec{f} \in X^n \right\rangle$$

is an indexed family of functions.

- Φ is *alternating* if

$$\varphi_{\pi(\vec{f})} = \text{sgn}(\pi) \varphi_{\vec{f}}$$

for every $\vec{f} \in X^n$ and every permutation π of $(0, \dots, n-1)$.

- Φ is *n-coherent* if it is alternating and

$$\sum_{i=0}^n (-1)^i \varphi_{\vec{f}^i} =^* 0$$

for all $\vec{f} \in X^{n+1}$.

- If $n = 1$, then Φ is *n-trivial* (i.e., *1-trivial*) if there exists a $\tau : \omega^2 \rightarrow \mathbb{Z}$ such that

$$\tau =^* \varphi_f$$

for all $f \in X$. If $n > 1$, then Φ is *n-trivial* if there exists an alternating family

$$\mathcal{T} = \left\langle \tau_{\vec{f}} : I(\wedge \vec{f}) \rightarrow \mathbb{Z} \mid \vec{f} \in X^{n-1} \right\rangle$$

such that

$$\sum_{i=0}^{n-1} (-1)^i \tau_{\vec{f}^i} =^* \varphi_{\vec{f}}$$

for all $\vec{f} \in X^n$. We term such a τ or \mathcal{T} an *n-trivialization* of Φ .

When it is clear from context, we will frequently omit the prefix *n-* when speaking of triviality. Lastly, if $A \subseteq X$, then $\Phi \upharpoonright A$ denotes $\langle \varphi_{\vec{f}} \mid \vec{f} \in A^n \rangle$.

Because the equalities in the definitions of *n-coherent* and *n-trivial* are all *mod finite*, it follows that the *n-coherence* or *n-triviality* of a family are not affected by altering finitely many values of each of its functions. In particular, we obtain the following useful observation.

Observation 2.8. Fix an $\ell \in \omega$ and a family of functions $\Phi = \langle \varphi_{\vec{f}} : I(\wedge \vec{f}) \rightarrow \mathbb{Z} \mid \vec{f} \in X^n \rangle$. Let $\tilde{\Phi} = \langle \tilde{\varphi}_{\vec{f}} \mid \vec{f} \in X^n \rangle$ where

$$\tilde{\varphi}_{\vec{f}}(j, k) = \begin{cases} 0 & \text{if } j \leq \ell \\ \varphi_{\vec{f}}(j, k) & \text{if } j > \ell \end{cases}$$

for all $(j, k) \in I(\wedge \vec{f})$. Then Φ is trivial if and only if $\tilde{\Phi}$ is.

By the following equivalence, our main theorem is, equivalently, a statement about the n -triviality of all n -coherent families of functions $\Phi = \langle \varphi_{\vec{f}} : I(\wedge \vec{f}) \rightarrow \mathbb{Z} \mid \vec{f} \in (\omega^\omega)^n \rangle$.

Fact 2.9 ([3, 6]). *For all positive integers n , $\lim^n \mathbf{A} = 0$ if and only if every n -coherent family of functions $\langle \varphi_{\vec{f}} : I(\wedge \vec{f}) \rightarrow \mathbb{Z} \mid \vec{f} \in (\omega^\omega)^n \rangle$ is trivial.*

Our overall argument's strategy is to arrange this fact's latter condition; in such an approach, homological algebraic considerations appear as essentially external, or preliminary, to our main work. Instrumental in our forcing arguments, however, will be a more locally finitary characterization of n -triviality, one connecting to that of Definition 2.7 via a long exact sequence of higher derived limits of \mathbf{A} -related inverse systems. As mentioned in the introduction, readers interested primarily in the set theoretic aspects of this work may, after reading the statement of Proposition 2.10 and the remark immediately thereafter, safely skip the remainder of this section, which is largely algebraic in nature, and proceed directly to Section 3.

Proposition 2.10. *Fix $X \subseteq {}^\omega\omega$ and a positive integer n and let $\Phi = \langle \varphi_{\vec{f}} \mid \vec{f} \in X^n \rangle$ be an n -coherent family of functions. Then Φ is trivial if and only if there exists an $\ell < \omega$ and an alternating family of finitely supported functions $\Psi = \langle \psi_{\vec{f}} : I(\wedge \vec{f}) \rightarrow \mathbb{Z} \mid \vec{f} \in X^n \rangle$ such that*

$$\sum_{i=0}^n (-1)^i \varphi_{\vec{f}_i}(j, k) = \sum_{i=0}^n (-1)^i \psi_{\vec{f}_i}(j, k)$$

for all $\vec{f} \in X^{n+1}$ and all $(j, k) \in I(\wedge \vec{f})$ with $j \geq \ell$.

Remark 2.11. When there is a possibility of confusion, we will refer to a Ψ as in the statement of Proposition 2.10 as a *type II trivialization* and a \mathcal{T} or τ as in Definition 2.7 as a *type I trivialization*. By and large, however, these two sorts of trivializations correspond to two distinct phases of our argument; in particular, the trivializations under discussion in Sections 3 and 6 are all of type I, while those under discussion in Sections 4 and 5 are of type II.

We need some preliminary definitions and results before continuing to the proof of Proposition 2.10.

Definition 2.12. The inverse systems

- $\mathbf{A} = (A_f, p_{fg}, {}^\omega\omega)$
- $\mathbf{B} = (B_f, q_{fg}, {}^\omega\omega)$
- $\mathbf{B}/\mathbf{A} = ((B/A)_f, r_{fg}, {}^\omega\omega)$

are defined as follows: $A_f = \bigoplus_{I(f)} \mathbb{Z}$, $B_f = \prod_{I(f)} \mathbb{Z}$, and $(B/A)_f = B_f/A_f$, for all f in ${}^\omega\omega$. For all $f \leq g$ in ${}^\omega\omega$, the bonding maps $p_{fg} : A_g \rightarrow A_f$ are simply the projection maps; similarly for the bonding maps of \mathbf{B} and \mathbf{B}/\mathbf{A} . These systems assemble in a short exact sequence

$$(1) \quad 0 \longrightarrow \mathbf{A} \longrightarrow \mathbf{B} \longrightarrow \mathbf{B}/\mathbf{A} \longrightarrow 0$$

which induces, in turn, a long exact sequence of derived limits:

$$(2) \quad 0 \longrightarrow \lim \mathbf{A} \longrightarrow \lim \mathbf{B} \longrightarrow \lim \mathbf{B}/\mathbf{A} \xrightarrow{\partial^0} \lim^1 \mathbf{A} \longrightarrow \lim^1 \mathbf{B} \longrightarrow \lim^1 \mathbf{B}/\mathbf{A} \xrightarrow{\partial^1} \dots$$

For any $X \subseteq {}^\omega\omega$ we write $\mathbf{A} \upharpoonright X$ for the restriction of \mathbf{A} to the index-set X ; similarly for \mathbf{B} and \mathbf{B}/\mathbf{A} . Observe that for any such X , short and long exact sequences just as above exist for $\mathbf{A} \upharpoonright X$, $\mathbf{B} \upharpoonright X$, and $\mathbf{B}/\mathbf{A} \upharpoonright X$.

The expressions \lim^n denote the *derived limits* of the inverse limit functor \lim ; they are functors taking, in our context, inverse systems to the category of abelian groups. A standard heuristic for these functors is that, in aggregate, they at least potentially recover the data of an inverse system that the \lim functor alone might lose. The exact sequences above are a main instance of this dynamic: \lim alone applied to the sequence (1) may fail to conserve its exactness, but when applied in combination with the higher derived limits \lim^n , as in (2), it does transmit exactness, as desired.

Our characterizations of the vanishing of $\lim^n \mathbf{A}$ derive from the isomorphism $\lim^n \mathbf{B}/\mathbf{A} \cong \lim^{n+1} \mathbf{A}$ for all $n > 0$; the *mod finite* relations of Definition 2.7, for example, are an artifact of the modulus \mathbf{A} on the left-hand side of this isomorphism. This isomorphism, in turn, is an effect of the following lemma within the long exact sequence (2).

Lemma 2.13. $\lim^n(\mathbf{B} \upharpoonright X) = 0$ for all downwards closed $X \subseteq {}^\omega\omega$ and $n > 0$.

We argue this fact via more concrete characterizations of $\lim^n(\mathbf{B} \upharpoonright X)$; these are essentially those given by [6, §2.2].

Definition 2.14. For all downwards closed $X \subseteq {}^\omega\omega$ and $n \geq 0$, the group $\lim^n(\mathbf{B} \upharpoonright X)$ is the cohomology of the cochain complex

$$\dots \xrightarrow{d^{n-1}} K^n(\mathbf{B} \upharpoonright X) \xrightarrow{d^n} K^{n+1}(\mathbf{B} \upharpoonright X) \xrightarrow{d^{n+1}} \dots$$

where $K^n(\mathbf{B} \upharpoonright X)$ denotes the subgroup of

$$\prod_{\vec{f} \in X^{n+1}} B_{\wedge \vec{f}}$$

whose elements c satisfy

$$c(\pi(\vec{f})) = \text{sgn}(\pi)c(\vec{f})$$

for all $\vec{f} \in X^{n+1}$ and permutations π of $(0, \dots, n)$. The differentials d^n are defined as usual: for any $c \in K^n(\mathbf{B} \upharpoonright X)$,

$$(3) \quad d^n c(\vec{f}) = \sum_{i=0}^{n+1} (-1)^i \left(c(\vec{f}^i) \upharpoonright I(\wedge \vec{f}) \right)$$

for each $\vec{f} \in X^{n+2}$.

$\lim^n(\mathbf{A} \upharpoonright X)$ and $\lim^n(\mathbf{B}/\mathbf{A} \upharpoonright X)$ are defined analogously.¹

Proof of Lemma 2.13. Fix a downwards closed $X \subseteq {}^\omega\omega$ and an $n > 0$ along with a $c \in K^n(\mathbf{B} \upharpoonright X)$ for which $d^n c = 0$. We will define a $b \in K^{n-1}(\mathbf{B} \upharpoonright X)$ with $d^{n-1} b = c$. To that end, for each $x \in \bigcup_{f \in X} I(f)$ fix an $f_x \in X$ such that $x \in I(f_x)$. Then, for each $\vec{f} \in X^n$ and $x \in I(\wedge \vec{f})$, let

$$b(\vec{f})(x) = (-1)^n c(f_0, \dots, f_{n-1}, f_x)(x).$$

¹The significance of downward closure here is that it is only, in general, in such contexts that our alternating cochain definition of \lim^n is equivalent to the more usual definition in terms of ordered cochains (for more on these matters, see [6, §2.2] and the references cited therein).

For all $\vec{f} \in X^{n+1}$ and $x \in I(\wedge \vec{f})$, we then have

$$\begin{aligned} d^{n-1}b(\vec{f})(x) &= \sum_{i=0}^n (-1)^i b(\vec{f}^i)(x) \\ &= \sum_{i=0}^n (-1)^{n+i} c(f_0, \dots, \hat{f}_i, \dots, f_n, f_x)(x) \\ &= (-1)^n d^n c(f_0, \dots, f_n, f_x)(x) - (-1)^{2n+1} c(\vec{f})(x) \\ &= c(\vec{f})(x), \end{aligned}$$

as desired. \square

As noted, together with the X -indexed variants of the long exact sequence (2), Lemma 2.13 has as consequences isomorphisms

$$\partial^n : \lim^n (\mathbf{B}/\mathbf{A} \upharpoonright X) \xrightarrow{\cong} \lim^{n+1} (\mathbf{A} \upharpoonright X)$$

for each $n > 0$, as well as the isomorphism

$$\partial^0 : \frac{\lim (\mathbf{B}/\mathbf{A} \upharpoonright X)}{\text{im}(\lim (\mathbf{B} \upharpoonright X))} \xrightarrow{\cong} \lim^1 (\mathbf{A} \upharpoonright X).$$

As shown in [6], for all $n \geq 0$ these isomorphisms ∂^n may be defined via the following procedure:

- (1) To define $\partial^n[c]$, fix a cocycle $c \in K^n(\mathbf{B}/\mathbf{A} \upharpoonright X)$ representing the cohomology class $[c]$.
- (2) Fix then a $\Phi = \langle \varphi_{\vec{f}} : I(\wedge \vec{f}) \rightarrow \mathbb{Z} \mid \vec{f} \in X^{n+1} \rangle$ representing c in the sense that each $\varphi_{\vec{f}}$ falls in the $A_{\vec{f}}$ -coset $c(\vec{f})$. Observe that such a Φ may be chosen to be alternating, and that in this case it will be an $(n+1)$ -coherent family of functions. (In fact, for all $n \geq 0$ the left-hand side of the above isomorphisms is naturally viewed as the quotient of $(n+1)$ -coherent families of functions indexed by X by the $(n+1)$ -trivial families of functions indexed by X , as the reader may verify.)
- (3) Let $\partial^n[c]$ be the cohomology class of the cocycle $\mathbf{d}^n \Phi \in K^{n+1}(\mathbf{A} \upharpoonright X)$, where

$$\mathbf{d}^n \Phi(\vec{f}) = \sum_{i=0}^{n+1} (-1)^i \varphi_{\vec{f}^i}$$

for all $\vec{f} \in X^{n+2}$.

We are now ready for the proof of Proposition 2.10.

Proof of Proposition 2.10. We will establish the equivalence for $\ell = 0$; its extension to higher $\ell < \omega$ will then follow from Observation 2.8.

Suppose first that X is downwards closed. For all $n \geq 1$, the fact that ∂^{n-1} defined above is an isomorphism implies that an n -coherent family Φ indexed by a downwards closed $X \subseteq {}^\omega \omega$ is n -trivial if and only if $\mathbf{d}^{n-1} \Phi$ is a coboundary in $K^n(\mathbf{A} \upharpoonright X)$, i.e., if and only if there exists a family of *finitely supported* functions $\Psi = \langle \psi_{\vec{f}} : I(\wedge \vec{f}) \rightarrow \mathbb{Z} \mid \vec{f} \in X^n \rangle$ such that

$$\mathbf{d}^{n-1} \Phi(\vec{f}) = \sum_{i=0}^n (-1)^i \psi_{\vec{f}^i}$$

for all $\vec{f} \in X^{n+1}$. Since

$$\mathbf{d}^{n-1}\Phi(\vec{f}) = \sum_{i=0}^n (-1)^i \varphi_{\vec{f}^i},$$

this establishes the proposition in case X is downwards closed.

