SIMULTANEOUSLY NONVANISHING HIGHER DERIVED LIMITS

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ABSTRACT. The derived functors $\lim^n$ of the inverse limit find many applications in algebra and topology. In particular, the vanishing of certain derived limits $\lim^n \mathbf{A}[H]$, parametrized by an abelian group H, has implications for strong homology and condensed mathematics. In this paper, we prove that if $\mathfrak{d} = \omega_n$, then $\lim^n \mathbf{A}[H] \neq 0$ holds for $H = \mathbb{Z}^{(\omega_n)}$ (i.e. the direct sum of ω_n-many copies of $\mathbb{Z}$). The same holds for $H = \mathbb{Z}$ under the assumption that $\mathrm{w}\diamondsuit(S^{k+1}_k)$ holds for all $k < n$. In particular, this shows that if $\lim^n \mathbf{A}[H] = 0$ holds for all $n \geq 1$ and all abelian groups H, then $2^{\aleph_0} \geq \aleph_{\omega+1}$, thus answering a question of Bannister. Finally, we prove some consistency results regarding simultaneous nonvanishing of derived limits, again in the case of $H = \mathbb{Z}$. In particular, we show the consistency, relative to ZFC, of $\bigwedge_{2 \leq k < \omega} \lim^k \mathbf{A} \neq 0$.

1. INTRODUCTION

The past few years have seen considerable progress in the application of set-theoretic tools to the study of homological algebra, and in particular to the study of the derived functors of the inverse limit functor. This paper is a contribution to this line of research. In particular, we isolate a number of situations in which certain derived limits are provably nonzero.

The inverse systems that we are primarily concerned with here are of the form $\mathbf{A}_{\mathcal{I}}[H]$, where $\mathcal{I}$ is a $\subseteq$-directed collection of sets and H is an arbitrary abelian group. These systems are indexed along the directed partial order $(\mathcal{I}, \subseteq)$.[1] We are especially interested in the case in which $\mathcal{I} = \{I_f \mid f \in {}^\omega\omega\}$, where

$$I_f = \{(k, m) \in \omega \times \omega \mid m < f(k)\}.$$

We will omit $\mathcal{I}$ from the notation $\mathbf{A}_{\mathcal{I}}[H]$ when $\mathcal{I}$ has this value; similarly, we will omit H from the notation when $H = \mathbb{Z}$.

The derived limits of these systems, and questions about their vanishing, show up in a variety of contexts in various fields of mathematical research. To give two prominent examples:

- In [15], Mardešić and Prasolov prove that, if strong homology is additive on the class of all closed subsets of Euclidean space, then $\lim^n \mathbf{A} = 0$ for all $1 \leq n < \omega$.

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[1] We refer the reader to Subsection 2.2 for precise definitions.

- Clausen and Scholze showed that the assertion that $\lim^n \mathbf{A}[H] = 0$ for all $1 \leq n < \omega$ and all abelian groups H is equivalent to a useful statement about the calculation of derived functors in the category of condensed abelian groups (for a precise statement, see [8, Lecture 4] or [6]).

In the late 1980s and early 1990s, a number of works were published applying set-theoretic tools to the study of the *first* derived limit of the system $\mathbf{A}$ (cf. [15, 9, 14, 18], and see the introduction of [5] for a summary of the contents of these works). It was not until roughly ten years ago that the *higher* derived limits of $\mathbf{A}$ and its relatives began to be explored. This was begun in [2] and has continued in a number of works published since then. We now survey some of the relevant recent highlights of this research program, beginning with results about the consistent *vanishing* of higher derived limits.

- In [5], Bergfalk and Lambie-Hanson prove that, relative to the consistency of the existence of certain large cardinals, the statement "for all $1 \leq n < \omega$, $\lim^n \mathbf{A} = 0$" is consistent with ZFC. In particular, they prove that, in any forcing extension obtained by adding weakly compact-many Hechler reals, $\lim^n \mathbf{A} = 0$ for all $1 \leq n < \omega$.
- In [4], Bergfalk, Hrušák, and Lambie-Hanson remove the large cardinal assumptions from the main result of [5]. In particular, they prove that, in any forcing extension obtained by adding $\beth_\omega$-many Cohen reals, $\lim^n \mathbf{A} = 0$ for all $1 \leq n < \omega$.
- In [1], Bannister sharpens the arguments from [5] and [4] to build models where derived limits vanish simultaneously for a broader class of systems. In particular, it follows from his work that, in the model of [4], we in fact have $\lim^n \mathbf{A}[H] = 0$ for all $1 \leq n < \omega$ and all abelian groups H.

In the other direction, there have been some recent results about the consistent *nonvanishing* of the derived limits of $\mathbf{A}$.

- In [19], Veličković and Vignati prove that, for each $1 \leq n < \omega$, it is consistent with ZFC that $\lim^n \mathbf{A} \neq 0$. In particular, they prove that, if $\mathfrak{b} = \mathfrak{d} = \aleph_n$ and $\mathrm{w}\diamondsuit(S^{k+1}_k)$ holds for all $k < n$, then $\lim^n \mathbf{A} \neq 0$.[2]
- In [7], Casarosa shows that the assumption of $\mathfrak{b} = \mathfrak{d}$ is not necessary in the main result from [19]. In particular, he proves that, for all $1 \leq k \leq n < \omega$, it is consistent that $\mathfrak{b} = \aleph_k$, $\mathfrak{d} = \aleph_n$, and $\lim^n \mathbf{A} \neq 0$.

A number of questions remained open in the wake of the work described above. Let us now recall two prominent such questions, which together comprise the primary motivation behind this paper.

Question 1. *What is the minimum value of* $2^{\aleph_0}$ *compatible with the statement "for all* $1 \leq n < \omega$, $\lim^n \mathbf{A} = 0$*"? What is the minimum value of* $2^{\aleph_0}$ *compatible with the statement "for all* $1 \leq n < \omega$ *and all abelian groups* H, $\lim^n \mathbf{A}[H] = 0$*"?*

The first half of Question 1 was asked in [5], and also in [4] and [1]. The second half of Question 1 is closely related to Bannister's [1, Question 7.7], which asks an analogous question about a slighly broader class of inverse systems. The first half of the question remains open; we provide here an answer to the second half and, in the process, to [1, Question 7.7]. In particular, we prove the following theorem.

[2] $\mathfrak{b}$ and $\mathfrak{d}$ are the *bounding* and *dominating* numbers, respectively. See definition 3.4 for the definition of $\mathrm{w}\diamondsuit(S^{k+1}_k)$.

Theorem A. *Suppose that $1 \leq n < \omega$ and $\mathfrak{d} = \omega_n$. Then*

(1) $\lim^n \mathbf{A}[\mathbb{Z}^{(\omega_n)}] \neq 0$, *where $\mathbb{Z}^{(\omega_n)}$ denotes the direct sum of ω_n-many copies of $\mathbb{Z}$;*

(2) *if, in addition, $\mathrm{w}\diamondsuit(S^{k+1}_k)$ holds for all $k < n$, then $\lim^n \mathbf{A} \neq 0$.*

In particular, if $\lim^n \mathbf{A}[H] = 0$ for all $1 \leq n < \omega$ and all abelian groups H, then $2^{\aleph_0} \geq \aleph_{\omega+1}$.

[1, Question 7.7] asks about the minimum value of the continuum compatible with additivity of derived limits for Ω_ω-systems. We refer the reader to [1] for the relevant definitions; we just note here that a statement of the form $\lim^n \mathbf{A}[H] \neq 0$ provides a counterexample to the additivity of derived limits for Ω_ω-systems, and Bannister showed in [1] that additivity of derived limits for Ω_ω-systems is compatible with $2^{\aleph_0} = \aleph_{\omega+1}$. Therefore, Theorem A provides an answer to [1, Question 7.7].