We now turn to the general case. For the forward direction, let $\Phi = \langle \varphi_{\vec{f}} \mid \vec{f} \in X^n \rangle$ be n -coherent and trivial, with a type I trivialization $\mathcal{T} = \langle \tau_{\vec{f}} \mid \vec{f} \in X^{n-1} \rangle$ (or $\tau : \omega^2 \rightarrow \mathbb{Z}$ if $n = 1$). Let Y denote the downwards closure of X and extend $\text{id} : X \rightarrow X$ to a $g : Y \rightarrow X$ for which $f \leq g(f)$ for all $f \in Y$; let $h : Y^n \rightarrow X^n$ denote the map given by coordinatewise applications of g and, if $n > 1$, define $h' : Y^{n-1} \rightarrow X^{n-1}$ in the same way. Letting $\tilde{\varphi}_{\vec{f}} = \varphi_{h(\vec{f})} \upharpoonright I(\wedge \vec{f})$ for all $\vec{f} \in Y^n$ then determines an n -coherent family $\Phi^h = \langle \tilde{\varphi}_{\vec{f}} \mid \vec{f} \in Y^n \rangle$ extending Φ that is also trivial, as witnessed by $\mathcal{T}^{h'} = \langle \tau_{h'(\vec{f})} \upharpoonright I(\vec{f}) \mid \vec{f} \in Y^{n-1} \rangle$ if $n > 1$ or simply by τ if $n = 1$. By the special case above, there then exists a $\Psi = \langle \psi_{\vec{f}} : I(\wedge \vec{f}) \rightarrow \mathbb{Z} \mid \vec{f} \in Y^n \rangle$ satisfying the equality of Proposition 2.10 for all $\vec{f} \in Y^{n+1}$. The restriction of this Ψ to X is as desired. The backwards implication of Proposition 2.10 follows from a straightforward reversal of this procedure. \square

3. THE CASE OF $n = 1$

We now argue the base case of our main theorem. The main result of this section is essentially due to Kamo [10]. In fact, the result in [10] is superior to the one presented here in that Kamo proves that $\lim^1 \mathbf{A} = 0$ in any extension obtained by adding ω_2 -many Cohen reals, whereas our hypothesis is that we have added at least $(\beth_1^+)^V$ -many Cohen reals. Our reason for presenting this slightly suboptimal proof is simply that many of the ideas of the proof of the general case appear here in a significantly simplified setting; this section therefore serves as an introduction to some of the techniques and ideas that will make an appearance in a more complicated guise later in the paper.

Theorem 3.1. *Let $\mathbb{P} = \text{Add}(\omega, \chi)$ for a cardinal $\chi > \beth_1$. The following then holds in $V^{\mathbb{P}}$: For any set $X \subseteq {}^\omega\omega$ containing at least $(\beth_1^+)^V$ -many of the Cohen reals added by \mathbb{P} , every 1-coherent family $\Phi = \langle \varphi_f \mid f \in X \rangle$ indexed by X is trivial.*

Proof. Fix a condition $p \in \mathbb{P}$ and \mathbb{P} -names \dot{X} and $\dot{\Phi} = \langle \dot{\varphi}_{\dot{f}} \mid \dot{f} \in \dot{X} \rangle$ such that

- $p \Vdash “|\{\alpha < \chi \mid \dot{f}_\alpha \in \dot{X}\}| \geq (\beth_1^+)^V”$, and
- $p \Vdash “\dot{\Phi} \text{ is a 1-coherent family}”$.

We will produce a condition $q \leq p$ forcing $\dot{\Phi}$ to be trivial.

Begin by letting Y be the set of $\alpha < \chi$ for which there is a condition $p_\alpha \leq p$ such that $p_\alpha \Vdash “\dot{f}_\alpha \in \dot{X}”$; observe that $|Y| \geq \beth_1^+$ by assumption. For each $\alpha \in Y$, fix such a condition p_α . Since \mathbb{P} is \beth_1^+ -Knaster, there exists a set $Y' \subseteq Y$ of size \beth_1^+ such that $\{p_\alpha \mid \alpha \in Y'\}$ consists of pairwise compatible conditions.

For each $(\alpha, \beta) \in [Y']^2$, fix a condition $q_{\alpha, \beta}$ extending both p_α and p_β and deciding the value of $\{(j, k) \in I(\dot{f}_\alpha \wedge \dot{f}_\beta) \mid \dot{\varphi}_{\dot{f}_\alpha}(j, k) \neq \dot{\varphi}_{\dot{f}_\beta}(j, k)\}$ to be equal to some set $\mathbf{e}_{\alpha, \beta} \in [\omega \times \omega]^{<\omega}$. By extending $q_{\alpha, \beta}$ if necessary, we may assume that $\{\alpha, \beta\} \subseteq u(q_{\alpha, \beta})$ (see again Section 2.1 for this notation). Let $u_{\alpha, \beta} = u(q_{\alpha, \beta})$.

By Fact 2.5, there exists a set $H \in [Y']^{\aleph_1}$ and a \bar{q} , \mathbf{e} , and i^* such that

- $\langle u_{\alpha, \beta} \mid (\alpha, \beta) \in [H]^2 \rangle$ is a uniform 2-dimensional Δ -system;
- $(\bar{q}_{\alpha, \beta}, \mathbf{e}_{\alpha, \beta}) = (\bar{q}, \mathbf{e})$ for all $(\alpha, \beta) \in [H]^2$;

- $\beta = u_{\alpha,\beta}(i^*)$ for all $(\alpha, \beta) \in [H]^2$.

By shrinking H if necessary, we may assume that $\text{otp}(H) = \omega_1$. Now let $\langle r_{\mathbf{m}} \mid \mathbf{m} \subseteq 2 \rangle$ witness that $\langle u_{\alpha,\beta} \mid (\alpha, \beta) \in [H]^2 \rangle$ is a uniform 2-dimensional Δ -system, and let $\langle u_\alpha \mid \alpha \in H \rangle$ and u_\emptyset be as given by Lemma 2.3. For each $\alpha \in H$, define a condition $q_\alpha \in \mathbb{P}$ by choosing a $\beta \in H \setminus (\alpha + 1)$ and letting $q_\alpha = q_{\alpha,\beta} \upharpoonright (u_\alpha \times \omega)$. We claim that this definition is independent of our choice of β . Indeed, suppose that $\beta < \beta'$ are elements of $H \setminus (\alpha + 1)$. Then $u_\alpha = u_{\alpha,\beta} \cap u_{\alpha,\beta'} = u_{\alpha,\beta}[\mathbf{r}_1] = u_{\alpha,\beta'}[\mathbf{r}_1]$, hence if $(\delta, m) \in \text{dom}(q_{\alpha,\beta}) \cap (u_\alpha \times \omega)$ then there is an $i \in \mathbf{r}_1$ such that $u_{\alpha,\beta}(i) = \delta = u_{\alpha,\beta'}(i)$. Since $\bar{q}_{\alpha,\beta} = \bar{q} = \bar{q}_{\alpha,\beta'}$, it follows that $q_{\alpha,\beta}(\delta, m) = \bar{q}(i, m) = q_{\alpha,\beta'}(\delta, m)$, so $q_{\alpha,\beta} \upharpoonright (u_\alpha \times \omega) \subseteq q_{\alpha,\beta'} \upharpoonright (u_\alpha \times \omega)$. A symmetric argument yields the reverse inclusion, showing that our definition of q_α is indeed independent of our choice of β . Observe that $q_\alpha = \bigcap \{q_{\alpha,\beta} \mid \beta \in H \setminus (\alpha + 1)\}$; as each $q_{\alpha,\beta}$ extends p_α , it follows that $q_\alpha \leq p_\alpha$ and hence that $q_\alpha \Vdash \dot{f}_\alpha \in \dot{X}$.

Similarly, define a condition q_\emptyset by choosing $(\alpha, \beta) \in [H]^2$ and letting $q_\emptyset = q_{\alpha,\beta} \upharpoonright (u_\emptyset \times \omega)$. By an argument exactly as in the previous paragraph, this definition is independent of our choice of (α, β) . Note that $q_\emptyset = \bigcap \{q_{\alpha,\beta} \mid (\alpha, \beta) \in [H]^2\}$; in consequence, since each $q_{\alpha,\beta}$ extends p , we have $q_\emptyset \leq p$.

We claim that q_\emptyset forces that $\dot{\Phi}$ is trivial. Let \dot{A} be a \mathbb{P} -name for $\{\alpha \in H \mid q_\alpha \in \dot{G}\}$, where \dot{G} is the canonical \mathbb{P} -name for the generic filter.

Claim 3.2. $q_\emptyset \Vdash |\dot{A}| = \aleph_1$.

Proof. Suppose for sake of contradiction that $r_0 \leq q_\emptyset$ is a condition in \mathbb{P} such that $r_0 \Vdash |\dot{A}| < \aleph_1$. Since $\text{otp}(H) = \omega_1$, we can therefore find a further extension r of r_0 and an $\eta \in H$ such that $r \Vdash \dot{A} \subseteq \eta$. Since $\langle u_\alpha \mid \alpha \in H \setminus \eta \rangle$ is an infinite Δ -system with root u_\emptyset , and since u_r is finite, there exists an $\alpha \in H \setminus \eta$ such that $u_\alpha \setminus u_\emptyset$ is disjoint from u_r . Decompose q_α as $q_\emptyset \cup (q_\alpha \upharpoonright (u_\alpha \setminus u_\emptyset) \times \omega)$ and observe that $r \leq q_\emptyset$ and $\text{dom}(r) \cap ((u_\alpha \setminus u_\emptyset) \times \omega) = \emptyset$. It follows that r and q_α are compatible. However, if s extends both r and q_α , then we have $s \Vdash \dot{A} \setminus \eta = \emptyset$ by virtue of s extending r and $s \Vdash \alpha \in \dot{A} \setminus \eta$ by virtue of s extending q_α , which forms a contradiction. \square

Now recall that $\beta = u_{\alpha,\beta}(i^*)$ for all $(\alpha, \beta) \in [H]^2$. Notice that $i^* \notin \mathbf{r}_1$, since the alternative would imply that $\beta \in u_\alpha = u_{\alpha,\beta}[\mathbf{r}_1]$ for all $(\alpha, \beta) \in [H]^2$, contradicting the fact that u_α is finite. Hence $\beta \in u_{\alpha,\beta} \setminus u_\alpha$ for all $(\alpha, \beta) \in [H]^2$. Let ℓ be the least natural number j such that

- $\mathbf{e} \subseteq j \times \omega$ and;
- $\{j' \mid (i^*, j') \in \text{dom}(\bar{q})\} \subseteq j$.

We then have $\{j' \mid (\beta, j') \in \text{dom}(q_{\alpha,\beta})\} \subseteq \ell$ for each $(\alpha, \beta) \in [H]^2$.

Claim 3.3. q_\emptyset forces

$$\{(j, k) \in I(\dot{f}_\alpha \wedge \dot{f}_{\alpha'}) \mid \dot{\varphi}_{\dot{f}_\alpha}(j, k) \neq \dot{\varphi}_{\dot{f}_{\alpha'}}(j, k)\} \subseteq \ell \times \omega$$

for all $\alpha, \alpha' \in \dot{A}$.

Proof. If not, then there exist an $r \leq q_\emptyset$, a pair of ordinals $\alpha < \alpha'$ in H , and a $(j, k) \in \omega \times \omega$ such that

- $r \leq q_\alpha, q_{\alpha'}$;
- $j \geq \ell$;
- $r \Vdash "(j, k) \in I(\dot{f}_\alpha \wedge \dot{f}_{\alpha'}) \text{ and } \dot{\varphi}_{\dot{f}_\alpha}(j, k) \neq \dot{\varphi}_{\dot{f}_{\alpha'}}(j, k)"$.

Both $\langle u_{\alpha,\beta} \mid \beta \in H \setminus (\alpha + 1) \rangle$ and $\langle u_{\alpha',\beta} \mid \beta \in H \setminus (\alpha + 1) \rangle$ are infinite Δ -systems with roots u_α and $u_{\alpha'}$, respectively; as u_r is finite, there therefore exists a $\beta \in H$ such that both $u_{\alpha,\beta} \setminus u_\alpha$ and $u_{\alpha',\beta} \setminus u_{\alpha'}$ are disjoint from u_r .

By Lemma 2.6, the conditions $q_{\alpha,\beta}$ and $q_{\alpha',\beta}$ are compatible. Observe also that $q_{\alpha,\beta} = q_\alpha \cup (q_{\alpha,\beta} \upharpoonright (u_{\alpha,\beta} \setminus u_\alpha) \times \omega)$. Since $r \leq q_\alpha$ and $\text{dom}(r) \cap ((u_{\alpha,\beta} \setminus u_\alpha) \times \omega) = \emptyset$, the conditions r and $q_{\alpha,\beta}$ are compatible. Similarly, r and $q_{\alpha',\beta}$ are compatible, and therefore $r^* = r \cup q_{\alpha,\beta} \cup q_{\alpha',\beta}$ is a condition in \mathbb{P} . Notice also that $\beta \notin u_r$. By the paragraph preceding Claim 3.3 and the fact that $j \geq \ell$, it follows that $(\beta, j) \notin \text{dom}(r^*)$, so we may extend r^* to a condition r^{**} such that $(\beta, j) \in \text{dom}(r^{**})$ and $r^{**}(\beta, j) = k$. In particular, r^{**} will force “ $(j, k) \in I(\dot{f}_\beta)$ ”.

Recall that $j \geq \ell$ implies $(j, k) \notin e$. Therefore, since it extends both $q_{\alpha,\beta}$ and $q_{\alpha',\beta}$ and forces “ $(j, k) \in I(\dot{f}_\alpha \wedge \dot{f}_{\alpha'} \wedge \dot{f}_\beta)$ ”, the condition r^{**} will force

$$“\dot{\varphi}_{\dot{f}_\alpha}(j, k) = \dot{\varphi}_{\dot{f}_\beta}(j, k) = \dot{\varphi}_{\dot{f}_{\alpha'}}(j, k)”,$$

contradicting the fact that $r^{**} \leq r$ and $r \Vdash “\dot{\varphi}_{\dot{f}_\alpha}(j, k) \neq \dot{\varphi}_{\dot{f}_{\alpha'}}(j, k)”$. \square

Now let G be \mathbb{P} -generic over V with $q_\emptyset \in G$. Let $\Phi = \langle \varphi_f \mid f \in X \rangle$ and A denote the realizations in $V[G]$ of $\dot{\Phi}$ and \dot{A} , respectively. Define a function $\tau : \omega^2 \rightarrow \mathbb{Z}$ as follows. For any $(j, k) \in \omega \times \omega$ with $j \geq \ell$, if $(j, k) \in I(f_\alpha)$ for some $\alpha \in A$ then let $\tau(j, k) = \varphi_{f_\alpha}(j, k)$ (by Claim 3.3, this definition is independent of our choice of α). In all other cases, let $\tau(j, k) = 0$.

We claim that τ witnesses that Φ is trivial. Assume for contradiction that it does not, so that for some $f \in X$ the set $E_f := \{(j, k) \in I(f) \mid \varphi_f(j, k) \neq \tau(j, k)\}$ is infinite. Since $I(f) \cap (\ell \times \omega)$ is finite, the set $E_f^* = E_f \cap ([\ell, \omega) \times \omega)$ is then infinite and there are infinitely many $j < \omega$ for which $E_f^* \cap (\{j\} \times \omega) \neq \emptyset$. As \mathbb{P} has the countable chain condition, $E_f \in V[G_W]$ for some countable set $W \subseteq \chi$ in V . Fix $\alpha \in A \setminus W$. By genericity, $I(f_\alpha) \cap E_f^*$ is infinite. It follows from our definition of τ that $\tau(j, k) = \varphi_{f_\alpha}(j, k)$ for all $(j, k) \in I(f_\alpha) \cap E_f^*$. By the coherence of Φ , we have $\varphi_f =^* \varphi_{f_\alpha}$, and hence $\varphi_f \upharpoonright (I(f_\alpha) \cap E_f^*) =^* \tau \upharpoonright (I(f_\alpha) \cap E_f^*)$, contradicting the definition of E_f^* and the fact that $I_{f_\alpha} \cap E_f^*$ is infinite. This shows that q_\emptyset forces $\dot{\Phi}$ to be trivial, concluding the proof. \square

4. DEFINING TRIVIALIZATIONS

We are now ready to begin the proof of the general case of our main theorem. Before we do so, we pause to give a rough general overview of the shape of the argument. First recall that, in light of Fact 2.9, it will suffice to prove the following theorem.

Theorem 4.1. *Let $\mathbb{P} = \text{Add}(\omega, \beth_\omega)$. Then in $V^\mathbb{P}$, for every $n > 0$, every n -coherent family of functions $\langle \varphi_{\vec{f}} \mid \vec{f} \in (\omega^\omega)^n \rangle$ is trivial.*

The proof will consist of three main steps, the last two of which were already legible in the $n = 1$ case, spread out over this and the next two sections. In the first step, we introduce some abstract algebraic machinery, developed in [6], that will aid us in defining trivializations for subfamilies of n -coherent families. In the second step, we will use this machinery to prove a lemma indicating that, after adding sufficiently many Cohen reals, every n -coherent family indexed by a set containing sufficiently many of these Cohen reals has a large subfamily that is trivial (this statement will be made more precise shortly). In the third and final step, we will prove that the triviality of these large subfamilies can be propagated to the entire n -coherent family, thus proving its triviality as well.

With this in mind, we turn now to describing the machinery of [6] that will be used to define type II trivializations of subfamilies of n -coherent families of functions. We remark that, already in [6], this apparatus takes on a certain opacity; as this is, if anything, even more the case for the variations listed here, some heuristic remarks appear at the end of this section.

Definition 4.2. Suppose that b is a finite set of ordinals. A *subset-final segment* of b of length m is a sequence $\vec{a} = \langle a_i \mid 1 \leq i \leq m \rangle$ such that

- $m \leq |b|$,
- $a_1 \subseteq \cdots \subseteq a_m = b$, and
- $|a_i| = |a_1| + i - 1$ for all i with $1 \leq i \leq m$.

If \vec{a} is a subset-final segment of b and $|a_1| = 1$, then we say that \vec{a} is a *long string* or a *long string for* b . Notice that in this case $m = |b|$. If \vec{a} is not long, then it is *short*.

Suppose now that X is a set of ordinals and we are working with an injective sequence $\langle f_\alpha \mid \alpha \in X \rangle$ of elements of ${}^\omega\omega$. (In the present context, this will always be a sequence of Cohen reals, but that is not important for the results in this section.) For each nonempty $\vec{\alpha} = \langle \alpha_k \mid k < n \rangle$ in $X^{<\omega}$, let $I(\vec{\alpha})$ denote $\bigcap_{k < n} I(f_{\alpha_k})$. Suppose that for each positive integer n the family $\Phi_n = \langle \varphi_{\vec{\alpha}} : I(\vec{\alpha}) \rightarrow \mathbb{Z} \mid \vec{\alpha} \in X^n \rangle$ is n -coherent, and let $\vec{\Phi}$ denote the family $\langle \varphi_{\vec{\alpha}} \mid \vec{\alpha} \in X^{<\omega} \rangle$. Suppose also that to each nonempty $a \in [X]^{<\omega}$ we have assigned an ordinal $\varepsilon_a \in X$ in such a way that

- if $a = \{\alpha\}$, then $\varepsilon_a = \alpha$;
- if $a \subsetneq b$, then $\varepsilon_a < \varepsilon_b$.

Now, given a nonempty $b \in [X]^{<\omega}$ and a subset-final segment $\vec{a} = \langle a_k \mid 1 \leq k \leq m \rangle$ of b , define the set $d_{\vec{a}}^{\vec{\varepsilon}}$ as follows. If $|a_1| = 1$, then let $d_{\vec{a}}^{\vec{\varepsilon}} = \{\varepsilon_{a_k} \mid 1 \leq k \leq m\}$. Note that, in this case, $d_{\vec{a}}^{\vec{\varepsilon}} \in [X]^{|b|}$. If $|a_1| > 1$, then let $d_{\vec{a}}^{\vec{\varepsilon}} = a_1 \cup \{\varepsilon_{a_k} \mid 1 \leq k \leq m\}$. Note that, in this case, $d_{\vec{a}}^{\vec{\varepsilon}} \in [X]^{|b|+1}$.

For $\vec{\alpha} \in X^{<\omega}$ of length at least two, let

$$e^{\vec{\Phi}}(\vec{\alpha}) = \sum_{i < |\vec{\alpha}|} (-1)^i \varphi_{\vec{\alpha}^i}.$$

When the family $\vec{\Phi}$ is clear from context, we will omit it from the superscript; similarly for the superscript of $d_{\vec{a}}^{\vec{\varepsilon}}$. Recall also our habit of viewing finite sets $a \in [X]^{<\omega}$ as sequences enumerated in increasing order; expressions like $e(a)$ should be interpreted on this principle. Since each Φ_n is n -coherent, $e(\vec{\alpha})$ is finitely supported for each $\vec{\alpha} \in X^{<\omega}$. Let $e(\vec{\alpha})$ denote the restriction of $e(\vec{\alpha})$ to its support.

We will be interested in linear combinations L of the form

$$\sum_{i < \ell} c_i e(\vec{\alpha}_i),$$

where $\ell < \omega$, each c_i is an integer, and each $\vec{\alpha}_i$ is an element of $X^{<\omega}$ of length at least two.² Given such a linear combination L and an ordinal $\varepsilon \in X$, we let the expression $L * \varepsilon$ denote

$$\sum_{i < \ell} c_i e(\vec{\alpha}_i \frown \langle \varepsilon \rangle).$$

For integers $n \geq 2$, we now define interrelated

- linear combinations $\mathcal{A}_n^{\vec{\Phi}, \vec{\varepsilon}}(a)$, parametrized by $a \in [X]^n$, and
- linear combinations $\mathcal{C}_n^{\vec{\Phi}, \vec{\varepsilon}}(b)$, parametrized by $b \in [X]^{n+1}$.

We again omit the superscripts $\vec{\Phi}$ and $\vec{\varepsilon}$ and restriction-notations whenever they are contextually clear; as elsewhere, sums of functions in expressions like $\mathcal{C}_n(b)$ below should always be understood to be taken on the intersection of those functions' domains.

²In settings like these, notational choices are simply of the lesser evil. We will write $\vec{\alpha}_i(j)$ for the j^{th} element of $\vec{\alpha}_i$.

We begin our definitions by letting

$$\mathcal{A}_2(a) = e(a \frown \langle \varepsilon_a \rangle).$$

for each $a \in [X]^2$. Next, suppose that $2 \leq n < \omega$ and $\mathcal{A}_n(a)$ has been defined for all $a \in [X]^n$. Given $b \in [X]^{n+1}$, let

$$(4) \quad \mathcal{C}_n(b) = e(b) - \sum_{i=0}^n (-1)^i \mathcal{A}_n(b^i), \text{ and}$$

$$(5) \quad \mathcal{A}_{n+1}(b) = (-1)^{n+1} \mathcal{C}_n(b) * \varepsilon_b.$$

The following lemma is easily verified by induction on n ; at the suggestion of the referee, we provide a sketch of the proof.

Lemma 4.3. *For all $b \in [X]^2$, letting $\vec{a} = \langle b \rangle$, we have $\mathcal{A}_2(b) = e(d_{\vec{a}})$.*

For all n with $2 \leq n < \omega$ and all $b \in [X]^{n+1}$, we have:

(1) $\mathcal{C}_n(b)$ is of the form

$$e(b) + \sum_{i < \ell} c_i e(d_{\vec{a}_i}),$$

where $\ell < \omega$ and, for each $i < \ell$, c_i is an integer and \vec{a}_i is a short subset-final segment of some element of $[b]^n$.

(2) $\mathcal{A}_{n+1}(b)$ is of the form

$$\sum_{i < \ell} c_i e(d_{\vec{a}_i}),$$

where $\ell < \omega$ and, for each $i < \ell$, c_i is an integer and \vec{a}_i is a short subset-final segment of b .

Proof. The first sentence of the lemma is immediate from the definition of $\mathcal{A}_2(b)$. For the rest, we prove (1) and (2) simultaneously by induction on n . So suppose that $2 \leq n < \omega$ and $b \in [X]^{n+1}$. Recall that

$$\mathcal{C}_n(b) = e(b) - \sum_{i=0}^n (-1)^i \mathcal{A}_n(b^i),$$

For each $i \leq n$, we know by the induction hypothesis (or the definition of \mathcal{A}_2 if $n = 2$) that $\mathcal{A}_n(b^i)$ is of the form

$$\sum_{j < \ell} c_j e(d_{\vec{a}_j}),$$

where each c_j is an integer and each \vec{a}_j is a short subset-final segment of b^i (and in particular is a short subset-final segment of some element of $[b]^n$). It follows that $\mathcal{C}_n(b)$ is of the desired form.

Next, recall that $\mathcal{A}_{n+1}(b) = (-1)^{n+1} \mathcal{C}_n(b) * \varepsilon_b$. By (1), we can express $\mathcal{C}_n(b)$ as

$$e(b) + \sum_{i < \ell} c_i e(d_{\vec{a}_i}),$$

where each \vec{a}_i is a short subset-final segment of some element of $[b]^n$. Note first that $e(b \frown \langle \varepsilon_b \rangle) = e(d_{\vec{a}})$, where \vec{a} is the short subset-final segment $\langle b \rangle$ of b . Next, for each $i < \ell$, since \vec{a}_i is a short subset-final segment of some element of $[b]^n$, and since $|b| = n+1$, it follows that $\vec{a} = \vec{a}_i \frown \langle b \rangle$ is a short subset-final segment of b , and moreover that $e(d_{\vec{a}_i} \frown \langle \varepsilon_b \rangle) = e(d_{\vec{a}})$. Putting this all together, we see that \mathcal{A}_{n+1} is of the desired form. \square

One consequence of this lemma is that, while the expressions defining $\mathcal{A}_n(a)$ and $\mathcal{C}_n(b)$ are constructed via a recursion involving $\vec{\Phi}$, the actual values of \mathcal{A}_n and \mathcal{C}_n are only dependent on Φ_n , so we can meaningfully speak of them in situations in which we have only Φ_n , and not Φ_m for any $m \neq n$, before us. In addition, the values of $\mathcal{A}_n(a)$ and $\mathcal{C}_n(b)$ are only dependent on ordinals ε_c for nonempty $c \subseteq a$ or $c \subseteq b$, respectively, so, when working just with the expressions $\mathcal{A}_n(a)$ or $\mathcal{C}_n(b)$, we need not require that ε_c is defined for any c that is not a subset of a or b , respectively. Finally, if $b \in [X]^{n+1}$ and $(j, k) \in \omega \times \omega$ is an element of $I(f_{\varepsilon_a})$ for all nonempty $a \subseteq b$, then (j, k) is in the domain of $\mathcal{C}_n(b)$.

The following is a consequence of [6, Lemma 6.4].

Lemma 4.4. *Suppose that $2 \leq n < \omega$, $b \in [X]^{n+1}$, $(j, k) \in \omega \times \omega$, and the following two statements hold.*

- *There exists a single integer w such that $e(d_{\vec{a}})(j, k) = w$ for every long string \vec{a} for b .*
- *$(j, k) \in I(f_{\varepsilon_a})$ for all nonempty $a \subseteq b$.*

Then $\mathcal{C}_n(b)(j, k) = 0$.

Remark 4.5. We briefly describe how Lemma 4.4 follows from the argument of [6, Lemma 6.4]. The set b here corresponds to τ in that result. The expression $\mathcal{S}_n(\tau)$ in [6] is an auxiliary expression that always equals 0. The two assumptions in Lemma 4.4 play the role of the statement $\mathfrak{u}_n(\tau)$ from [6].

As we will see, the significance of these definitions is the following: the expressions $\mathcal{A}_n(a)$ will correspond to the elements of candidate type II trivializations of $\Phi \restriction A$. Under the conditions of Lemma 4.4, the expressions $\mathcal{C}_n(b)$ amount to verifications that these families of expressions $\mathcal{A}_n(a)$ do indeed trivialize $\Phi \restriction A$. These relations, together with the necessity of working coordinatewise, as in Lemma 4.4, are points we expand on in the following heuristic remark. Though not strictly needed for following our argument, it is hoped that it may be clarifying.

Remark 4.6. The variability in $|d_{\vec{a}}|$ noted above, depending on whether $|a_1| = 1$, underscores the unique status of the sets $d_{\vec{a}}$ which are indexed by long strings; it is on the cancellations between their associated terms $e(d_{\vec{a}})$ that the desired relations between the other terms in $\mathcal{C}_n(b)$ depend. This we hope to illuminate by the following diagram and discussion.

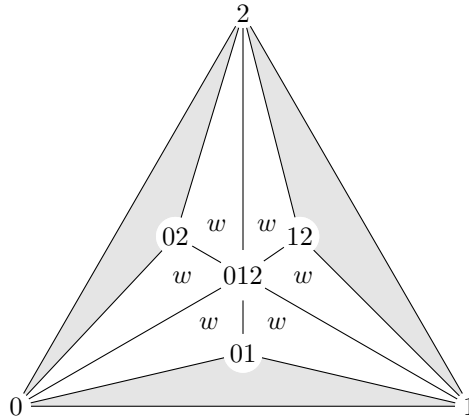


FIGURE 1. The subdivision organizing the $n = 2$ case.

The n -tuples \vec{f} structuring our various coherent families Φ are naturally viewed as simplices, with $(n-1)$ -tuples \vec{f}^i ($i < n$) as faces. Within this view, strings of increasing subsets $\cdots \subsetneq a_i \subsetneq \cdots$ of $b = \vec{f}$ correspond to simplices in the *barycentric subdivision* of the simplex b (see, e.g., [20, Chapter 3.3]). For example, the white inner region of Figure 1 corresponds to the barycentric subdivision of the triangle with vertices 0, 1, and 2 (we bend that region's edges for reasons soon to be made clear): the 2-faces of that subdivided region are $\{0, 01, 012\}$, $\{1, 01, 012\}$, etc., each corresponding to inclusion-increasing sequences of nonempty subsets of $\{0, 1, 2\}$.

Fix now a 2-coherent family of functions Φ . Let $a = \{\alpha_0, \alpha_1, \alpha_2\} \in [X]^3$ and suppose that the hypotheses of Lemma 4.4 hold. For readability, we let, for example, $\varepsilon_{0,1}$ denote $\varepsilon_{\alpha_0, \alpha_1}$ (in particular, $\varepsilon_\ell = \alpha_\ell$ for $\ell < 3$). Writing $\varphi_{0,1}$ for the function indexed by $(f_{\varepsilon_0}, f_{\varepsilon_1})$, and so on, what are wanted are *finitely supported* functions $\psi_{0,1}$, and so on, whose differences reproduce those among the corresponding functions of Φ , as described in Proposition 2.10. The idea of the above machinery is to derive these finitely supported functions from the coherence of Φ itself, as the differences between carefully chosen families of functions φ ; at the stage $n = 2$, for example, we will have

$$(6) \quad \psi_{0,1} := \mathcal{A}_2(\varepsilon_0, \varepsilon_1) = \varphi_{1,01} - \varphi_{0,01} + \varphi_{0,1}.$$

Visually, this definition corresponds to the grey triangle at the base of Figure 1, under the natural association of the functions $\varphi_{1,01}$, $\varphi_{0,01}$, and $\varphi_{0,1}$ with the edges $\{1, 01\}$, $\{0, 01\}$, and $\{0, 1\}$, respectively. Under this correspondence, the desired relation

$$(7) \quad \varphi_{1,2} - \varphi_{0,2} + \varphi_{0,1} = \psi_{1,2} - \psi_{0,2} + \psi_{0,1}$$

may be viewed as asserting the equality of the oriented sum of the functions φ associated to the boundary of the triangle $\{0, 1, 2\}$ with that of the functions φ associated to the boundary of the grey region. (As should be clear, this is a deliberately schematic discussion; we return to the question of the argument of these functions below.) This holds precisely because of the first bulleted “long string” condition listed in Lemma 4.4, which amounts in the present context to the boundary sums associated to the triangles $\{0, 01, 012\}$, $\{1, 01, 012\}$, etc., all equaling w . As these are oriented sums, they entail cancellations, so that first, the boundary sum associated to $\{0, 1, 01\}$ may be identified with that associated to $\{0, 1, 012\}$, and second, such identifications for each of the grey triangles cancel inside the triangle $\{0, 1, 2\}$, leaving nothing summed but its boundary, just as equation (7) requires.