We now turn to the second question motivating this paper.

Question 2. *Let X be an arbitrary set of positive integers. Is there a model of* ZFC *in which $\lim^n \mathbf{A} = 0$ if and only if $n \in X$? In particular, is the statement "for all $1 \leq n < \omega$, $\lim^n \mathbf{A} \neq 0$" consistent with* ZFC*?*

Question 2 was asked in [4], and we make some partial progress towards it here. Notably, prior to the present paper, there was only one instance in which the simultaneous *nonvanishing* of $\lim^n \mathbf{A}$ for multiple values of $1 \leq n < \omega$ was known to be consistent. Namely, it follows from results in [18] and [2] that it is consistent with ZFC that $\lim^1 \mathbf{A} \neq 0$ and $\lim^2 \mathbf{A} \neq 0$ simultaneously (in the model witnessing this, $\lim^n \mathbf{A} = 0$ for all $3 \leq n < \omega$). Here, we construct models witnessing the consistency of the simultaneous nonvanishing of $\lim^n \mathbf{A}$. In particular, we prove the following theorem.

Theorem B. (1) *Fix $2 \leq n < \omega$. There exists a model of* ZFC *in which $\mathfrak{b} = \mathfrak{d} = \aleph_n$ and $\bigwedge_{2 \leq k \leq n} \lim^k \mathbf{A} \neq 0$.*

(2) *There exists a model of* ZFC *in which $\bigwedge_{2 \leq k < \omega} \lim^k \mathbf{A} \neq 0$.*

Note that, by a theorem of Goblot [10], if $1 \leq n < \omega$ and $\mathfrak{d} = \aleph_n$, then $\lim^k \mathbf{A} = 0$ for all $k > n$. Thus, clause (1) of Theorem B is in one sense optimal. In our model for clause (2) of Theorem B, we have $\mathfrak{b} = \mathfrak{d} = \aleph_{\omega+2}$. By the aforementioned result of Goblot, $\bigwedge_{2 \leq k < \omega} \lim^k \mathbf{A} \neq 0$ implies that $\mathfrak{d}$ is at least $\aleph_{\omega+1}$. Our result is thus almost optimal; it remains open whether one can obtain the same conclusion with $\mathfrak{d} = \aleph_{\omega+1}$. Also, the models that we construct in proving Theorem B will satisfy $\lim^1 \mathbf{A} = 0$. We therefore fall just short of answering the "in particular" clause of Question 2, which remains open.

We now briefly discuss some of the methods underlying our results and the structure of the paper. As has become standard in this area of research, given $1 \leq n < \omega$, our verifications that a derived limit $\lim^n \mathbf{A}_{\mathcal{I}}[H]$ is nonzero in a particular model will reduce to the construction of a combinatorial object known as a *nontrivial coherent n-family*. In Section 2, we recall some of the basic definitions and facts surrounding nontrivial coherent n-families and their connections with derived limits.

Our proof of Theorem A builds upon the work of Veličković and Vignati in [19]. In that paper, the assumption that $\mathfrak{b} = \mathfrak{d}$ yields the existence of a $\subseteq^*$-increasing cofinal sequence from $\{I_f \mid f \in {}^\omega\omega\}$ along which to perform recursive constructions of objects witnessing instances of $\lim^n \mathbf{A} \neq 0$. In the absence of the assumption that $\mathfrak{b} = \mathfrak{d}$, such sequences need not exist. As shown in [7], the weaker assumption of the existence of an unbounded chain is sufficient. To generalize these results even further, in Section 3 we introduce the notion of an *ascending sequence* of sets. "Ascending" is a weakening of "$\subseteq^*$-increasing", but we prove that it is strong enough to allow one to carry out modified versions of the constructions from [19].

In Section 4 we apply the technical results of Section 3 to obtain nonvanishing results for derived limits. In particular, we isolate natural conditions on an ideal $\mathcal{I}$ that imply the existence of an ascending $\subseteq^*$-cofinal sequence of elements of $\mathcal{I}$ to which one can apply the methods developed in Section 3. We then apply these ideas to the specific ideal $\emptyset \times \mathrm{Fin}$, which is the ideal generated by the sets $\{I_f \mid f \in {}^\omega\omega\}$, to obtain a proof of Theorem A.

Finally, Section 5 contains our proof of Theorem B. This proof again builds on the techniques of [19], combining them with the essential use of certain square sequences to enable the recursive construction of nontrivial coherent n-families of length greater than ω_n. The heart of the section is a technical stepping-up lemma allowing us to use an appropriate combination of square principles and weak diamonds to construct nontrivial coherent $(n+1)$-families out of shorter nontrivial coherent n-families. This lemma is then applied in various forcing extensions to yield a proof of Theorem B.

1.1. **Notation and conventions.** Our notation is for the most part standard. For undefined notions in set theory, we refer the reader to [12], and in homological algebra, to [20].

We identify an ordinal with the set of all ordinals strictly less than it. In particular, we identify the natural number n with the set $\{0, 1, \ldots, n-1\}$. If C is a set of ordinals, then we let $\mathrm{Lim}(C)$ denote the set $\{\alpha \in C \setminus \{0\} \mid \sup(C \cap \alpha) = \alpha\}$. If a and b are sets of ordinals, then we let $a < b$ denote the assertion $(\forall \alpha \in A)(\forall \beta \in B)(\alpha < \beta)$. In particular, for any set a of ordinals, we have $\emptyset < a$ and $a < \emptyset$, so this is only a partial order when restricted to nonempty sets.

If α is an ordinal, then $\mathrm{cf}(\alpha)$ denotes its cofinality. If λ is a regular infinite cardinal and δ is an ordinal, then $S^\delta_\lambda = \{\alpha < \delta \mid \mathrm{cf}(\alpha) = \lambda\}$. If $m < n < \omega$, then we let S^n_m denote $S^{\omega_n}_{\omega_m}$.

Given a positive integer n, we will identify functions with domain n and sequences of length n, i.e., we will make no distinction between a function u with domain n and the sequence $\langle u(0), u(1), \ldots, u(n-1)\rangle$. Given such a u and a natural number $i < n$, we let u^i denote the sequence of length $n-1$ formed by removing $u(i)$ from u. This is often denoted

$$\langle u(0), u(1), \ldots, \widehat{u(i)}, \ldots, u(n-1)\rangle;$$

formally, it is the function u^i with domain $n-1$ defined by

$$u^i(j) = \begin{cases} u(j) & \text{if } j < i \\ u(j+1) & \text{if } j \geq i \end{cases}$$

for all $j < n-1$. If $n < \omega$ and $\sigma : n \to n$ is a permutation, then $\mathrm{sgn}(\sigma) \in \{-, +\}$ denotes the parity of σ.

Given a class X and a cardinal κ, $[X]^\kappa$ denotes the class of all subsets of X of cardinality κ. Given a set of ordinals a, we let $\mathrm{otp}(a)$ denote its order-type. We will customarily identify sets of ordinals with the functions enumerating them in increasing order, i.e., if $a \subseteq \mathrm{Ord}$ has order-type δ then, for all $i < \delta$, we let $a(i)$ denotes the unique $\beta \in a$ such that $\mathrm{otp}(a \cap \beta) = i$. We will also regularly identify a sequence of length 1 with its unique value, e.g., we will write φ_α instead of $\varphi_{\langle \alpha \rangle}$. Similarly, we may write, e.g., $\varphi_{\alpha\beta}$ instead of $\varphi_{\langle \alpha, \beta \rangle}$. If σ is a sequence indexed by an ordinal, then $\mathrm{lh}(\sigma)$ denotes its domain. If $\alpha < \beta$ are ordinals and σ and τ are sequences of length α and β, respectively, then $\sigma \sqsubseteq \tau$ denotes the assertion that $\tau \restriction \alpha = \sigma$.

Given two functions φ_0 and φ_1, we write $\varphi_0 =^* \varphi_1$ to denote the assertion that, on their common domains, φ_0 and φ_1 agree at all but finitely many places. More formally, this is the assertion that the set

$$\{x \in \mathrm{dom}(\varphi_0) \cap \mathrm{dom}(\varphi_1) \mid \varphi_0(x) \neq \varphi_1(x)\}$$

is finite. As a special case, if j is an element of the codomain of φ, then $\varphi =^* j$ is the assertion that $\{x \in \mathrm{dom}(\varphi_0) \mid \varphi(x) \neq j\}$ is finite. Similarly, if u and v are two sets, then we let $u \subseteq^* v$ denote the assertion that $u \setminus v$ is finite. If φ is a function from a set x into an abelian group A, then the *support* of φ, denoted $\mathrm{supp}(\varphi)$, is $\{i \in x \mid \varphi(i) \neq 0\}$. If f is a function and $X \subseteq \mathrm{dom}(f)$, then $f[X]$ denotes the pointwise image of X under f. Similarly, if X is a subset of the codomain of f, then $f^{-1}[X]$ denotes the preimage of X under f.

We let Ab denote the category of abelian groups. We will be interested in functions mapping into abelian groups. For a set u and an abelian group H, the set of functions from u to H itself has a natural abelian group structure defined by pointwise addition. For improved readability, we will slightly abuse notation in the following way: given sets u_0 and u_1, an abelian group H, and functions $\varphi_i : u_i \to H$ for $i < 2$, we will write $\varphi_0 + \varphi_1$ instead of the more precise $\varphi_0 \restriction (u_0 \cap u_1) + \varphi_1 \restriction (u_0 \cap u_1)$. This generalizes straightforwardly to longer finite sums: if n is a positive integer and $\langle \varphi_i \mid i < n \rangle$ is a family of functions mapping into H, then by convention the domain of the function $\sum_{i<n} \varphi_i$ is $\bigcap_{i<n} \mathrm{dom}(\varphi_i)$.

2. Coherence, triviality, and derived limits

In this section, we introduce the primary objects of study of this paper and review some of their basic properties.

2.1. Nontrivial coherent families.

Definition 2.1. Suppose that $\mathcal{I}$ is a collection of sets, n is a positive integer, and H is an abelian group. An *H-valued n-family indexed along $\mathcal{I}$* is a family of functions of the form

$$\Phi = \left\langle \varphi_u : \bigcap_{i<n} u(i) \to H \;\middle|\; u \in \mathcal{I}^n \right\rangle.$$

If H and $\mathcal{I}$ are either clear from context or irrelevant, then we may refer to such an object simply as an *n-family.* An n-family is said to be

- *alternating* if, for every $u \in \mathcal{I}^n$ and every permutation $\sigma : n \to n$, we have
$$\varphi_{u \circ \sigma} = \mathrm{sgn}(\sigma) \varphi_u;$$
- *coherent* if it is alternating and, for all $v \in \mathcal{I}^{n+1}$, we have
$$\sum_{i<n+1} (-1)^i \varphi_{v^i} =^* 0;$$

- *trivial* if
  - $n = 1$ and there is a function $\psi : \bigcup \mathcal{I} \to H$ such that, for all $u \in \mathcal{I}$, we have $\psi \restriction u =^* \varphi_u$; or
  - $n > 1$ and there is an alternating $(n-1)$-family

$$\Psi = \left\langle \psi_t : \bigcap_{i<n-1} t(i) \to H \;\middle|\; t \in \mathcal{I}^{n-1} \right\rangle$$

    such that, for all $u \in \mathcal{I}^n$, we have

$$\varphi_u =^* \sum_{i<n} (-1)^i \psi_{u^i}.$$

  In this case, we say that ψ or Ψ *trivializes* Φ.

Remark 2.2. It is easily verified that a trivial n-family is always coherent. The question of when the converse holds is the primary motivating question of this work.

Moreover, if an n-family $\langle \varphi_u \mid u \in \mathcal{I}^n \rangle$ is alternating, then, for every non-injective $u \in \mathcal{I}^n$ we must have $\varphi_u = 0$. Therefore, when constructing alternating n-families indexed along $\mathcal{I}$, it suffices to specify φ_u for injective $u \in \mathcal{I}^n$. In practice, we will often be working with sets $\mathcal{I}$ equipped with a natural linear order $<_{\mathcal{I}}$. In this setting, when constructing an alternating n-family indexed along $\mathcal{I}$, it suffices to specify φ_u for functions $u \in \mathcal{I}^n$ that are strictly increasing with respect to $<_{\mathcal{I}}$, since the requirement that the family be alternating will then determine all other values. Similarly, when verifying that such a family is either coherent or trivial, it suffices to consider functions $v \in \mathcal{I}^{n+1}$ or $u \in \mathcal{I}^n$, respectively, that are strictly increasing with respect to $<_{\mathcal{I}}$. In particular, given a sequence of sets $\vec{x} = \langle x_\alpha \mid \alpha < \delta \rangle$, by an *$n$-family indexed along $\vec{x}$* we mean a family of the form

$$\Phi = \langle \varphi_a : \bigcup_{i<n} x_{a(i)} \mid a \in \delta^n \rangle \qquad \text{or} \qquad \Phi = \langle \varphi_a : \bigcup_{i<n} x_{a(i)} \mid a \in [\delta]^n \rangle$$

Definition 2.3. Suppose that $\Phi = \langle \varphi_u \mid u \in \mathcal{I}^n \rangle$ is an n-family. If $\mathcal{J} \subseteq \mathcal{I}$, then we let $\Phi \restriction \mathcal{J}$ denote $\langle \varphi_u \mid u \in \mathcal{J}^n \rangle$.

If $\Phi = \langle \varphi_u \mid u \in \mathcal{I}^n \rangle$ and $\Psi = \langle \psi_u \mid u \in \mathcal{J}^n \rangle$ are n-families, then $\Psi \sqsubseteq \Phi$ denotes the assertion that $\Psi = \Phi \restriction \mathcal{J}$ (we will sometimes say in this situation that Φ *extends* Ψ).

Note that, if $\mathcal{J} \subseteq \mathcal{I}$ and Φ as above is coherent (*resp.* trivial), then $\Phi \restriction \mathcal{J}$ is also coherent (*resp.* trivial). The following partial converse to this follows from the arguments of [5, Lemma 2.7]; we provide a proof for completeness.

Proposition 2.4. *Suppose that $\Phi = \langle \varphi_u \mid u \in \mathcal{I}^n \rangle$ is a coherent H-valued n-family, $\mathcal{J}$ is a $\subseteq^*$-cofinal subset of $\mathcal{I}$, and $\Phi \restriction \mathcal{J}$ is trivial. Then Φ is trivial.*

Proof. We assume that $n > 1$. The case in which $n = 1$ is easier and left to the reader. Let $\Psi = \langle \psi_t \mid t \in \mathcal{J}^{n-1} \rangle$ trivialize $\Phi \restriction \mathcal{J}$. For each $x \in \mathcal{I}$, choose an $x^+ \in \mathcal{J}$ such that $x \subseteq^* x^+$. Given $m < \omega$ and $v \in \mathcal{I}^m$, define

- $v^+ \in \mathcal{J}^m$ by letting $v^+(i) = v(i)^+$ for all $i < m$; and
- for $i < m$, define $\pi_i(v) \in \mathcal{I}^{m+1}$ as

$$\pi_i(v) = \langle v(0), \ldots, v(i), v(i)^+, \ldots, v(m-1)^+ \rangle.$$

Now define an alternating H-valued $(n-1)$-family $\Psi' = \langle \psi'_t \mid t \in \mathcal{I}^{n-1} \rangle$ by letting

$$\psi'_t = \psi_{t^+} - \sum_{i=0}^{n-2} (-1)^i \varphi_{\pi_i(t)}$$

for all $t \in \mathcal{I}^{n-1}$ (the sum on the right is defined on all but finitely many elements of $\bigcap_{i<n-1} t(i)$; ψ'_t is defined more precisely by extending that sum to the domain $\bigcap_{i<n-1} t(i)$ by setting it equal to 0 on all otherwise undefined arguments).