At the arithmetic level, all of this manifests (with only minor notational adjustments) as exactly the two types of cancellations in the summed equations concluding Section 6 of [6]. The ensuing simplification of that sum is an instance of what Lemma 4.4 records as $\mathcal{C}_n(b) = 0$, which translates, in turn, to the equation (7) we had desired.

The subdivision perspective sketched above is valuable for returning sense to what appear here or in [6] as rather opaque and complicated algebraic identities: the meaning of those identities is that, in forcing extensions, higher-dimensional Δ -systems can determine trivializing structures within n -coherent families by uniformizing the boundary sums associated to the n -faces of the barycentric subdivision of any $(n+1)$ -tuple of indices, viewed as a simplex. This perspective clarifies the passage from one dimension to the next, as well; as the interested reader may verify, in the $n = 3$ case, along with the face $\{0, 1, 2\}$, the 2-faces pictured in Figure 1 play within a tetrahedron exactly the role that the boundaries of the grey faces had played within a triangle in the case of $n = 2$. Put differently, the way verification-expressions \mathcal{C}_n figure in the trivializing expressions \mathcal{A}_{n+1} of the next level, as in equation (5), amounts to little other than the fact that the restriction of the barycentric subdivision of an $(n+1)$ -simplex to any n -face is a barycentric subdivision of that face.

Complicating the above considerations, however, is the issue of domain: for the right-hand side of equations like (7) to truly be trivializing in the sense of Proposition 2.10, the domain of $\psi_{0,1}$, like that of $\varphi_{0,1}$, must be $I(f_{\varepsilon_0} \wedge f_{\varepsilon_1})$, and similarly for the functions $\psi_{0,2}$ and $\psi_{1,2}$. If $\psi_{0,1}$ is defined as in equation (6), then this amounts to a requirement that $f_{\varepsilon_{01}} \geq f_{\varepsilon_0} \wedge f_{\varepsilon_1}$, which, if the functions indexed are Cohen reals, can never be the case. This is a requirement we can only meet locally, choosing for each $(j, k) \in I(f_{\varepsilon_0} \wedge f_{\varepsilon_1})$ an $\varepsilon_{01}^{j,k}$ such that $(j, k) \in I(f_{\varepsilon_{01}^{j,k}})$. This is the approach we take, and this is the meaning of the parameter (j, k) appearing in Lemma 4.4. The good news in this approach is that equations like (7) hold if and only if they hold coordinatewise, so that our arithmetic is essentially unaffected. The bad news is that functions like $\psi_{0,1}$ may now fail to be finitely supported, and much of the work of the following section is towards ensuring that they will be.

5. TRIVIALIZING SUBFAMILIES

In this section, we state and prove a lemma indicating that, after adding sufficiently many Cohen reals, every n -coherent family indexed by a set containing sufficiently many of these Cohen reals has a large subfamily that is trivial. To make this statement precise, we begin by defining a sequence of cardinals $\langle \lambda_n \mid 1 \leq n < \omega \rangle$. First, as in Theorem 3.1, let $\lambda_1 = \beth_1^+$; then, for all $n > 1$, let $\lambda_n = \sigma(\lambda_{n-1}^+, 2n+1)$ (again see Definition 2.4 for the notation $\sigma(\cdot, \cdot)$). Note that $\sup\{\lambda_n \mid n < \omega\} = \beth_\omega$. It is also readily verified that each λ_n is $<\aleph_1$ -inaccessible, as defined in Section 2.1; in consequence, since λ_n is a successor cardinal, λ_n^+ is also $<\aleph_1$ -inaccessible.

We remark at the outset that this section is the most technically demanding of the paper; on first pass, readers who are looking for a high-level overview of the proof of our main result may want to read the statement of Lemma 5.1 and merely skim its proof before continuing to the conclusion of the argument in Section 6, returning later to this section to fill in the missing details if desired.

Lemma 5.1. *Let $2 \leq n < \omega$, let $\chi \geq \lambda_n$ be a cardinal, and let $\mathbb{P} = \text{Add}(\omega, \chi)$. The following then holds in $V^{\mathbb{P}}$: For every set $X \subseteq {}^\omega\omega$ containing at least λ_n -many of the Cohen reals added by \mathbb{P} , and for every n -coherent family $\Phi = \langle \varphi_{\vec{f}} \mid \vec{f} \in X^n \rangle$ indexed by X , there is $B \subseteq X$ such that B contains at least λ_{n-1} -many of the Cohen reals added by \mathbb{P} and $\Phi \upharpoonright B$ is trivial.*

Proof. Let $\langle \dot{f}_\alpha \mid \alpha < \chi \rangle$ be the sequence of canonical names for the Cohen reals added by \mathbb{P} . Fix \mathbb{P} -names \dot{X} and $\dot{\Phi} = \langle \dot{\varphi}_{\dot{f}} \mid \dot{f} \in \dot{X}^n \rangle$, and a condition $p \in \mathbb{P}$ such that

- $p \Vdash “|\{\alpha < \chi \mid \dot{f}_\alpha \in \dot{X}\}| \geq \lambda_n”$, and
- $p \Vdash “\dot{\Phi} \text{ is an } n\text{-coherent family}”$.

We will find a $q \leq p$ and a \mathbb{P} -name \dot{A} such that q forces the following statements:

- $|\dot{A}| \geq \lambda_{n-1}$;
- $\{\dot{f}_\alpha \mid \alpha \in \dot{A}\} \subseteq \dot{X}$;
- $\dot{\Phi} \upharpoonright \dot{A}$ is trivial.

(Here and below we use $\dot{\Phi} \upharpoonright \dot{A}$ as shorthand for the more formally correct $\dot{\Phi} \upharpoonright \{\dot{f}_\alpha \mid \alpha \in \dot{A}\}$). As before, begin by letting Y be the set of $\alpha < \chi$ for which there is a condition $p_\alpha \leq p$ such that $p_\alpha \Vdash “\dot{f}_\alpha \in \dot{X}”$; observe that $|Y| \geq \lambda_n$ by assumption. For each $\alpha \in Y$, fix such a condition p_α . Since \mathbb{P} is λ_n -Knaster, there exists a set $Y' \subseteq Y$ of size λ_n such that $\{p_\alpha \mid \alpha \in Y'\}$ consists of pairwise compatible conditions. Note that, for all $a \in [Y']^{<\omega}$, we have $\bigcup_{\alpha \in a} p_\alpha \in \mathbb{P}$.

Given a \mathbb{P} -name $\dot{h} = \langle \dot{h}_0, \dots, \dot{h}_n \rangle$ for an element of $({}^\omega\omega)^{n+1}$, let $\dot{e}(\dot{h})$ be a \mathbb{P} -name that is forced to be equal to

$$\sum_{i=0}^n (-1)^i \varphi_{\dot{h}^i}$$

if $\dot{h} \in \dot{X}^{n+1}$ and is forced to be 0 otherwise. Since p forces that Φ is n -coherent, any extension of p will force that $\dot{e}(\dot{h})$ is a finitely-supported function from a subset of $\omega \times \omega$ into \mathbb{Z} . Let $\dot{e}(\dot{h})$ be a \mathbb{P} -name for the restriction of $\dot{e}(\dot{h})$ to its support. For all $a \in [Y']^{n+1}$, let $\dot{e}(a)$ denote $\dot{e}(\langle \dot{f}_\alpha \mid \alpha \in a \rangle)$.

For each $a \in [Y']^{n+1}$ let $\langle q_{a,\ell} \mid \ell < \omega \rangle$ enumerate a maximal antichain \mathcal{A}_a of conditions in \mathbb{P} below $\bigcup_{\alpha \in a} p_\alpha$ such that each $q_{a,\ell}$ decides the value of $\dot{e}(a)$ to be equal to some finite partial function $\mathbf{e}_{a,\ell} \in V$. Recall that, for $p \in \mathbb{P}$, $u(p)$ is the set $\{\alpha < \chi \mid \text{dom}(p) \cap (\{\alpha\} \times \omega) \neq \emptyset\}$. For readability, let $u(a, \ell)$ denote $u(q_{a,\ell})$.

For each $b \in [Y']^{2n+1}$ let $v_b = \bigcup \{u(a, \ell) \mid a \in [b]^{n+1}, \ell < \omega\}$. Define a “coding” function $F : [Y']^{2n+1} \rightarrow H(\omega_1)$ as follows. First, for each $b \in [Y']^{2n+1}$, each $\mathbf{m} \in [2n+1]^{n+1}$ and each $\ell < \omega$, let

$$w_{\mathbf{m},\ell}^b = \{\eta < \text{otp}(v_b) \mid v_b(\eta) \in u(b[\mathbf{m}], \ell)\}.$$

(Note that $w_{\mathbf{m},\ell}^b \in [\text{otp}(v_b)]^{<\omega}$). Then, for each $b \in [Y']^{2n+1}$, let

$$F(b) = \langle \langle \bar{q}_{b[\mathbf{m}],\ell}, w_{\mathbf{m},\ell}^b, \mathbf{e}_{b[\mathbf{m}],\ell} \rangle \mid \mathbf{m} \in [2n+1]^{n+1}, \ell < \omega \rangle.$$

Recall that $\lambda_n = \sigma(\lambda_{n-1}^+, 2n+1)$ and λ_{n-1}^+ is $<\aleph_1$ -inaccessible. Therefore, by Fact 2.5, there exists $H \in [Y']^{\lambda_{n-1}^+}$ such that

- F is constant on $[H]^{2n+1}$, taking value $\langle \langle \bar{q}_{\mathbf{m},\ell}, w_{\mathbf{m},\ell}, \mathbf{e}_{\mathbf{m},\ell} \rangle \mid \mathbf{m} \in [2n+1]^{n+1}, \ell < \omega \rangle$, and
- $\langle v_b \mid b \in [H]^{2n+1} \rangle$ is a uniform $(2n+1)$ -dimensional Δ -system.

By taking an initial segment of H if necessary, we can assume that $\text{otp}(H) = \lambda_{n-1}^+$. Let ρ and $\langle \mathbf{r}_{\mathbf{m}} \mid \mathbf{m} \in [2n+1] \rangle$ witness that $\langle v_b \mid b \in [H]^{2n+1} \rangle$ is a uniform $(2n+1)$ -dimensional Δ -system.

Let $\langle v_a \mid a \in [H]^{<2n+1} \rangle$ be given by Lemma 2.3 applied to $\langle v_b \mid b \in [H]^{2n+1} \rangle$. We will actually need slightly more than what Lemma 2.3 gives us. Given $a \in [H]^n$, $k \leq n$, and $\alpha \in H$, we say that α is k -addable for a if $\alpha \notin a$ and $|a \cap \alpha| = k$. In other words, α is k -addable to a if, letting $a' = a \cup \{\alpha\}$, we have $|a'| = n+1$ and $a'(k) = \alpha$. Given an $a \in [H]^n$ and a $k \leq n$ such that there is at least one $\alpha \in H$ that is k -addable for a , define $v_{a,k}$ as follows. Let $\alpha \in H$ be such that α is addable for a , let $b \in [H]^{2n+1}$ be such that $b[n+1] = a \cup \{\alpha\}$, and let $v_{a,k} = v_b[\mathbf{r}_{(n+1) \setminus \{k\}}]$.

Claim 5.2. *For each $a \in [H]^n$ and $k \leq n$ for which $v_{a,k}$ is defined, the value of $v_{a,k}$ is independent of our choice of α and b .*

Proof. Suppose that $\alpha, \alpha' \in H$ are both k -addable for a and $b, b' \in [H]^{2n+1}$ are such that $b[n+1] = a \cup \{\alpha\}$ and $b'[n+1] = a \cup \{\alpha'\}$. We will show that $v_b[\mathbf{r}_{(n+1) \setminus \{k\}}] = v_{b'}[\mathbf{r}_{(n+1) \setminus \{k\}}]$.

First, fix $c \in [H]^n$ such that $\min(c) > \max(b \cup b')$, let $d = a \cup \{\alpha\} \cup c$, and let $d' = a \cup \{\alpha'\} \cup c$. Then b and d are aligned, with $\mathbf{r}(b, d) = n+1$, and b' and d' are aligned, with $\mathbf{r}(b', d') = n+1$. Also, d and d' are aligned, with either $\mathbf{r}(d, d') = 2n+1$ (if $\alpha = \alpha'$) or $\mathbf{r}(d, d') = (2n+1) \setminus \{k\}$ (if $\alpha \neq \alpha'$). Altogether, it follows that

$$v_b[\mathbf{r}_{(n+1) \setminus \{k\}}] = v_d[\mathbf{r}_{(n+1) \setminus \{k\}}] = v_{d'}[\mathbf{r}_{(n+1) \setminus \{k\}}] = v_{b'}[\mathbf{r}_{(n+1) \setminus \{k\}}],$$

as desired. \square

Claim 5.3. *Suppose that $a \in [H]^n$ and $k \leq n$ are such that there are at least two elements of H that are k -addable for a . Then the collection*

$$\{v_{a \cup \{\alpha\}} \mid \alpha \in H \text{ is } k\text{-addable for } a\}$$

is a (1-dimensional) Δ -system, with root $v_{a,k}$.

Proof. Fix $\alpha < \alpha'$ in H such that both α and α' are k -addable for a . Fix $c \in [H]^n$ such that $\min(c) > \max(a \cup \{\alpha, \alpha'\})$. Let $b = a \cup \{\alpha\} \cup c$ and $b' = a \cup \{\alpha'\} \cup c$. Then $v_{a \cup \{\alpha\}} = v_b[\mathbf{r}_{n+1}]$, $v_{a \cup \{\alpha'\}} = v_{b'}[\mathbf{r}_{n+1}]$, and $v_{a,k} = v_b[\mathbf{r}_{(n+1) \setminus \{k\}}] = v_{b'}[\mathbf{r}_{(n+1) \setminus \{k\}}]$. Moreover, b and b' are aligned with $\mathbf{r}(b, b') = (2n+1) \setminus \{k\}$, so $v_b \cap v_{b'} = v_b[\mathbf{r}_{(2n+1) \setminus \{k\}}] = v_{b'}[\mathbf{r}_{(2n+1) \setminus \{k\}}]$. Altogether, this implies

$$\begin{aligned} v_{a \cup \{\alpha\}} \cap v_{a \cup \{\alpha'\}} &= v_b[\mathbf{r}_{n+1}] \cap v_{b'}[\mathbf{r}_{n+1}] \\ &= v_b[\mathbf{r}_{n+1}] \cap v_{b'}[\mathbf{r}_{n+1}] \cap v_b[\mathbf{r}_{(2n+1) \setminus \{k\}}] \cap v_{b'}[\mathbf{r}_{(2n+1) \setminus \{k\}}] \\ &= v_b[\mathbf{r}_{(n+1) \setminus \{k\}}] \cap v_{b'}[\mathbf{r}_{(n+1) \setminus \{k\}}] \\ &= v_{a,k} \cap v_{a,k} = v_{a,k}. \end{aligned}$$

Therefore, $\{v_{a \cup \{\alpha\}} \mid \alpha \in H \text{ is addable for } a\}$ is a Δ -system with root $v_{a,k}$. \square

Claim 5.4. *Let $a \in [H]^n$, $k \leq n$, and $\ell < \omega$, and suppose that $\alpha, \alpha' \in H$ are both k -addable for a . Then $q_{a \cup \{\alpha\}, \ell} \upharpoonright (v_{a,k} \times \omega) = q_{a \cup \{\alpha'\}, \ell} \upharpoonright (v_{a,k} \times \omega)$.*

Proof. We can assume that $\alpha \neq \alpha'$, as otherwise the claim is trivial. Fix $c \in [H]^n$ with $\min(c) > \max(a \cup \{\alpha, \alpha'\})$, and let $b = a \cup \{\alpha\} \cup c$ and $b' = a \cup \{\alpha'\} \cup c$. By Claim 5.2, we have $v_{a,k} = v_b[\mathbf{r}_{(n+1) \setminus \{k\}}] = v_{b'}[\mathbf{r}_{(n+1) \setminus \{k\}}]$.