We claim that Ψ' trivializes Φ. To verify this, fix $u \in \mathcal{I}^n$, and consider

$$\sum_{i=0}^{n-1} (-1)^i \psi'_{u^i} = \sum_{i=0}^{n-1} (-1)^i \left(\psi_{(u^i)^+} - \sum_{j=0}^{n-2} (-1)^j \varphi_{\pi_j(u^i)} \right). \tag{1}$$

Note first that, since Ψ trivializes $\Phi \restriction \mathcal{J}$, we have

$$\sum_{i=0}^{n-1} (-1)^i \psi_{(u^i)^+} =^* \varphi_{u^+},$$

and therefore the sum in (1) reduces to

$$\varphi_{u^+} + \sum_{i=0}^{n-1} \sum_{j=0}^{n-2} (-1)^{i+j+1} \varphi_{\pi_j(u^i)}. \tag{2}$$

For each $\ell < n$, let

$$A_\ell = \{(i,j) \in (n-1) \times (n-2) \mid \ell = j < i \text{ or } i \leq j = \ell - 1\}.$$

Note that A_ℓ consists precisely of those pairs (i,j) for which the formal definition of the sequence $\pi_j(u^i)$ contains u_ℓ and $(u_\ell)^+$ as consecutive elements. Now the sum in (2) becomes

$$\varphi_{u^+} + \sum_{\ell<n} \sum_{(i,j) \in A_\ell} (-1)^{i+j+1} \varphi_{\pi_j(u^i)}.$$

By the coherence of Φ, we have

$$\varphi_{u^+} + \sum_{(i,j) \in A_0} (-1)^{i+j+1} \varphi_{\pi_j(u^i)} =^* \varphi_{\langle u(0), u(1)^+, \ldots, u(n-1)^+ \rangle}.$$

More generally, for $\ell < n$, we have

$$\varphi_{\langle u(0), \ldots, u(\ell-1), u(\ell)^+, \ldots, u(n-1)^+ \rangle} + \sum_{(i,j) \in A_\ell} (-1)^{i+j+1} \varphi_{\pi_j(u^i)} =^* \varphi_{\langle u(0), \ldots, u(\ell), u(\ell+1)^+, \ldots, u(n-1)^+ \rangle}.$$

By repeatedly applying these equalities, we obtain:

$$\begin{aligned}\sum_{i=0}^{n-1}(-1)^i\psi'_{\varphi_{u^i}} &= u^+ + \sum_{\ell<n}\sum_{(i,j)\in A_\ell}(-1)^{i+j+1}\varphi_{\pi_j(u^i)} \\ &=^* \varphi_{\langle u(0),u(1)^+,\ldots,u(n-1)^+\rangle} + \sum_{\ell=1}^{n-1}\sum_{(i,j)\in A_\ell}(-1)^{i+j+1}\varphi_{\pi_j(u^i)} \\ &=^* \varphi_{\langle u(1),u(1),u(2)^+,\ldots,u(n-1)^+\rangle} + \sum_{\ell=2}^{n-1}\sum_{(i,j)\in A_\ell}(-1)^{i+j+1}\varphi_{\pi_j(u^i)} \\ &=^* \cdots \\ &=^* \varphi_{\langle u(1),\ldots,u(n-2),u(n-1)^+\rangle} + \sum_{(i,j)\in A_{n-1}}(-1)^{i+j+1}\varphi_{\pi_j(u^i)} \\ &=^* \varphi_u.\end{aligned}$$

Thus, Ψ' does trivialize Φ, as desired. □

We can also lift coherence from a $\subseteq^*$-cofinal subset of $\mathcal{I}$ to all of $\mathcal{I}$, in the following sense.

Proposition 2.5. *Suppose that $\mathcal{J}$ is a $\subseteq^*$-cofinal subset of $\mathcal{I}$ and $\Phi = \langle \varphi_u \mid u \in \mathcal{J}^n\rangle$ is a coherent H-valued n-family. Then there is a coherent H-valued n-family $\Phi' = \langle \varphi'_u \mid u \in \mathcal{I}^n\rangle$ such that $\Phi' \restriction \mathcal{J} = \Phi$.*

Proof. For each $x \in \mathcal{I}$, choose an $x^+ \in \mathcal{J}$ such that $x \subseteq^* x^+$. If $x \in \mathcal{J}$, then choose $x^+ = x$. Given $m < \omega$ and $v \in \mathcal{I}^m$, define $v^+ \in \mathcal{J}^m$ by letting $v^+(i) = v(i)^+$ for all $i < m$. Now, for all $u \in \mathcal{I}^n$, define $\varphi'_u : \bigcap_{i<n} u(i) \to H$ by setting $\varphi'_u = \varphi_{u^+}$. Note that φ_{u^+} is a function defined on all but finitely many elements of $\bigcap_{i<n} u(i)$; φ'_u is defined more precisely by extending that sum to the domain $\bigcap_{i<n} u(i)$ by setting it equal to 0 on all otherwise undefined arguments.

It is now immediate from the construction that Φ' is a coherent n-family with $\Phi' \restriction \mathcal{J} = \Phi$. □

Remark 2.6. In light of Proposition 2.5, if we are given $0 < n < \omega$, an abelian group H, a collection $\mathcal{I}$ of sets, and a $\subseteq^*$-cofinal subset $\mathcal{J} \subseteq \mathcal{I}$ and we seek to construct a nontrivial coherent H-valued n-family indexed along $\mathcal{I}$, then it suffices to construct such a family indexed along $\mathcal{J}$. Indeed, by Proposition 2.5, this family extends to one indexed along $\mathcal{I}$, and the nontriviality of the original family clearly implies the nontriviality of the extension.

The following is a variation on Goblot's vanishing theorem [10] indicating that if the cofinality of $(\mathcal{I}, \subseteq^*)$ is less than ω_n, then every coherent n-family indexed along $\mathcal{I}$ is trivial.

Proposition 2.7. *Suppose that H is an abelian group, $0 < n < \omega$, $\mathcal{I}$ is a collection of sets with $\mathrm{cf}(\mathcal{I}, \subseteq^*) < \omega_n$, and $\Phi = \langle \varphi_u \mid u \in \mathcal{I}^n\rangle$ is an H-valued coherent n-family. Then Φ is trivial.*

Proof. For concreteness, assume that $\mathrm{cf}(\mathcal{I}, \subseteq^*)$ is infinite; the case in which it is finite is much easier. Let $\kappa = \mathrm{cf}(\mathcal{I}, \subseteq^*)$. By replacing $\mathcal{I}$ with some $\subseteq^*$-cofinal subset and invoking Proposition 2.4, we can assume that $|\mathcal{I}| = \kappa$. Enumerate $\mathcal{I}$ as

$\langle x_\eta \mid \eta < \kappa \rangle$. For ease of notation, and recalling Remark 2.2, we will think of Φ as $\langle \varphi_b \mid b \in [\kappa]^n \rangle$, where elements of $[\kappa]^n$ are thought of as strictly increasing n-tuples from κ and, formally, given $b \in [\kappa]^n$, $\varphi_b = \varphi_{\langle x_{b(i)} \mid i < n \rangle}$. Note that this does not involve any loss of information due to the fact that Φ is assumed to be alternating.

The proof is by induction on n. Suppose first that $n = 1$, and hence $\kappa = \omega$. We will define a function $\psi : \bigcup \mathcal{I} \to H$ witnessing that Φ is trivial. For each $k < \omega$, let $y_k = x_k \setminus (\bigcup\{x_j \mid j < k\})$; note that $\{y_k \mid k < \omega\}$ is a partition of $\bigcup \mathcal{I}$. Now, for each $k < \omega$, set $\psi \restriction y_k = \varphi_k \restriction y_k$. Using the coherence of Φ, it is straightforward to verify that ψ thus defined witnesses that Φ is trivial.

Now suppose that $n > 1$. We must construct an alternating $(n-1)$-family

$$\Psi = \langle \psi_t \mid t \in \mathcal{I}^{n-1} \rangle$$

witnessing that Φ is trivial. Precisely as above, using the assumption that our families are alternating we will in fact construct a family of the form

$$\Psi = \langle \psi_a \mid a \in [\kappa]^{n-1} \rangle$$

such that, for all $b \in [\kappa]^n$, we have

$$\varphi_b =^* \sum_{i<n} (-1)^i \psi_{b^i}.$$

The construction is slightly different depending on whether $n = 2$ or $n > 2$. Suppose first that $n = 2$, in which case $\kappa \leq \omega_1$. We will construct $\psi_\eta : x_\eta \to H$ for $\eta < \kappa$ by recursion on η.

Suppose that $\xi < \kappa$ and we have constructed $\langle \psi_\eta \mid \eta < \xi \rangle$ such that, for all $\eta_0 < \eta_1 < \xi$, we have $\varphi_{\eta_0\eta_1} =^* \psi_{\eta_1} - \psi_{\eta_0}$. Define a 1-family of functions

$$T_\xi = \langle \tau_\eta : x_\eta \cap x_\xi \to H \mid \eta < \xi \rangle$$

by letting $\tau_\eta = \psi_\eta + \varphi_{\eta\xi}$.

Claim 2.8. *T_ξ is coherent.*

Proof. Fix $\eta_0 < \eta_1 < \xi$. Then

$$\begin{aligned} \tau_{\eta_1} - \tau_{\eta_0} &= (\psi_{\eta_1} + \varphi_{\eta_1\xi}) - (\psi_{\eta_0} + \varphi_{\eta_0\xi}) \\ &=^* \varphi_{\eta_1\xi} - \varphi_{\eta_0\xi} + \varphi_{\eta_0\eta_1} \\ &=^* 0, \end{aligned}$$

where all functions are restricted to the domain $x_{\eta_0} \cap x_{\eta_1} \cap x_\xi$, the passage from the first line to the second follows from our assumptions about $\langle \psi_\eta \mid \eta < \xi \rangle$, and the passage from the second line to the third follows from the coherence of Φ. □

Since $|\xi| < \omega_1$, the inductive hypothesis implies that T_ξ is trivial. We can therefore fix a function $\psi_\xi : x_\xi \to H$ such that, for all $\eta < \xi$, we have $\psi_\xi \restriction (x_\eta \cap x_\xi) =^* \tau_\eta$. Note that, for all $\eta < \xi$, we have

$$\psi_\xi - \psi_\eta =^* \tau_\eta - \psi_\eta = (\psi_\eta + \varphi_{\eta\xi}) - \psi_\eta = \varphi_{\eta\xi},$$

so this choice satisfies the requirements of the construction. This completes the case $n = 2$.

Suppose now that $n > 2$. We will construct $\psi_a : \bigcap_{i<n-1} x_{a(i)} \to H$ for $a \in [\kappa]^{n-1}$ by recursion on $\max(a)$.

Suppose that $\xi < \kappa$ and we have constructed $\langle \psi_a \mid a \in [\xi]^{n-1} \rangle$ such that, for all $b \in [\xi]^n$, we have

$$\varphi_b =^* \sum_{i<n} (-1)^i \psi_{b^i}.$$

We now describe how to define $\varphi_{d \cup \{\xi\}}$ for $d \in [\xi]^{n-2}$. First, define an $(n-1)$-family of functions

$$T_\xi = \langle \tau_a : x_\xi \cap \bigcap \{x_{a(i)} \mid i < n-1\} \to H \mid a \in [\xi]^{n-1} \rangle$$

by letting $\tau_a = (-1)^n \psi_a + \varphi_{a \cup \{\xi\}}$.

Claim 2.9. *T_ξ is coherent.*

Proof. Fix $b \in [\xi]^n$, and let $c = b \cup \{\xi\}$. Note that, for $i < n-1$, we have $c^i = b^i \cup \{\xi\}$, and $c^{n-1} = b$. Then

$$\begin{aligned}
\sum_{i<n} (-1)^i \tau_{b^i} &= (-1)^n \sum_{i<n} (-1)^i \psi_{b^i} + \sum_{i<n} (-1)^i \varphi_{b^i \cup \{\xi\}} \\
&=^* (-1)^n \varphi_b + \sum_{i<n} (-1)^i \varphi_{b^i \cup \{\xi\}} \\
&= (-1)^n \varphi_{c^{n-1}} + \sum_{i<n} (-1)^i \varphi_{c^i} \\
&= \sum_{i<n+1} (-1)^i \varphi_{c^i} =^* 0,
\end{aligned}$$

where all functions are restricted to the domain $x_\xi \cap \bigcap \{x_{b(i)} \mid i < n\}$, the passage from the first line to the second follows from our assumptions about $\langle \psi_a \mid a \in [\xi]^{n-1} \rangle$, the passage from the second line to the third follows from the observations at the beginning of this proof, the passage from the third line to the fourth is a simple rearranging of terms, and the final equality (mod finite) follows from the coherence of Φ. □

Since $|\xi| < \kappa \leq \omega_{n-1}$, the inductive hypothesis implies that T_ξ is trivial. We can thus fix an $(n-2)$-family of functions

$$\langle \psi_{d \cup \{\xi\}} : x_\xi \cap \bigcap \{x_{d(i)} \mid i < n-2\} \to H \mid d \in [\xi]^{n-2} \rangle$$

such that, for all $a \in [\xi]^{n-1}$, we have

$$\tau_a =^* \sum_{i<n-1} (-1)^i \psi_{a^i \cup \{\xi\}}.$$

We claim that this assignment of $\psi_{d \cup \{\xi\}}$ works. To check this, fix an arbitrary $a \in [\xi]^{n-1}$, and let $b = a \cup \{\xi\}$. Then

$$\begin{aligned}
\sum_{i<n} (-1)^i \psi_{b^i} &= (-1)^{n-1} \psi_a + \sum_{i<n-1} (-1)^i \psi_{a^i \cup \{\xi\}} \\
&=^* (-1)^{n-1} \psi_a + \tau_a \\
&= (-1)^{n-1} \psi_a + (-1)^n \psi_a + \varphi_{a \cup \{\xi\}} \\
&= \varphi_b.
\end{aligned}$$

Thus, we can carry out the construction of a family $\langle \psi_a \mid a \in [\kappa]^{n-1} \rangle$ witnessing that Φ is trivial, thus completing the proof. □

Definition 2.10. Suppose that $\vec{x} = \langle x_\alpha \mid \alpha < \delta \rangle$ is a sequence of sets, $1 \leq n < \omega$, H is an abelian group, and $\Phi = \langle \varphi_a : \bigcap_{i<n} x_{a(i)} \to H \mid u \in \delta^n \rangle$ is an n-family of functions indexed along $\vec{x}$. If E is a set, then $\Phi \restriction\restriction E$ denotes the n-family

$$\left\langle \varphi_a \restriction E \cap \bigcap_{i<n} x_{a(i)} \,\middle|\, a \in \delta^n \right\rangle$$

indexed along $\vec{x} \restriction\restriction E := \langle x_\alpha \cap E \mid \alpha < \delta \rangle$.