Now suppose that $(\gamma, j) \in \text{dom}(q_{a \cup \{\alpha\}, \ell}) \cap (v_{a,k} \times \omega)$. Then there is $\eta \in \mathbf{r}_{(n+1) \setminus \{k\}}$ such that $\gamma = v_b(\eta)$; moreover, $\eta \in w_{n+1, \ell}$ and, since b and b' are aligned with $\mathbf{r}(b, b') \supseteq (n+1) \setminus \{k\}$, we have $\gamma = v_{b'}(\eta)$. Let $i < \omega$ be such that $\eta = w_{n+1, \ell}(i)$. Then, since $\bar{q}_{a \cup \{\alpha\}, \ell} = \bar{q}_{a \cup \{\alpha'\}, \ell} = \bar{q}_{n+1, \ell}$, we have $(\gamma, j) \in \text{dom}(q_{a \cup \{\alpha'\}, \ell})$ and

$$q_{a \cup \{\alpha'\}, \ell}(\gamma, j) = \bar{q}_{n+1, \ell}(i, j) = q_{a \cup \{\alpha\}, \ell}(\gamma, j).$$

A symmetric argument shows that, if $(\gamma, j) \in \text{dom}(q_{a \cup \{\alpha'\}, \ell}) \cap (v_{a,k} \times \omega)$, then we have $(\gamma, j) \in \text{dom}(q_{a \cup \{\alpha\}, \ell})$ and $q_{a \cup \{\alpha\}, \ell}(\gamma, j) = q_{a \cup \{\alpha'\}, \ell}(\gamma, j)$. It follows that $q_{a \cup \{\alpha\}, \ell} \upharpoonright (v_{a,k} \times \omega) = q_{a \cup \{\alpha'\}, \ell} \upharpoonright (v_{a,k} \times \omega)$. \square

We next note that the values of $\bar{q}_{\mathbf{m}, \ell}$ and $\mathbf{e}_{\mathbf{m}, \ell}$ are independent of $\mathbf{m} \in [2n+1]^{n+1}$. Indeed, suppose that $b \in [H]^{2n+1}$, $\mathbf{m} \in [2n+1]^{n+1}$, and $a = b[\mathbf{m}]$. Then we can find a $b^* \in [H]^{2n+1}$ for which $a = b^*[n+1]$ (i.e., a is an initial segment of b^*). But then

$$\bar{q}_{\mathbf{m}, \ell} = \bar{q}_{b[\mathbf{m}], \ell} = \bar{q}_{a, \ell} = \bar{q}_{b^*[n+1], \ell} = \bar{q}_{n+1, \ell},$$

and similarly for $\mathbf{e}_{\mathbf{m}, \ell}$. Hence we may in fact fix a sequence $\langle (\bar{q}_\ell, \mathbf{e}_\ell) \mid \ell < \omega \rangle$ such that $(\bar{q}_{\mathbf{m}, \ell}, \mathbf{e}_{\mathbf{m}, \ell}) = (\bar{q}_\ell, \mathbf{e}_\ell)$ for all $\mathbf{m} \in [2n]^{n+1}$.

Now let H_0, H_1, \dots, H_n be subsets of H such that $\text{otp}(H_k) = \lambda_{n-1}$ and $H_k < H_{k'}$ for all $k < k' \leq n$. For each $k \leq n$ let $\delta_k = \min(H_k)$, and let $d = \{\delta_k \mid k \leq n\}$. Let $q = q_{d, 0}$; note that $q \leq p$. Let $d^+ = \{\delta_k \mid 1 \leq k \leq n\}$ and for all $\alpha \in H_0$ let $d_\alpha = \{\alpha\} \cup d^+$ be the result of replacing δ_0 by α in d . Let \dot{A} be a \mathbb{P} -name for the set of $\alpha \in H_0$ such that $q_{d_\alpha, 0} \in \dot{G}$. We claim that q and \dot{A} are as desired; more precisely, we claim that q forces that

- (i) $|\dot{A}| \geq \lambda_{n-1}$,
- (ii) $\{\dot{f}_\alpha \mid \alpha \in \dot{A}\} \subseteq \dot{X}$, and
- (iii) $\dot{\Phi} \upharpoonright \dot{A}$ is trivial.

We can dispense with the first two of these items fairly quickly.

Claim 5.5. $q \Vdash \dot{A} \text{ is unbounded in } H_0$.

Proof. Fix an arbitrary $\gamma \in H_0$ and an arbitrary condition $r \leq q$. We will find an $\alpha \in H_0 \setminus \gamma$ such that $q_{d_\alpha,0}$ is compatible with r . A routine density argument will then establish the claim.

By Claim 5.3, $\{v_{d_\alpha} \mid \alpha \in H_0 \setminus \gamma\}$ is an infinite (1-dimensional) Δ -system with root $v_{d^+,0}$. Therefore, there exists an $\alpha \in H_0 \setminus \gamma$ such that $v_{d_\alpha} \setminus v_{d^+,0}$ is disjoint from $u(r)$. By Claim 5.4 $q_{d_\alpha,0} \restriction (v_{d^+,0} \times \omega) = q_{d,0} \restriction (v_{d^+,0} \times \omega)$; in consequence, since r extends $q_{d,0} = q$, it is compatible with $q_{d_\alpha,0}$, as desired. \square

It follows immediately that q forces \dot{A} to have cardinality λ_{n-1} . To see item (ii) above, notice that $q_{d_\alpha,0} \Vdash \dot{f}_\alpha \in \dot{X}$ for all $\alpha \in H_0$, since $q_{d_\alpha,0} \leq p_\alpha$. It follows that $\{\dot{f}_\alpha \mid \alpha \in \dot{A}\}$ is forced to be a subset of \dot{X} .

We now show that q forces that $\dot{\Phi} \restriction \dot{A}$ is trivial; this verification will be considerably more involved. Let G be \mathbb{P} -generic over V with $q \in G$. We will begin by working in $V[G]$. To denote the interpretations of \mathbb{P} -names in $V[G]$ we simply remove their dots, e.g., the interpretation of \dot{A} is A , the interpretation of $\dot{\Phi}$ is Φ , etc. Much as above, for $a \in [\chi]^{<\omega}$, let $I(a)$ denote $I(\wedge \{f_\alpha \mid \alpha \in a\})$.

For each $a \in [A]^{\leq n}$ with $|a| > 1$ let $d_a = a \cup \{\delta_i \mid |a| - 1 \leq i < n\}$. Notice that $d_a \in [H]^{n+1}$ (we emphasize that $\delta_n \notin d_a$ in this case). Since $p_\alpha \in G$ for all $\alpha \in d_a$, there is some unique $\ell_a < \omega$ for which $q_{d_a, \ell_a} \in G$.

For each $a \in [A]^{\leq n}$ with $|a| = 1$, i.e., if $a = \{\alpha\}$ for some $\alpha \in A$, let $d_a = d_\alpha$. Again we then have $d_a \in [H]^{n+1}$; since $q_{d_\alpha,0} \in G$ for all $\alpha \in A$, we let $\ell_a = 0$ for any such singleton a .

For each nonempty $a \in [A]^{\leq n+1}$ let j_a equal

$$\max\{j < \omega \mid \text{there exists a nonempty } a' \in [a]^{\leq n} \text{ such that } (\gamma, j) \in \text{dom}(q_{d_{a'}, \ell_{a'}}) \text{ for some } \gamma < \chi\}.$$

In particular, there is a single $j^* < \omega$ such that

$$j_{\{\alpha\}} = j^* = \max\{j < \omega \mid \exists \eta < \omega (\eta, j) \in \text{dom}(\bar{q}_0)\}$$

for every $\alpha \in A$. For ease of notation, let $I(a)_{>j^*}$ denote $\{(j, k) \in I(a) \mid j > j^*\}$. Notice that the j_a terms are monotonic: if $a' \subseteq a$, then $j_{a'} \leq j_a$.

We now recursively define ordinals

$$\langle \varepsilon_a^{j,k} \mid a \in [A]^{\leq n+1} \text{ is nonempty and } (j, k) \in I(a)_{>j^*} \rangle.$$

These ordinals will play, for each $(j, k) \in I(a)_{>j^*}$, the role of the sequence $\vec{\varepsilon}$ described in Section 4. They will satisfy the following properties:

- (1) $\varepsilon_a^{j,k} \in H_{|a|-1}$ for all nonempty $a \in [A]^{\leq n+1}$ and all $(j, k) \in I(a)_{>j^*}$;
- (2) $\varepsilon_{\{\alpha\}}^{j,k} = \alpha$ for all $\alpha \in A$ and all $(j, k) \in I(f_\alpha)_{>j^*}$;
- (3) $(j, k) \in I(f_{\varepsilon_a^{j,k}})$ for all nonempty $a \in [A]^{\leq n+1}$ and all $(j, k) \in I(a)_{>j^*}$.

The ordinals $\varepsilon_a^{j,k}$ will satisfy an additional property as well; to state it, we will need some further notation. Suppose that $b \in [A]^{\leq n+1}$ is a nonempty set and $\vec{a} = \langle a_i \mid 1 \leq i \leq m \rangle$ is a subset-final segment of b . Suppose also that $(j, k) \in I(b)_{>j^*}$. Then, recalling the notation $d_{\vec{a}}^{\vec{\varepsilon}}$ (or $d_{\vec{a}}$, simply) from Section 4, we define $d_{\vec{a}}^{j,k} \in [H]^{n+1}$ as follows, splitting into cases depending on whether or not $|a_1| > 1$. First, if $|a_1| = 1$ (and hence if $|b| = m$), then let

$$d_{\vec{a}}^{j,k} = \{\varepsilon_{a_i}^{j,k} \mid 1 \leq i \leq m\} \cup \{\delta_i \mid m \leq i \leq n\}.$$

Notice that in this case $|d_{\vec{a}}^{j,k} \cap H_i| = 1$ for all $i \leq n$.

Next, if $|a_1| > 1$ and $|b| \leq n$, then let

$$d_a^{j,k} = a_1 \cup \{\varepsilon_{a_i}^{j,k} \mid 1 \leq i \leq m\} \cup \{\delta_i \mid |a_1| + m - 1 \leq i < n\}.$$

Notice that in this case $|d_a^{j,k} \cap H_0| = |a_1|$, $d_a^{j,k} \cap H_i = \emptyset$ for $0 < i < |a_1| - 1$ and $i = n$, and $|d_a^{j,k} \cap H_i| = 1$ for $|a_1| - 1 \leq i < n$. If $|a_1| > 1$ and $|b| = n + 1$, then leave $d_a^{j,k}$ undefined. In any case, if $d_a^{j,k}$ is defined, then $d_a^{j,k} \in [H]^{n+1}$.

We may now state our final requirement for the ordinals $\varepsilon_a^{j,k}$:

- (4) For every nonempty $b \in [A]^{\leq n+1}$, every subset-final segment $\vec{a} = \langle a_i \mid 1 \leq i \leq m \rangle$ of b , and every $(j, k) \in I(b)_{>j^*}$,
- (a) if $|a_1| = 1$, then $q_{d_a^{j,k}, 0} \in G$;
 - (b) if $|a_1| > 1$ and $|b| \leq n$ and $j > j_b$, then $q_{d_a^{j,k}, \ell_{a_1}} \in G$.

A main resource for the construction of the ordinals $\varepsilon_a^{j,k}$ will be Claim 5.6 below. To facilitate its statement, we introduce the following terminology and convention: if $c \in [H]^{n+1}$, $i \leq n$, and $\alpha \in H$, then we say that α is *i-possible* for c if the following two statements hold:

- if $i > 0$, then $\alpha > c(i - 1)$;
- if $i < n$, then $\alpha < c(i + 1)$.

Intuitively, α is *i-possible* for c if $c(i)$ can be replaced by α without changing the positions of the other elements of c within the set. If α is *i-possible* for c , then $c[i \mapsto \alpha]$ denotes this replacement, i.e., it denotes the set $(c \setminus \{c(i)\}) \cup \{\alpha\}$.

Claim 5.6. *Suppose that $c_0, c_1 \in [H]^{n+1}$, $i_0, i_1 \leq n$, and $c_0(i_0) = c_1(i_1)$. Suppose also that $\alpha \in H$ is i_0 -possible for c_0 and i_1 -possible for c_1 and $\ell_0, \ell_1 < \omega$ are such that q_{c_0, ℓ_0} and q_{c_1, ℓ_1} are compatible in \mathbb{P} . Then $q_{c_0[i_0 \mapsto \alpha], \ell_0}$ and $q_{c_1[i_1 \mapsto \alpha], \ell_1}$ are also compatible in \mathbb{P} .*

Proof. If $c_0(i_0) = \alpha$, then there is nothing to prove, so assume that $c_0(i_0) \neq \alpha$. Let $c = c_0 \cup c_1$. Since c_0 and c_1 share at least one element, we know that $|c| \leq 2n + 1$. Let $b \in [H]^{2n+1}$ be a (possibly trivial) end-extension of c such that every element of $b \setminus c$ is greater than α . Let $i^* \leq 2n$ be such that $b(i^*) = c_0(i_0) = c_1(i_1)$, and let $b^* = (b \setminus \{c_0(i_0)\}) \cup \{\alpha\}$. Since α is i_0 -possible for c_0 and i_1 -possible for c_1 , and since all elements of $b \setminus c$ are greater than α , we know that b and b^* are aligned and $\mathbf{r}(b, b^*) = (2n + 1) \setminus \{i^*\} =: \mathbf{m}$. Let $\mathbf{m}_0, \mathbf{m}_1 \in [2n + 1]^{n+1}$ be such that $c_0 = b[\mathbf{m}_0]$ and $c_1 = b[\mathbf{m}_1]$, and hence such that $c_0[i_0 \mapsto \alpha] = b^*[\mathbf{m}_0]$ and $c_1[i_1 \mapsto \alpha] = b^*[\mathbf{m}_1]$.