Note that, if Φ is an alternating, coherent, or trivial n-family, then these properties are inherited by $\Phi \restriction\restriction E$ for every set E. In particular, if Φ is a coherent n-family and $\Phi \restriction\restriction E$ is nontrivial for some set E, then also Φ is nontrivial.

Definition 2.11. Suppose that $1 < n < \omega$, δ is an ordinal, H is an abelian group, $\langle x_\alpha \mid \alpha < \delta \rangle$ is a sequence of sets, and $\Phi = \langle \varphi_b : \bigcap_{i<n} x_{b(i)} \to H \mid b \in \delta^n \rangle$ is a coherent n-family. Let $\mathrm{Triv}_H(\Phi)$ be the set of all $(n-1)$-families $\Psi = \langle \psi_a : \bigcap_{i<n-1} x_{a(i)} \to H \mid a \in \delta^{n-1} \rangle$ that trivialize Φ (so $\mathrm{Triv}_H(\Phi)$ is empty if Φ is nontrivial).

If the group H is clear from context, we may omit it from the notation $\mathrm{Triv}_H(\Phi)$.

Proposition 2.12. *Suppose that $1 < n < \omega$, δ is an ordinal, H is an abelian group, $\langle x_\alpha \mid \alpha < \delta \rangle$ is a sequence of sets, and $\Phi = \langle \varphi_b : \bigcap_{i<n} x_{b(i)} \to H \mid b \in \delta^n \rangle$ is a coherent n-family. For all $\Psi^0, \Psi^1 \in \mathrm{Triv}_H(\Phi)$, the $(n-1)$-family*

$$\Psi^1 - \Psi^0 = \langle \psi^1_a - \psi^0_a \mid a \in \delta^{n-1} \rangle$$

is coherent.

Proof. Fix $\Psi^0, \Psi^1 \in \mathrm{Triv}_H(\Phi)$. The fact that $\Psi^1 - \Psi^0$ is alternating follows immediately from the fact that both Ψ^0 and Ψ^1 are alternating. To check coherence, fix $b \in \delta^n$. For each $\ell < 2$, the fact that $\Psi^\ell \in \mathrm{Triv}_H(\Phi)$ implies that $\sum_{i<n} \psi^\ell_{b^i} =^* \varphi_b$. Therefore, we have

$$\sum_{i<n} (\psi^1_{b^i} - \psi^0_{b^i}) = \sum_{i<n} \psi^1_{b^i} - \sum_{i<n} \psi^0_{b^i} =^* \varphi_b - \varphi_b = 0,$$

as desired. □

Definition 2.13. Given $1 < n < \omega$, an abelian group H, and an H-valued coherent n-family Φ, define an equivalence relation $\cong_{\Phi,H}$ on $\mathrm{Triv}_H(\Phi)$ by setting $\Psi^0 \cong_{\Phi,H} \Psi^1$ if and only if the coherent $(n-1)$-family $\Psi^1 - \Psi^0$ is trivial (via an H-valued trivialization).

The following basic facts are easily established; their proofs are left to the reader.

Proposition 2.14. *Suppose that $1 < n < \omega$, H is an abelian group, Φ is an H-valued coherent n-family, and $\Psi^0, \Psi^1 \in \mathrm{Triv}_H(\Phi)$.*

1. *For every set E, we have $\Psi^0 \restriction\restriction E, \Psi^1 \restriction\restriction E \in \mathrm{Triv}_H(\Phi \restriction\restriction E)$.*
2. *If $E \subseteq F$ are sets and $\Psi^0 \restriction\restriction F \cong_{\Phi \restriction\restriction F, H} \Psi^1 \restriction\restriction F$, then $\Psi^0 \restriction\restriction E \cong_{\Phi \restriction\restriction E, H} \Psi^1 \restriction\restriction E$.* □

Proposition 2.15. *Suppose that $1 < n < \omega$, δ is an ordinal, H is an abelian group, $\langle x_\alpha \mid \alpha < \delta \rangle$ is a sequence of sets, and $\Phi = \langle \varphi_b : \bigcap_{i<n} x_{b(i)} \to H \mid b \in \delta^n \rangle$ is a coherent n-family. Suppose also that $\Psi \in \mathrm{Triv}_H(\Phi)$ and $T = \langle \tau_a : \bigcap_{i<n-1} x_{a(i)} \to H \mid a \in \delta^{n-1} \rangle$ is a coherent $(n-1)$-family. Then the family*

$$\Psi + T = \langle \psi_v + \tau_v \mid v \in \delta^{n-1} \rangle$$

is in $\mathrm{Triv}_H(\Phi)$. *Moreover, we have* $\Psi + T \cong_{\Phi,H} \Psi$ *if and only if* T *is trivial.*

Proof. We first verify that $\Psi + T$ is in $\mathrm{Triv}_H(\Phi)$. To this end, fix $b \in \delta^n$. Then we have

$$\begin{aligned}\sum_{i<n}(\psi_{b^i} + \tau_{b^i}) &= \sum_{i<n}\psi_{b^i} + \sum_{i<n}\tau_{b^i} \\ &=^* \varphi_b + 0 = \varphi_b,\end{aligned}$$

as desired. The "moreover" clause follows immediately from the definition of the equivalence relation $\cong_{\Phi,H}$. □

Definition 2.16. Suppose that

- $1 < n < \omega$;
- δ is an ordinal;
- $\langle x_\alpha \mid \alpha < \delta + 1\rangle$ is a sequence of sets;
- H is an abelian group;
- $\Phi = \langle \varphi_b : \bigcap_{i<n} x_{b(i)} \to H \mid b \in \delta^n\rangle$ is a coherent n-family;
- $\Psi = \langle \psi_a : x_\delta \cap \bigcap_{i<n-1} a_{a(i)} \to H \mid a \in \delta^{n-1}\rangle \in \mathrm{Triv}_H(\Phi \restriction\restriction x_\delta)$.

Then let $\Phi^\frown\langle\Psi\rangle$ denote the coherent n-family $\langle \varphi_b : \bigcap_{i<n} x_{b(i)} \to H \mid b \in (\delta+1)^n\rangle$ extending Φ defined by letting $\varphi_{a^\frown\langle\delta\rangle} = (-1)^{n+1}\psi_a$ for all $a \in \delta^{n-1}$.

Remark 2.17. Implicit in Definition 2.16 is the assertion that the n-family $\Phi^\frown\langle\Psi\rangle$ is indeed coherent. The verification of this fact is routine and left to the reader.

Proposition 2.18. *Suppose that*

- $1 < n < \omega$;
- δ *is an ordinal;*
- $\langle x_\alpha \mid \alpha < \delta + 1\rangle$ *is a sequence of sets;*
- H *is an abelian group;*
- $\Phi = \langle \varphi_b : \bigcap_{i<n} x_{b(i)} \to H \mid b \in \delta^n\rangle$ *is a coherent n-family;*
- $\Psi \in \mathrm{Triv}_H(\Phi \restriction\restriction x_\delta)$, $\Psi' \in \mathrm{Triv}_H(\Phi)$, *and* $\Psi \not\cong_{\Phi\restriction\restriction x_\delta, H} \Psi' \restriction\restriction x_\delta$.

Then Ψ' *does not extend to a trivialization of* $\Phi^\frown\langle\Psi\rangle$.

Proof. Assume that $n > 2$. The case $n = 2$ is similar but easier, and left to the reader. Let $\Phi^\frown\langle\Psi\rangle = \langle\varphi_b \mid b \in (\delta+1)^n\rangle$. Suppose for the sake of contradiction that $\langle\psi'_a \mid a \in (\delta+1)^{n-1}\rangle$ extends Ψ' and trivializes $\Phi^\frown\langle\Psi\rangle$. Then, for all $a \in \delta^{n-1}$, we have

$$\begin{aligned}(-1)^{n+1}\psi_a &= \varphi_{a^\frown\langle\delta\rangle} \\ &=^* \sum_{i<n}(-1)^i\psi'_{(a^\frown\langle\delta\rangle)^i} \\ &= (-1)^{n-1}\psi'_a + \sum_{i<n-1}\psi'_{a^i{}^\frown\langle\delta\rangle}.\end{aligned}$$

Rearranging the above equation, we see that, for all $a \in \delta^{n-1}$, we have

$$\psi_a - \psi'_a =^* (-1)^{n+1}\sum_{i<n-1}\psi'_{a^i{}^\frown\langle\delta\rangle}.$$

In particular, the family $\langle(-1)^{n+1}\psi'_{e^\frown\langle\delta\rangle} \mid e \in \delta^{n-2}\rangle$ trivializes $\Psi - \Psi' \restriction\restriction x_\delta$, contradicting the fact that $\Psi \not\cong_{\Phi\restriction\restriction x_\delta, H} \Psi' \restriction\restriction x_\delta$. □

2.2. **Derived limits.** One of the reasons for our interest in coherent n-families is their connection with the calculation of the derived limits of certain natural inverse systems of abelian groups.

Definition 2.19. If $(\Lambda, \leq_\Lambda)$ is a directed partial order, then an *inverse system of abelian groups indexed by* Λ is a structure of the form

$$\mathbf{B} = \langle B_u, \pi_{uv} \mid u, v \in \Lambda,\ u \leq_\Lambda v\rangle$$

such that

(1) for all $u \in \Lambda$, B_u is an abelian group;
(2) for all $u \leq_\Lambda v$, $\pi_{uv} : B_v \to B_u$ is a group homomorphism;
(3) for all $u \leq_\Lambda v \leq_\Lambda w$, $\pi_{uw} = \pi_{uv} \circ \pi_{vw}$.

Given an inverse system $\mathbf{B}$, one can form the *inverse limit* $\lim \mathbf{B} \in \mathsf{Ab}$. Concretely, $\lim \mathbf{B}$ can be represented as

$$\{x \in \prod_{u\in\Lambda} B_u \mid \forall u \leq_\Lambda v,\ x_u = \pi_{uv} x(v)\}.$$

Given a directed set Λ, one can form the category $\mathsf{Ab}^{\Lambda^{\mathrm{op}}}$ of all inverse systems of abelian groups indexed by Λ. (Since the details of this and what follows will not be necessary for us in this paper, we refer the reader to [16, Section III] for precise definitions and proofs of facts presented in the remainder of this section.) The inverse limit operation then induces a functor $\lim : \mathsf{Ab}^{\Lambda^{\mathrm{op}}} \to \mathsf{Ab}$. This functor is left exact but in general not exact; it therefore has (right) derived functors $\lim^n : \mathsf{Ab}^{\Lambda^{\mathrm{op}}} \to \mathsf{Ab}$ for $1 \leq n < \omega$. The nontrivial coherent families of functions introduced earlier in this section play a key role in the computation of the derived limits of a particular class of inverse systems, which we now introduce.

Definition 2.20. Suppose that H is an abelian group.

(1) Given a set x, let $A_x[H]$ denote the group $\bigoplus_x H$. Concretely, this is the group of all finitely-supported functions from x to H.
(2) Given sets $x \subseteq y$, let $\pi^H_{xy} : A_y[H] \to A_x[H]$ be the natural restriction map. More precisely, if $f : y \to H$ is in $A_y[H]$, then $\pi^H_{xy}(f) = f \restriction x$. Note that this is an abelian group homomorphism. If the group H is clear from context, then we will omit it from the notation.
(3) Suppose that $\mathcal{I}$ is a collection of sets that is $\subseteq$-directed, i.e., for all $x, y \in \mathcal{I}$, there is $z \in \mathcal{I}$ such that $x \cup y \subseteq z$. Then $\mathbf{A}_{\mathcal{I}}[H]$ denotes the inverse system of abelian groups

$$\langle A_x[H], \pi^H_{xy} \mid x, y \in \mathcal{I},\ x \subseteq y\rangle.$$

There is a particular $\subseteq$-directed collection of sets in which we will be especially interested in this paper. Given a function $f : \omega \to \omega$, let $I_f = \{(k, m) \in \omega \times \omega \mid m < f(k)\}$; i.e., I_f is the region "under the graph of f" in the plane $\omega \times \omega$. If we omit $\mathcal{I}$ from the notation $\mathbf{A}_{\mathcal{I}}[H]$, then $\mathcal{I}$ should be understood to be the set $\{I_f \mid f \in {}^\omega\omega\}$. Moreover, given $f \leq g \in {}^\omega\omega$, we will typically write $A_f[H]$ in place of $A_{I_f}[H]$ and π_{fg} in place of $\pi_{I_f I_g}$, i.e., the inverse system $\mathbf{A}[H]$ is the system

$$\langle A_f[H], \pi_{fg} \mid f \leq g \in {}^\omega\omega\rangle.$$

We will also omit mention of the group H in case $H = \mathbb{Z}$, e.g., we will write $\mathbf{A}$ for $\mathbf{A}[\mathbb{Z}]$.

We can now state the connection between nontrivial coherent n-families and the derived limits of these inverse systems. For a proof of the fact, see [16, Section III] or [5, Section 2.2].[3]

Fact 2.21. *Suppose that $1 \leq n < \omega$, $\mathcal{I}$ is a $\subseteq$-directed collection of sets, and H is an abelian group. Then the following are equivalent:*

(1) $\lim^n \mathbf{A}_{\mathcal{I}}[H] = 0$*;*
(2) every coherent H-valued n-family indexed along $\mathcal{I}$ is trivial.

Fact 2.21, together with Propositions 2.4 and 2.5 yield the following observation.

Fact 2.22. *Suppose that $1 \leq n < \omega$, $\mathcal{I}$ is a $\subseteq$-directed collection of sets, $\mathcal{J}$ is a $\subseteq^*$-cofinal subset of $\mathcal{J}$, and H is an abelian group. Then the following are equivalent:*

(1) $\lim^n \mathbf{A}_{\mathcal{I}}[H] = 0$*;*
(2) $\lim^n \mathbf{A}_{\mathcal{J}}[H] = 0$.

3. Ascending sequences and nontrivial coherence

In this section, we isolate the notion of an *ascending sequence* of sets and prove that under certain assumptions one can recursively construct nontrivial coherent n-families along them.

For a nonempty set Y, we let $\mathrm{Fn}^+(Y)$ denote the collection of all finite partial functions w from Y to 2 such that $w^{-1}\{1\} \neq \emptyset$.

Definition 3.1. Suppose that $\vec{x} = \langle x_\alpha \mid \alpha < \delta \rangle$ is a sequence of sets, $1 \leq n < \omega$, H is an abelian group, and $\Phi = \langle \varphi_a : \bigcap_{i<n} x_{a(i)} \to H \mid a \in \delta^n \rangle$ is an alternating n-family.