Suppose for the sake of contradiction that $q_{c_0[i_0 \mapsto \alpha], \ell_0}$ and $q_{c_1[i_1 \mapsto \alpha], \ell_1}$ are incompatible in \mathbb{P} . Then there is a (γ^*, j) in the intersection of their domains such that

$$q_{c_0[i_0 \mapsto \alpha], \ell_0}(\gamma^*, j) \neq q_{c_1[i_1 \mapsto \alpha], \ell_1}(\gamma^*, j).$$

Suppose that $\eta < \rho$ is such that $\gamma^* = u_{b^*}(\eta)$. Let $\gamma = u_b(\eta)$. Since $F(b^*) = F(b)$, and in particular since $w_{\mathbf{m}_0, \ell_0}^b = w_{\mathbf{m}_0, \ell_0}^{b^*}$, $w_{\mathbf{m}_1, \ell_1}^b = w_{\mathbf{m}_1, \ell_1}^{b^*}$, $\bar{q}_b[\mathbf{m}_0, \ell_0] = \bar{q}_{b^*}[\mathbf{m}_0, \ell_0]$, and $\bar{q}_b[\mathbf{m}_1, \ell_1] = \bar{q}_{b^*}[\mathbf{m}_1, \ell_1]$, we know that (γ, j) is in the domain of both q_{c_0, ℓ_0} and q_{c_1, ℓ_1} and also that $q_{c_0, \ell_0}(\gamma, j) = q_{c_0[i_0 \mapsto \alpha], \ell_0}(\gamma^*, j)$ and $q_{c_1, \ell_1}(\gamma, j) = q_{c_1[i_1 \mapsto \alpha], \ell_1}(\gamma^*, j)$. It follows that $q_{c_0, \ell_0}(\gamma, j) \neq q_{c_1, \ell_1}(\gamma, j)$, but this contradicts our assumption that q_{c_0, ℓ_0} and q_{c_1, ℓ_1} are compatible in \mathbb{P} . \square

We turn now more directly to the construction of the family of ordinals

$$\langle \varepsilon_a^{j,k} \mid a \in [A]^{\leq n+1} \text{ is nonempty and } (j, k) \in I(a)_{>j^*} \rangle$$

satisfying the requirements (1)–(4) listed above. The construction is by recursion on $|a|$. If $\alpha \in A$ and $(j, k) \in I(f_\alpha)_{>j^*}$, then condition (2) dictates that $\varepsilon_{\{\alpha\}}^{j,k} = \alpha$. Conditions (1)–(3) are then trivially satisfied. To see condition (4), observe that the only subset-final segment of $\{\alpha\}$ for any $\alpha \in A$ is

$\vec{a} = \langle \{\alpha\} \rangle$; by the definition of A , we then have $q_{d_{\vec{a}}^{j,k},0} = q_{d_{\alpha},0} \in G$, just as required by condition (4a). This concludes the cases in which $|a| = 1$.

Next suppose that $b \in [A]^{\leq n}$ and $|b| \geq 2$; fix $(j,k) \in I(b)_{>j^*}$ and suppose also that we have defined $\varepsilon_a^{j,k}$ for all nonempty $a \subsetneq b$. We will define an $\varepsilon_b^{j,k} \in H_{|b|-1}$. Suppose that $\vec{a} = \langle a_i \mid 1 \leq i \leq m \rangle$ is a subset-final segment of b . If $m = 1$, then define $d_{\vec{a}}^-$ to be d_b (see again the third paragraph after Claim 5.5 for the definition of d_b). If $m > 1$, then notice that $\vec{a}^- := \langle a_i \mid 1 \leq i < m \rangle$ is a subset-final segment of a_{m-1} , and define $d_{\vec{a}}^-$ to be $d_{\vec{a}^-}^{j,k}$. Observe that $q_{d_{\vec{a}}^-, \ell_{a_1}} \in G$ in either case: in the $m = 1$ case, this follows from the definition of $\ell_{a_1} = \ell_b$. In the $m > 1$ case, this follows from our inductive hypothesis that statement (4) holds when applied to a_{m-1} . If \vec{a} is a long string (i.e., if $|a_1| = 1$), then let $i_{\vec{a}} = |b| - 1$, and if \vec{a} is a short string (i.e., if $|a_1| > 1$), then let $i_{\vec{a}} = |b|$. Notice that, in any case, we have $d_{\vec{a}}^-(i_{\vec{a}}) = \delta_{|b|-1}$ and, once we have defined $\varepsilon_b^{j,k}$, we will have $d_{\vec{a}}^{j,k} = d_{\vec{a}}^-[i_{\vec{a}} \mapsto \varepsilon_b^{j,k}]$.

Now, to see that we can find an ordinal $\varepsilon_b^{j,k}$ satisfying conditions (1)–(4), move back to V and fix an arbitrary $r \in \mathbb{P}$ extending

$$q^* := \bigcup \{q_{d_{\vec{a}}^-, \ell_{a_1}} \mid \vec{a} = \langle a_i \mid i \leq m \rangle \text{ is a subset-final segment of } b\},$$

which we know to be in G . We will find a condition $s \leq r$ and an ordinal ε such that s forces ε to be a valid choice for $\varepsilon_b^{j,k}$. By the preceding paragraph and Claim 5.3, we know that for every subset-final segment \vec{a} of b , the collection $\{v_{d_{\vec{a}}^-(i_{\vec{a}}) \mapsto \varepsilon} \mid \varepsilon \in H_{|b|-1}\}$ is an infinite (1-dimensional) Δ -system with root $v_{d_{\vec{a}}^-\setminus\{\delta_{|b|-1}\}, i_{\vec{a}}}$. Therefore we can fix an $\varepsilon \in H_{|b|-1}$ such that $\varepsilon \notin u(r)$ and such that $v_{d_{\vec{a}}^-[i_{\vec{a}} \mapsto \varepsilon]} \setminus v_{d_{\vec{a}}^-\setminus\{\delta_{|b|-1}\}, i_{\vec{a}}}$ is disjoint from $u(r)$ for every subset-final segment \vec{a} of b . For each such \vec{a} , Claim 5.4 implies that

$$q_{d_{\vec{a}}^-[i_{\vec{a}} \mapsto \varepsilon], \ell_{a_1}} \restriction (v_{d_{\vec{a}}^-\setminus\{\delta_{|b|-1}\}, i_{\vec{a}}} \times \omega) = q_{d_{\vec{a}}^-, \ell_{a_1}} \restriction (v_{d_{\vec{a}}^-\setminus\{\delta_{|b|-1}\}, i_{\vec{a}}} \times \omega),$$

and we know that r extends $q_{d_{\vec{a}}^-, \ell_{a_1}}$. Therefore, $q_{d_{\vec{a}}^-[i_{\vec{a}} \mapsto \varepsilon], \ell_{a_1}}$ is compatible with r . Moreover, for all pairs \vec{a} and \vec{a}^* of subset-final segments of b , since $q_{d_{\vec{a}}^-, \ell_{a_1}}$ and $q_{d_{\vec{a}^*}^-, \ell_{a_1^*}}$ are both in G and are therefore compatible, Claim 5.6 implies that $q_{d_{\vec{a}}^-[i_{\vec{a}} \mapsto \varepsilon], \ell_{a_1}}$ and $q_{d_{\vec{a}^*}^-[i_{\vec{a}^*} \mapsto \varepsilon], \ell_{a_1^*}}$ are compatible.

We now split into two cases. Suppose first that $j \leq j_b$, so that we need to satisfy requirement (4a) but not (4b). Let

$$s_0 = r \cup \bigcup \{q_{d_{\vec{a}}^-[i_{\vec{a}} \mapsto \varepsilon], \ell_{a_1}} \mid \vec{a} \text{ is a subset-final segment of } b \text{ and } |a_1| = 1\}.$$

By the previous paragraph, s_0 is a condition in \mathbb{P} . Notice also that $\ell_{a_1} = 0$ for all such \vec{a} in the above union, since a_1 is a singleton. By the definition of j^* , the fact that $j > j^*$, and the fact that $\varepsilon \notin u(r)$, we know that $(\varepsilon, j) \notin \text{dom}(s_0)$. Therefore we can extend s_0 to a condition s such that $(\varepsilon, j) \in \text{dom}(s)$ and $s(\varepsilon, j) \geq k$, i.e., $s \Vdash "(j, k) \in I(f_\varepsilon)"$. This s in fact forces that letting $\varepsilon_a^{j,k} = \varepsilon$ satisfies requirements (1)–(4), as the reader may easily verify.

If, on the other hand, $j > j_b$, then we need to satisfy both the conditions (4a) and (4b). Let

$$s_0 = r \cup \bigcup \{q_{d_{\vec{a}}^-[i_{\vec{a}} \mapsto \varepsilon], \ell_{a_1}} \mid \vec{a} \text{ is a subset-final segment of } b\}.$$

As in the previous case, s_0 is a condition in \mathbb{P} . By the definition of j_b , the fact that $j > j_b$, and the fact that $\varepsilon \notin u(r)$, we know that $(\varepsilon, j) \notin \text{dom}(s_0)$. Just as in the previous case, we can extend s_0 to a condition s such that $(\varepsilon, j) \in \text{dom}(s)$ and $s(\varepsilon, j) \geq k$. Also as in the previous case, this s forces that letting $\varepsilon_a^{j,k} = \varepsilon$ satisfies requirements (1)–(4), as desired.

By genericity, our analysis in V shows that we may choose in $V[G]$ an $\varepsilon_b^{j,k}$ satisfying requirements (1)–(4), and thereby continue with our construction.

Finally, suppose that $b \in [A]^{n+1}$ and fix $(j, k) \in I(b)_{>j^*}$, and suppose that we have defined $\varepsilon_a^{j,k}$ for all nonempty $a \subsetneq b$. We will define an $\varepsilon_b^{j,k} \in H_n$; this will be similar to the previous case, but we no longer need to satisfy requirement (4b) and can therefore focus exclusively on long strings. Suppose that $\vec{a} = \langle a_i \mid 1 \leq i \leq n \rangle$ is a long string for b . Set $\vec{a}^- := \langle a_i \mid 1 \leq i < n \rangle$ and $d_{\vec{a}}^- := d_{\vec{a}^-}^{j,k}$, and note that \vec{a}^- is a long string for a_{n-1} . As in the previous case, we have $q_{d_{\vec{a}}^-,0} \in G$, $d_{\vec{a}}^-(n) = \delta_n$ and, once we have defined $\varepsilon_b^{j,k}$, we will have $d_{\vec{a}}^{j,k} = d_{\vec{a}}^-[n \mapsto \varepsilon_b^{j,k}]$.

The argument that we can find an ordinal $\varepsilon_b^{j,k}$ satisfying requirements (1)–(4) repeats many of the steps of the analogous argument in the case of $b \in [A]^{\leq n}$, but we include it for completeness. Move back to V and fix an arbitrary $r \in \mathbb{P}$ extending

$$q^* := \bigcup \{q_{d_{\vec{a}}^-,0} \mid \vec{a} \text{ is a long string for } b\},$$

which we know to be in G . We will find $s \leq r$ and an ordinal ε such that s forces ε to be a valid choice for $\varepsilon_b^{j,k}$. By the preceding paragraph and Claim 5.3, we know that, for every long string \vec{a} for b , the collection $\{v_{d_{\vec{a}}^-[n \mapsto \varepsilon]} \mid \varepsilon \in H_n\}$ is an infinite Δ -system with root $v_{d_{\vec{a}}^-\setminus\{\delta_n\},n}$. Therefore, we can fix an $\varepsilon \in H_n$ such that $\varepsilon \notin u(r)$ and such that $v_{d_{\vec{a}}^-[n \mapsto \varepsilon]} \setminus v_{d_{\vec{a}}^-\setminus\{\delta_n\},n}$ is disjoint from $u(r)$ for every long string \vec{a} for b . For each such \vec{a} , Claim 5.4 implies that

$$q_{d_{\vec{a}}^-[n \mapsto \varepsilon],0} \restriction (v_{d_{\vec{a}}^-\setminus\{\delta_n\},n} \times \omega) = q_{d_{\vec{a}}^-,0} \restriction (v_{d_{\vec{a}}^-\setminus\{\delta_n\},n} \times \omega),$$

and we know that r extends $q_{d_{\vec{a}}^-,0}$. Therefore, $q_{d_{\vec{a}}^-[n \mapsto \varepsilon],0}$ is compatible with r . Moreover, for all pairs \vec{a} and \vec{a}^* of long strings for b , since $q_{d_{\vec{a}}^-,0}$ and $q_{d_{\vec{a}^*}^-,0}$ are both in G and are therefore compatible, Claim 5.6 implies that $q_{d_{\vec{a}}^-[n \mapsto \varepsilon],0}$ and $q_{d_{\vec{a}^*}^-[n \mapsto \varepsilon],0}$ are compatible.

Now let

$$s_0 = r \cup \bigcup \{q_{d_{\vec{a}}^-[n \mapsto \varepsilon],0} \mid \vec{a} \text{ is a long string for } b\}.$$

By the previous paragraph, s_0 is a condition in \mathbb{P} . By the definition of j^* , the fact that $j > j^*$, and the fact that $\varepsilon \notin u(r)$, we know that $(\varepsilon, j) \notin \text{dom}(s_0)$. Therefore we can extend s_0 to a condition s such that $(\varepsilon, j) \in \text{dom}(s)$ and $s(\varepsilon, j) \geq k$, i.e., $s \Vdash "(j, k) \in I(\dot{f}_\varepsilon)"$. This s in fact forces that letting $\varepsilon_a^{j,k} = \varepsilon$ satisfies requirements (1)–(4), as the reader may easily verify.

Suppose now that the construction of the ordinals

$$\langle \varepsilon_a^{j,k} \mid a \in [A]^{\leq n+1} \text{ is nonempty and } (j, k) \in I(a)_{>j^*} \rangle$$

is completed. For all $a \in [A]^n$ and all $(j, k) \in I(a)_{>j^*}$, let $\mathcal{A}_n^{j,k}(a)$ be defined as in Section 4, using the n -coherent family Φ and the ordinals $\langle \varepsilon_{a'}^{j,k} \mid a' \in [A]^{\leq n} \rangle$. Similarly define $\mathcal{C}_n^{j,k}(b)$ for $b \in [A]^{n+1}$ and $(j, k) \in I(b)_{>j^*}$. For $a \in [A]^n$, define a function $\psi_a : I(a) \rightarrow \mathbb{Z}$ as follows. If $(j, k) \in I(a)$ and $j \leq j^*$, then let $\psi_a(j, k) = 0$. If $(j, k) \in I(a)_{>j^*}$, then let $\psi_a(j, k) = \mathcal{A}_n^{j,k}(a)(j, k)$. Define $\psi_{\vec{a}}$ for non-increasing $\vec{a} \in A^n$ in the unique way that renders $\Psi = \langle \psi_{\vec{a}} \mid \vec{a} \in A^n \rangle$ an alternating family. We claim that Ψ together with the natural number j^* witnesses the triviality of $\Phi \restriction A$ in the sense of Proposition 2.10.

We first show that each ψ_a is finitely supported. To see this, fix an arbitrary $a \in [A]^n$. For each nonempty $a' \subseteq a$, we have a finite partial function $\mathbf{e}_{\ell_{a'}}$ such that, if \vec{a} is a subset-final segment of a with $a_1 = a'$, then, for all $(j, k) \in I(a)_{>j^*}$, if $q_{d_{\vec{a}}^{j,k}, \ell_{a'}} \in G$, then $\mathbf{e}(d_{\vec{a}}^{j,k}) = \mathbf{e}_{\ell_{a'}}$. Fix a natural number $j_a^* \geq j_a$ such that, for all nonempty $a' \subseteq a$, we have $\text{dom}(\mathbf{e}_{\ell_{a'}}) \subseteq (j_a^* \times \omega)$.

We claim that $\psi_a(j, k) = 0$ for all $(j, k) \in I(a)_{>j_a^*}$. To see this, fix such a pair (j, k) . By the definition of $\psi_a(j, k)$, we know that $\psi_a(j, k) = \mathcal{A}_n^{j,k}(a)(j, k)$. By Lemma 4.3, we know that $\mathcal{A}_n^{j,k}(a)$

is of the form

$$\sum_{i < \ell} c_i e(d_{\vec{a}_i}^{j,k}),$$

where $\ell < \omega$ and each c_i is an integer and each \vec{a}_i is a subset-final segment of a with $|\vec{a}_i(1)| > 1$ (recall that $\vec{a}_i(1)$ denotes the first element of \vec{a}_i). Moreover, since $j > j_a^* \geq j_a$, we know by condition (4b) that $q_{d_{\vec{a}_i}^{j,k}, \ell_{\vec{a}_i(1)}} \in G$, and hence that $e(d_{\vec{a}_i}^{j,k}) = e_{\ell_{\vec{a}_i(1)}}$, for all $i < \ell$. In particular, $\text{dom}(e(d_{\vec{a}_i}^{j,k})) \subseteq (j_a^* \times \omega)$, so $e(d_{\vec{a}_i}^{j,k})(j, k) = 0$. It follows that $\psi_a(j, k) = 0$. In consequence, the support of ψ_a is a subset of $I(a) \cap ((j_a^* + 1) \times \omega)$, which is a finite set.

It now only remains to be shown that for all $\vec{\beta} \in A^{n+1}$ and all $(j, k) \in I(\vec{\beta})_{>j^*}$,

$$e(\vec{\beta})(j, k) = \sum_{i=0}^n (-1)^i \psi_{\vec{\beta}^i}(j, k).$$

Since Φ and Ψ are both alternating, it suffices to prove this for $b \in [A]^{n+1}$. Fix such a b and a coordinate-pair $(j, k) \in I(b)_{>j^*}$. Notice that for every long string \vec{a} for b , since $\varepsilon_{\vec{a}}^{j,k}$ satisfies requirement (4a), we have $q_{d_{\vec{a},0}^{j,k}} \in G$ and hence $e(d_{\vec{a}}^{j,k}) = e_0$. Thus we have $e(d_{\vec{a}}^{j,k})(j, k) = e_0(j, k)$, where $e_0 : \omega \times \omega \rightarrow \mathbb{Z}$ is the function whose restriction to its support is equal to e_0 . Moreover, by the construction of $\varepsilon_{\vec{a}}^{j,k}$ for nonempty $a \subseteq b$ and the assumption that $(j, k) \in I(b)_{>j^*}$, we know that $(j, k) \in I(f_{\varepsilon_{\vec{a}}^{j,k}})$ for all nonempty $a \subseteq b$. Therefore the hypotheses of Fact 4.4 hold, and consequently $\mathcal{C}_n^{j,k}(b)(j, k) = 0$. By the definition of $\mathcal{C}_n^{j,k}(b)$, we then have

$$0 = \mathcal{C}_n^{j,k}(b)(j, k) = e(b)(j, k) - \sum_{i=0}^n (-1)^i \mathcal{A}_n^{j,k}(b^i)(j, k) = e(b)(j, k) - \sum_{i=0}^n (-1)^i \psi_{b^i}(j, k),$$

implying that

$$e(b)(j, k) = \sum_{i=0}^n (-1)^i \psi_{b^i}(j, k),$$

as desired.

It follows that, in $V[G]$, the restricted family $\Phi \upharpoonright A$ is trivial. \square

6. PROPAGATING TRIVIALIZATIONS

We are now ready for the final step in the proof of our main result. The work remaining to be done consists primarily in showing how the triviality of restrictions of n -coherent families to domains constaining sufficiently many Cohen reals implies the triviality of the entire families. We note at the outset that all trivializing families in this section are type I trivializations, i.e., trivializing families as in Definition 2.7.

We first introduce a slight technical variation of n -coherent families.

Definition 6.1. Fix an $X \subseteq {}^\omega\omega$, a function $g \in {}^\omega\omega$, and a positive integer n . We say that a family of functions $\Phi = \langle \varphi_{\vec{f}} : I(\wedge \vec{f}) \cap I(g) \rightarrow \mathbb{Z} \mid \vec{f} \in X^n \rangle$ is *n -coherent below g* if it satisfies the first two bullet points of Definition 2.7, the only difference being that in this case the domain of $\varphi_{\vec{f}}$ is $I(\wedge \vec{f}) \cap I(g)$ rather than $I(\wedge \vec{f})$. We say that such a family is *n -trivial below g* if there is a τ or \mathcal{T} as in the third bullet point of Definition 2.7, again with the only difference being that, in case $n = 1$, we have $\tau : I(g) \rightarrow \mathbb{Z}$, and in case $n > 1$, we have $\tau_{\vec{f}} : I(\wedge \vec{f}) \cap I(g) \rightarrow \mathbb{Z}$ for all $\vec{f} \in X^{n-1}$.

The following proposition is a simple observation but will be needed in the arguments of this section.

Proposition 6.2. *Suppose that $X \subseteq {}^\omega\omega$, $g \in {}^\omega\omega$, n is a positive integer, and every n -coherent family of functions indexed by X^n is trivial. Then every n -coherent family of functions below g indexed by X^n is trivial below g .*

Proof. Let $\Phi = \langle \varphi_{\vec{f}} : I(\wedge \vec{f}) \cap I(g) \rightarrow \mathbb{Z} \mid \vec{f} \in X^n \rangle$ be n -coherent below g . Define an n -coherent family $\Phi^* = \langle \varphi_{\vec{f}}^* : I(\wedge \vec{f}) \rightarrow \mathbb{Z} \mid \vec{f} \in X^n \rangle$ by letting $\varphi_{\vec{f}}^*(j, k) = \varphi_{\vec{f}}(j, k)$ for all $(j, k) \in I(\wedge \vec{f}) \cap I(g)$ and $\varphi_{\vec{f}}^*(j, k) = 0$ for all $(j, k) \in I(\wedge \vec{f}) \setminus I(g)$. It is easily verified that Φ^* is n -coherent. By assumption, Φ^* is trivial, as witnessed by a single function τ^* if $n = 1$ or a family $\mathcal{T}^* = \langle \tau_{\vec{f}}^* \mid \vec{f} \in X^{n-1} \rangle$ if $n > 1$. If $n = 1$, then the function $\tau := \tau^* \upharpoonright I(g)$ witnesses that Φ is trivial below g , and if $n > 1$, then the family $\mathcal{T} := \langle \tau_{\vec{f}}^* \upharpoonright I(\wedge \vec{f}) \cap I(g) \mid \vec{f} \in X^{n-1} \rangle$ witnesses that Φ is trivial below g , as desired. \square

We now present the final piece in the proof of our main result. The proof will be by induction on n , the dimension of the coherent family under consideration; to make this induction possible, we will prove the following theorem, which immediately implies Theorem 4.1 and therefore our Main Theorem.

Theorem 6.3. *Let n be a positive integer, let $\chi \geq \lambda_n$ be a cardinal, and let $\mathbb{P} = \text{Add}(\omega, \chi)$. The following then holds in $V^{\mathbb{P}}$: For any set $X \subseteq {}^\omega\omega$ containing at least λ_n -many of the Cohen reals added by \mathbb{P} , every n -coherent family $\Phi = \langle \varphi_{\vec{f}} \mid \vec{f} \in X^n \rangle$ indexed by X is trivial.*

Proof. The proof is by induction on n . The case $n = 1$ was that of Theorem 3.1. Therefore fix an $n > 1$ and suppose the theorem proven for all positive $m < n$. Let G be \mathbb{P} -generic over V , and work in $V[G]$. Let $\langle f_\alpha \mid \alpha < \chi \rangle$ denote the Cohen reals added by G , and fix a set X and an n -coherent family Φ as in the statement of the theorem. We will show that Φ is trivial.

By Lemma 5.1, we can find $A \subseteq \chi$ such that

- $|A| \geq \lambda_{n-1}$;
- $\{f_\alpha \mid \alpha \in A\} \subseteq X$;
- $\Phi \upharpoonright A$ is trivial.

Fix a family $\mathcal{T}_1 = \langle \tau_{\vec{\alpha}} : I(\vec{\alpha}) \rightarrow \mathbb{Z} \mid \vec{\alpha} \in A^{n-1} \rangle$ that trivializes $\Phi \upharpoonright A$ (recall that $I(\vec{\alpha})$ denotes $I(\bigwedge_{k < n-1} f_{\alpha_k})$). We now show how to propagate this to a trivialization of all of Φ .

Because the argument in the general case is quite notationally involved, we begin by presenting the special case of $n = 2$, where the key underlying ideas are more readily apparent. So assume for now that $n = 2$, and therefore our family \mathcal{T}_1 has the form $\langle \tau_\alpha : I(f_\alpha) \rightarrow \mathbb{Z} \mid \alpha \in A \rangle$. We begin the process of propagating \mathcal{T}_1 to all of Φ as follows: for all $f \in X$ and $\alpha \in A$, let

$$\varsigma_\alpha^f = \varphi_{\alpha, f} + \tau_\alpha.$$

(Here and below, as usual, in subscripts we will tend to abbreviate Cohen reals by their indices, so $\varphi_{\alpha, f}$ denotes $\varphi_{f_\alpha, f}$; the expression $\Phi \upharpoonright A$ just above, which denotes $\langle \varphi_{\alpha, \beta} \mid (\alpha, \beta) \in A^2 \rangle$, is a related minor abuse. We note lastly that, in keeping with our restriction conventions, the domain of ς_α^f should be understood to be $\text{dom}(\varphi_{\alpha, f}) \cap \text{dom}(\tau_\alpha)$.)

Claim 6.4. *For each $f \in X$ the family $C_1^f := \langle \varsigma_\alpha^f : I(f \wedge f_\alpha) \rightarrow \mathbb{Z} \mid \alpha \in A \rangle$ is 1-coherent below f .*

Proof. This follows from the fact that, for all α and β in A , we have

$$\varsigma_\beta^f - \varsigma_\alpha^f = \varphi_{\beta, f} + \tau_\beta - \varphi_{\alpha, f} - \tau_\alpha =^* \varphi_{\beta, f} - \varphi_{\alpha, f} + \varphi_{\alpha, \beta} =^* 0,$$

where the first $=^*$ follows from the fact that $\langle \tau_\alpha \mid \alpha \in A \rangle$ trivializes $\Phi \upharpoonright A$ and hence $\tau_\beta - \tau_\alpha =^* \varphi_{\alpha, \beta}$, and the second $=^*$ follows from the 2-coherence of Φ . \square

As A contains more than \beth_1 -many Cohen reals, the induction hypothesis (which in this case is Theorem 3.1) and Proposition 6.2 imply that each such family C_1^f admits a trivialization $\tau^f : I(f) \rightarrow \mathbb{Z}$.

Claim 6.5. *The family $\mathcal{T}_2 := \langle \tau^f \mid f \in X \rangle$ trivializes Φ .*

Proof. Suppose for contradiction that it did not. Then for some $f, g \in X$ and infinite $E \subseteq I(f \wedge g)$,

$$\tau^g(j, k) - \tau^f(j, k) \neq \varphi_{f,g}(j, k)$$

for all $(j, k) \in E$. As $\text{Add}(\omega, \chi)$ has the countable chain condition, there exists a $W \in [\chi]^{\aleph_0}$ such that $E \in V[G_W]$. By genericity, for any $\beta \in A \setminus W$ the domain $I(f_\beta)$ then has infinite intersection with E . However,

$$\tau^g - \tau^f =^* \varsigma_\beta^g - \varsigma_\beta^f = \varphi_{\beta,g} + \tau_\beta - \varphi_{\beta,f} - \tau_\beta =^* \varphi_{f,g},$$

where, as indicated, the equalities should each be read as applying over the restricted domain $I(f \wedge g \wedge f_\beta)$. It follows that $\tau^g(j, k) - \tau^f(j, k) \neq \varphi_{f,g}(j, k)$ for only finitely many $(j, k) \in E \cap I(f_\beta)$, contradicting our assumption. \square

We now return to the general case, first giving a general overview of the structure of the argument. We will define families of functions $\langle \mathcal{T}_k \mid 1 \leq k \leq n \rangle$ and $\langle \mathcal{C}_k \mid 1 \leq k < n \rangle$ via the following sequence of steps:

- (1) \mathcal{T}_1 has already been defined as $\langle \tau_{\vec{\alpha}} \mid \vec{\alpha} \in A^{n-1} \rangle$, which trivializes $\Phi \restriction A$.
- (2) If $k < n$, then \mathcal{T}_k will induce a family of $(n - k)$ -coherent families of functions

$$\mathcal{C}_k = \langle \varsigma_{\vec{\alpha}}^{\vec{f}} \mid \vec{f} \in X^k \text{ and } \vec{\alpha} \in A^{n-k} \rangle.$$

To be precise, \mathcal{C}_k will be the union of the $(n - k)$ -coherent families of functions

$$C_k^{\vec{f}} = \langle \varsigma_{\vec{\alpha}}^{\vec{f}} \mid \vec{\alpha} \in A^{n-k} \rangle$$

as \vec{f} ranges through X^k .

- (3) Our inductive hypothesis ensures us trivializations $T_{k+1}^{\vec{f}} = \langle \tau_{\vec{\alpha}}^{\vec{f}} \mid \vec{\alpha} \in A^{n-k-1} \rangle$ of each $C_k^{\vec{f}}$. These serve then to define

$$\mathcal{T}_{k+1} := \bigcup_{\vec{f} \in X^k} T_{k+1}^{\vec{f}},$$

and repeated, alternating applications of this and the previous step cumulatively yield the sequence $\langle \mathcal{T}_k \mid 1 \leq k \leq n \rangle$, whose last element, \mathcal{T}_n , will trivialize Φ .

To carry out the above sequence of steps, we must accomplish the following tasks:

- (i) We must specify precisely how the $(n - k)$ -coherent families of functions $C_k^{\vec{f}}$ derive from the families \mathcal{T}_k , and verify that they are in fact $(n - k)$ -coherent.
- (ii) We must verify that \mathcal{T}_n does indeed trivialize Φ .

We begin with item (i). The families \mathcal{C}_k are inductively defined on positive integers $k < n$. The pattern when $k = 1$ is already visible in the case of $n = 1$: for each $f \in X$ the subclass $C_1^f = \langle \varsigma_{\vec{\alpha}}^f \mid \vec{\alpha} \in A^{n-1} \rangle$ of \mathcal{C}_1 is defined from $\mathcal{T}_1 = \langle \tau_{\vec{\alpha}} \mid \vec{\alpha} \in A^{n-1} \rangle$ by

$$(8) \quad \varsigma_{\vec{\alpha}}^f = \varphi_{\vec{\alpha}f} + (-1)^n \tau_{\vec{\alpha}}$$

for each $\vec{\alpha} \in A^{n-1}$. Observe then that for all $\vec{\alpha} \in A^n$,

$$\begin{aligned} \sum_{i=0}^{n-1} (-1)^i \varsigma_{\vec{\alpha}^i}^f &= \sum_{i=0}^{n-1} (-1)^i \varphi_{\vec{\alpha}^i f} + (-1)^n \sum_{i=0}^{n-1} (-1)^i \tau_{\vec{\alpha}^i} \\ &=^* \sum_{i=0}^{n-1} (-1)^i \varphi_{\vec{\alpha}^i f} + (-1)^n \varphi_{\vec{\alpha}} \\ &=^* 0, \end{aligned}$$

by the coherence of Φ . This shows that C_1^f is $(n-1)$ -coherent below f and hence, by the inductive hypothesis, $(n-1)$ -trivial.

For the more general inductive definition of \mathcal{C}_k , suppose that the family \mathcal{T}_k is defined; suppose also that the families \mathcal{C}_j and \mathcal{T}_j are defined for all positive $j < k$ and that each exhibits the coherence and trivialization features, respectively, described above. We then define \mathcal{C}_k by letting

$$(9) \quad \varsigma_{\vec{\alpha}}^{\vec{f}} = \varphi_{\vec{\alpha} \vec{f}} + (-1)^{n-k+1} \sum_{i=0}^{k-1} (-1)^i \tau_{\vec{\alpha}}^{\vec{f}^i}$$

for each $\vec{f} \in X^k$ and $\vec{\alpha} \in A^{n-k}$. Observe that equation (8) identifies naturally with the case of $k = 1$.

Claim 6.6. *For each $\vec{f} \in X^k$ the family $C_k^{\vec{f}} = \langle \varsigma_{\vec{\alpha}}^{\vec{f}} \mid \vec{\alpha} \in A^{n-k} \rangle$ is $(n-k)$ -coherent.*

Proof. The more formal statement of the claim is that for each $\vec{f} \in X^k$ and $\vec{\alpha} \in A^{n-k+1}$,

$$\sum_{i=0}^{n-k} (-1)^i \varsigma_{\vec{\alpha}^i}^{\vec{f}} =^* 0.$$

This is computationally verified as follows:

$$\begin{aligned}
\sum_{i=0}^{n-k} (-1)^i \varsigma_{\vec{\alpha}^i}^{\vec{f}} &= \sum_{i=0}^{n-k} (-1)^i \varphi_{\vec{\alpha}^i \vec{f}} + (-1)^{n-k+1} \sum_{i=0}^{n-k} (-1)^i \sum_{j=0}^{k-1} (-1)^j \tau_{\vec{\alpha}^i}^{\vec{f}^j} \\
&=^* \sum_{i=0}^{n-k} (-1)^i \varphi_{\vec{\alpha}^i \vec{f}} + (-1)^{n-k+1} \sum_{j=0}^{k-1} (-1)^j \varsigma_{\vec{\alpha}}^{\vec{f}^j} \\
&=^* \sum_{i=0}^{n-k} (-1)^i \varphi_{\vec{\alpha}^i \vec{f}} + (-1)^{n-k+1} \left(\sum_{j=0}^{k-1} (-1)^j \left(\varphi_{\vec{\alpha} \vec{f}^j} + (-1)^{n-k} \sum_{\ell=0}^{k-2} (-1)^\ell \tau_{\vec{\alpha}}^{(\vec{f}^j)^\ell} \right) \right) \\
&=^* \sum_{i=0}^n (-1)^i \varphi_{(\vec{\alpha} \vec{f})^i} + (-1)^{n-k+1} \left(\sum_{j=0}^{k-1} (-1)^{j+n-k} \sum_{\ell=0}^{k-2} (-1)^\ell \tau_{\vec{\alpha}}^{(\vec{f}^j)^\ell} \right) \\
&=^* - \left(\sum_{j=0}^{k-1} (-1)^j \sum_{\ell=0}^{k-2} (-1)^\ell \tau_{\vec{\alpha}}^{(\vec{f}^j)^\ell} \right) \\
&=^* - \left(\sum_{j \leq \ell \leq k-2} (-1)^{j+\ell} \tau_{\vec{\alpha}}^{(\vec{f}^j)^\ell} + \sum_{\ell < j \leq k-1} (-1)^{j+\ell} \tau_{\vec{\alpha}}^{(\vec{f}^j)^\ell} \right) \\
&=^* - \left(\sum_{j \leq \ell \leq k-2} (-1)^{j+\ell} \tau_{\vec{\alpha}}^{(\vec{f}^{\ell+1})^j} + \sum_{\ell < j \leq k-1} (-1)^{j+\ell} \tau_{\vec{\alpha}}^{(\vec{f}^j)^\ell} \right) \\
&=^* - \left(\sum_{\ell < j \leq k-1} (-1)^{j+\ell+1} \tau_{\vec{\alpha}}^{(\vec{f}^j)^\ell} + \sum_{\ell < j \leq k-1} (-1)^{j+\ell} \tau_{\vec{\alpha}}^{(\vec{f}^j)^\ell} \right) \\
&= 0.
\end{aligned}$$

The fact that the functions $\tau_{\vec{\alpha}^i}^{\vec{f}^j}$ trivialize the functions $\varsigma_{\vec{\alpha}^i}^{\vec{f}}$ underlies the passage from the first line to the second; replace $\tau_{\vec{\alpha}^i}^{\vec{f}^j}$ with its definition at (9) to pass from the second line to the third. Nothing more than a regrouping underlies the passage from the third line to the fourth, whereupon the first sum vanishes by the n -coherence of Φ . Simple bookkeeping converts the fifth line into the sixth, and the fact that $(\vec{f}^j)^\ell = (\vec{f}^{\ell+1})^j$ for all $j \leq \ell \leq k-2$ converts the sixth line into the seventh. A renaming of variables in the first sum then yields the eighth line, whose terms all plainly cancel. \square

As specified in the overview of the argument above, since $C_k^{\vec{f}}$ is $(n-k)$ -coherent, we can apply the inductive hypothesis to obtain a trivializing family $T_{k+1}^{\vec{f}} = \langle \tau_{\vec{\alpha}}^{\vec{f}} \mid \vec{\alpha} \in A^{n-k-1} \rangle$. We then let

$$\mathcal{T}_{k+1} = \bigcup_{\vec{f} \in X^k} T_{k+1}^{\vec{f}}.$$

At the end of the process, $C_{n-1}^{\vec{f}}$ is 1-coherent, so the trivializing family $T_n^{\vec{f}}$ is in fact a single function $\tau^{\vec{f}} : I(\bigwedge \vec{f}) \rightarrow \mathbb{Z}$. We therefore think of \mathcal{T}_n as the sequence $\langle \tau^{\vec{f}} \mid \vec{f} \in X^{n-1} \rangle$. We turn now to item (ii).

Claim 6.7. *The family $\mathcal{T}_n = \langle \tau^{\vec{f}} \mid \vec{f} \in X^{n-1} \rangle$ trivializes Φ .*

Proof. Suppose for contradiction that it did not. Then for some $\vec{f} \in X^n$ and infinite $E \subseteq I(\vec{f})$,

$$\sum_{i=0}^{n-1} (-1)^i \tau^{\vec{f}^i}(x) \neq \varphi_{\vec{f}}(x)$$

for all $x \in E$. As \mathbb{P} has the countable chain condition, there exists a $W \in [\chi]^{\aleph_0}$ such that $E \in V[G_W]$. For any $\beta \in A \setminus W$ the domain $I(f_\beta)$ then has infinite intersection with E . However,

$$\begin{aligned} \sum_{i=0}^{n-1} (-1)^i \tau^{\vec{f}^i} &=^* \sum_{i=0}^{n-1} (-1)^i \varsigma_\beta^{\vec{f}^i} \\ &=^* \sum_{i=0}^{n-1} (-1)^i \left(\varphi_{\beta \vec{f}^i} + \sum_{\ell=0}^{n-2} (-1)^\ell \tau_{\beta}^{(\vec{f}^i)^\ell} \right) \\ &=^* \sum_{i=0}^{n-1} (-1)^i \varphi_{\beta \vec{f}^i} \\ &=^* \varphi_{\vec{f}}. \end{aligned}$$

In consequence, $\sum_{i=0}^{n-1} (-1)^i \tau^{\vec{f}^i}(x) \neq \varphi_{\vec{f}}(x)$ for only finitely many $x \in E \cap I(f_\beta)$, contradicting our assumption. Above, the first equality simply records the fact that the functions $\tau^{\vec{f}^i}$ trivialize the families $\langle \varsigma_\beta^{\vec{f}^i} \mid \beta \in A \rangle$; for the second equality, replace $\varsigma_\beta^{\vec{f}^i}$ with its definition at (9). The third, as above, reflects alternating-sum cancellations, while the fourth follows from the coherence of Φ . \square

As we have shown that Φ is trivial, this completes the proof of the theorem and hence of our Main Theorem. \square

7. CONCLUSION

As noted in our introduction, this work fully answers the first question, and partially or potentially addresses the second question, appearing in [6]. We restate the latter:

Question 7.1. *What is the minimum value of the continuum that is compatible with the statement “ $\lim^n \mathbf{A} = 0$ for all $n > 0$ ”?*

By our Main Corollary, this question is tantamount to the following:

Question 7.2. *Does $2^{\aleph_0} < \aleph_\omega$ imply that $\lim^k \mathbf{A} \neq 0$ for some $k > 0$?*

A positive answer to this question would entail a positive answer to the following (a revision, in light of present knowledge, of one appearing in [16]):

Question 7.3. *Does $2^{\aleph_0} \leq \aleph_2$ imply that either $\lim^1 \mathbf{A} \neq 0$ or $\lim^2 \mathbf{A} \neq 0$?*

Of interest in its own right, but all the more so in light of Question 7.2, is:

Question 7.4. *What is the behavior of the groups $\lim^n \mathbf{A}$ in the standard forcing extensions in which $2^{\aleph_0} = \aleph_2$? By [3], of particular interest among them will be those models in which $\mathfrak{b} < \mathfrak{d}$; prominent among these is the Miller model.*

The fundamental reason that $2^{\aleph_0} = \aleph_n$ implies $\lim^n \mathbf{A} \neq 0$ when $n = 1$ is that the answer to the following question is yes when $n = 1$ as well.

Question 7.5. *Is it a ZFC theorem that any $F \subseteq {}^\omega\omega$ of $<^*$ -ordertype ω_n indexes a nontrivial n -coherent family?*

As noted, Veličković and Vignati [24] have recently obtained a positive answer to Question 7.5 in the presence of additional cardinal arithmetic assumptions. In particular, they prove that if $2^{\aleph_k} < 2^{\aleph_{k+1}}$ for all $1 \leq k < n$, then every $F \subseteq {}^\omega\omega$ of $<^*$ -ordertype ω_n indexes a nontrivial n -coherent family. A natural variation on these results and questions is the following:

Question 7.6. *Fix $X \subseteq \mathbb{N}$. Is there a model of ZFC in which $\lim^n \mathbf{A} = 0$ if and only if $n \in X$?*

A main way of seeing that the answer to Question 7.5 is yes when $n = 1$ applies walks techniques to transfer large portions of a nontrivial coherent family on ω_1 to any F as above [2, pp. 96-98]. Question 7.5 is more generally in large part a question about the combinatorics of the ordinals ω_n ($n \in \omega$). Here our researches link up with those of [5] and [4] in ways we may take the occasion to clarify. A central focus of both those works is nontrivial n -coherent families of functions indexed by ordinals ξ ; much as in the present work, such functions represent nonzero elements of \lim^n of an inverse system $\mathbf{C}(\xi, \mathbb{Z})$, which is defined as follows: for any ordinal ξ and abelian group A let $\mathbf{C}(\xi, A)$ denote the inverse system $(\oplus_\alpha A, p_{\alpha\beta}, \xi)$ in which the maps $p_{\alpha\beta} : \oplus_\beta A \rightarrow \oplus_\alpha A$ are projections for all $\alpha \leq \beta < \xi$. Highly relevant for Question 7.5 are the following facts:

- $\lim^m \mathbf{C}(\omega_n, \mathbb{Z}) \neq 0$ for all $n \geq m \geq 0$ in Gödel's constructible universe L , as shown in [5].
- There exists (in ZFC) an abelian group A such that $\lim^n \mathbf{C}(\omega_n, A) \neq 0$ for all $n \geq 0$, as shown in [4].

Against this background, one of the most central questions is surely the following:

Question 7.7. *Is it a ZFC theorem that $\lim^n \mathbf{C}(\omega_n, \mathbb{Z}) \neq 0$ for all $n \geq 0$? Put differently, do there exist height- ω_n nontrivial n -coherent families of functions mapping to \mathbb{Z} for all $n > 0$ in any model of the ZFC axioms?*

Broadly speaking, the argument of [4] is that the fundamental content of a main result from [15] is the existence of higher-dimensional variants of the walks apparatus first appearing in [22]. It seems likely that the answer to Question 7.7 will depend on a better understanding of these higher-dimensional walks, particularly if that answer is yes.

Question 7.8. *How much of the classical machinery of walks extends to the n -dimensional walks on ω_n of [4]?*

Question 7.5 may be viewed as a special case of Question 7.8. The prominence of classical coherence phenomena in infinitary combinatorics, as well as the growing prominence of their higher-dimensional variants, is partly explained in [5] by their connections both to the Čech cohomology groups of the ordinals and to the broader set-theoretic theme of incompactness. The project of understanding higher-dimensional coherence will in part entail understanding its relation to central incompactness principles like $\square(\kappa)$.

Question 7.9. *What are the behaviors of n -dimensional walks on cardinals $\kappa > \omega_n$, particularly under assumptions like $\square(\kappa)$?*

Complementary to the ZFC focus of Questions 5–7 above, in other words, are consistency questions. As the possible behaviors of $\lim^n \mathbf{C}(\omega_1, A)$ and $\lim^n \mathbf{C}(\omega_2, A)$ are either understood or subsumed by previous questions, the following is among the most immediate:

Question 7.10. *Is it consistent with the ZFC axioms that $\lim^2 \mathbf{C}(\omega_3, A) = 0$ for all abelian groups A ?*

Most of the above may be framed as questions about the possible “spectra” of nontrivial multi-dimensional coherence phenomena, or equivalently, of nonvanishing \lim^n , either of \mathbf{A} or of $\mathbf{C}(-, -)$. Bound up with these questions seems to be that of the relation of these inverse systems’ higher limits to each other. Several other families of inverse systems’ higher limits seem to be implicated in these behaviors as well; among the more obvious generalizations of the system \mathbf{A} , for example, are those which replace its index-set ${}^\omega\omega$ with ${}^\kappa\lambda$ for arbitrary cardinals κ and λ . As it happens, the vanishing of these systems’ higher limits carries implications within the framework of Clausen and Scholze’s *condensed mathematics* [17, 7]. If κ is infinite and λ is uncountable, then \lim^1 of the associated system is nonzero. The systems in which $\lambda = \omega$, on the other hand, are denoted \mathbf{A}_κ in [3]; there it is shown that $\lim^1 \mathbf{A} = 0$ if and only if $\lim^1 \mathbf{A}_\kappa = 0$ for all $\kappa \geq \omega$. Whether this holds for higher \lim^n is an interesting question, as is the following:

Question 7.11. *Let κ be an uncountable cardinal. Is it consistent that $\lim^n \mathbf{A}_\kappa = 0$ for all $n > 0$?*

A second generalization of the system \mathbf{A} retains the order ${}^\omega\omega$, but varies the groups which it indexes, as well as the homomorphisms connecting them. The work [1] isolates a class of such systems significant in strong homology computations; it then shows that arguments applied to \mathbf{A} in [6] in fact apply to this broader class of systems. This carries the consequence that it is consistent with the ZFC axioms that strong homology is additive on the category of locally compact separable metric spaces; notably, however, these arguments require the existence of a weakly compact cardinal. Somewhat surprisingly, and in contrast to [6] and [1], there is no straightforward adaptation of the present work’s argument to this wider class, for the simple reason that the equivalence of type I and type II triviality so essential to this paper’s argument no longer holds in that more general setting.

Question 7.12. *What is the consistency strength of the statement “strong homology is additive on the category of locally compact separable metric spaces”?*

A last context in which these questions are likely interesting is in the presence of determinacy hypotheses. Relatedly, one might ask how “definable” a nontrivial n -coherent family of functions indexed by ${}^\omega\omega$ (viewed as a set of real numbers) can be. When $n = 1$, such a family is necessarily nonanalytic [23]; the following question was communicated to the first author by Justin Tatch Moore in 2014.

Question 7.13. *Fix $n > 1$. Can a nontrivial n -coherent family of functions indexed by ${}^\omega\omega$ be analytic?*

ACKNOWLEDGEMENTS

The authors would like to thank both the anonymous referee and Nathaniel Bannister for their alert readings of, and many helpful comments on, the text.

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