(1) If E is a set, then we say that Φ is *supported on E* if $\mathrm{supp}(\varphi_a) \subseteq E$ for all $a \in \delta^n$.
(2) Suppose that $\vec{e} = \langle e_w \mid w \in \mathrm{Fn}^+(\delta) \rangle$ is a sequence of sets. For all $a \in \delta^n$, let $\beta_a := \max\{a(i) \mid i < n\}$. We say that Φ is *supported on $\vec{e}$* if for all $a \in \delta^n$, we have $\mathrm{supp}(\varphi_a) \subseteq \bigcup\{e_w \mid w \in \mathrm{Fn}^+(\beta_a)\}$.

The following proposition is immediate.

Proposition 3.2. *Suppose that $1 < n < \omega$, H is an abelian group, Φ is an H-valued coherent n-family, and $\Psi^0, \Psi^1 \in \mathrm{Triv}_H(\Phi)$. If E is a set, Ψ^0 and Ψ^1 are both supported on E, and $\Psi^0 \restriction\restriction E \cong_{\Phi\restriction\restriction E, H} \Psi^1 \restriction\restriction E$, then $\Psi^0 \cong_{\Phi, H} \Psi^1$.* □

Definition 3.3. Suppose that δ is a limit ordinal of uncountable cofinality and $\vec{x} = \langle x_\alpha \mid \alpha < \delta \rangle$ is a sequence of sets.

(1) For $w \in \mathrm{Fn}^+(\delta)$, we say that $\vec{x}$ *respects* w if the set
$$d_w^{\vec{x}} := \bigcap_{\beta \in w^{-1}\{1\}} x_\beta \setminus \bigcup_{\alpha \in w^{-1}\{0\}} x_\alpha$$
is infinite.
(2) We say that $\vec{x}$ is *ascending* if there is a sequence $\langle e_w \mid w \in \mathrm{Fn}^+(\delta) \rangle$ and a club $C \subseteq \delta$ satisfying the following requirements:
    (a) for all $w \in \mathrm{Fn}^+(\delta)$, if $\vec{x}$ respects w, then $e_w \in [d_w^{\vec{x}}]^{\aleph_0}$ (otherwise $e_w = \emptyset$);

[3]The contents of [5, Section 2.2] are specifically about the system $\mathbf{A}$, but the arguments therein straightforwardly generalize to $\mathbf{A}_{\mathcal{I}}[H]$ for any $\subseteq$-directed collection $\mathcal{I}$ and any abelian group H.

(b) for all $\beta \in C$ and all $a \in [\beta]^{<\omega}$, we have $x_\beta \not\subseteq^* \bigcup_{\alpha \in a} x_\alpha$;
(c) for all $a \in [C]^{<\omega}$ and all $w \in \mathrm{Fn}^+(\min(a))$, if $\vec{x}$ respects w, then the set

$$e_w \cap \bigcap_{\beta \in a} x_\beta$$

is infinite. In particular, for all $\beta \in C$ and all $w \in \mathrm{Fn}^+(\beta)$, if $\vec{x}$ respects w, then $\vec{x}$ respects $w \cup \{(\beta, 1)\}$.

We now prove that, for $1 \leq n < \omega$, ascending sequences of length ω_n carry nontrivial coherent n-families. We first prove the existence of n-families mapping into arbitrary sufficiently small nonzero abelian groups under an additional weak diamond assumption. We will later eliminate this weak diamond assumption, at the cost of increasing the size of the groups that we are mapping into. Before beginning, let us recall the definition of weak diamond.

Definition 3.4. Let λ be a regular uncountable cardinal and let $S \subseteq \lambda$ be stationary in λ. The principle $\mathrm{w}\diamondsuit(S)$ asserts that for every $F : 2^{<\lambda} \to 2$ there exists $g : \lambda \to 2$ such that for every $b : \lambda \to 2$ the set

$$\{\alpha \in S \mid g(\alpha) = F(b \restriction \alpha)\}$$

is stationary in λ.

We begin with the special case of $n = 1$.

Theorem 3.5. *Suppose that $\vec{x} = \langle x_\alpha \mid \alpha < \omega_1 \rangle$ is an ascending sequence of sets, H is a nonzero abelian group, and $\mathrm{w}\diamondsuit(\omega_1)$ holds. Then there is a nontrivial coherent 1-family $\Phi = \langle \varphi_\alpha : x_\alpha \to H \mid \alpha < \omega_1 \rangle$.*

Moreover, if $\vec{e} = \langle e_w \mid w \in \mathrm{Fn}^+(\omega_1) \rangle$ and C witness that $\vec{x}$ is ascending, then we can arrange so that Φ is supported on $\vec{e}$.

Proof. Fix for the duration of the proof an arbitrary nonzero element H, and denote it by 1_H. We will denote the zero element of H by 0_H. For each $\sigma \in {}^{<\omega_1}2$, we will construct a coherent H-valued n-family $\Phi^\sigma = \langle \varphi^\sigma_\alpha \mid \alpha < \mathrm{lh}(\sigma) \rangle$ in such a way that if $\sigma \sqsubseteq \tau \in {}^{<\omega_1}2$, then $\Phi^\sigma \sqsubseteq \Phi^\tau$. We will then find some $g \in {}^{\omega_1}2$ for which the family $\Phi^g := \bigcup_{\beta < \omega_1} \Phi^{g \restriction \beta}$ is nontrivial.

Let $\vec{e} = \langle e_w \mid w \in \mathrm{Fn}^+(\omega_1) \rangle$ and $C \subseteq \omega_1$ witness that $\vec{a}$ is ascending. Before we begin the construction of $\langle \Phi^\sigma \mid \sigma \in {}^{<\omega_1}2 \rangle$, we isolate, for each $\gamma \in \mathrm{Lim}(C)$, a relevant set $y_\gamma \in [x_\gamma]^{\aleph_0}$.

Fix $\gamma \in \mathrm{Lim}(C)$, as well as a bijection $\pi_\gamma : \gamma \to \omega$. Let $\langle \gamma_n \mid n < \omega \rangle$ be an increasing cofinal sequence from $C \cap \gamma$ such that $\langle \pi_\gamma(\gamma_n) \mid n < \omega \rangle$ is also increasing. For each $n < \omega$, define a function $w_{\gamma,n} \in \mathrm{Fn}^+(\gamma)$ by letting

$$\mathrm{dom}(w_{\gamma,n}) := \{\gamma_n\} \cup \{\alpha < \gamma_n \mid \pi_\gamma(\alpha) < \pi_\gamma(\gamma_n)\}$$

and setting $w_{\gamma,n}(\gamma_n) := 1$ and $w_{\gamma,n}(\alpha) := 0$ for all $\alpha \in \mathrm{dom}(w_n) \cap \gamma_n$. Then let $w^+_{\gamma,n} := w_{\gamma,n} \cup \{(\gamma, 1)\}$. The fact that $\gamma_n < \gamma$ are both in C implies that $\vec{x}$ respects both $w_{\gamma,n}$ and $w^+_{\gamma,n}$ and that $x_\gamma \cap e_{w_{\gamma,n}}$ is infinite. Note also that, for all $m < n < \omega$, we have $e_{w_{\gamma,m}} \cap e_{w_{\gamma,n}} = \emptyset$, since $w_{\gamma,n}(\gamma_m) = 0$. Now, for each $n < \omega$, choose a single element $i_{\gamma,n} \in x_\gamma \cap e_{w_{\gamma,n}}$, and let $y_\gamma := \{i_{\gamma,n} \mid n < \omega\}$. Note that, by construction, the set $y_\gamma \cap x_\alpha$ is finite for all $\alpha < \gamma$, since $w_{\gamma,n}(\alpha) = 0$ for all sufficiently large $n < \omega$.

We now turn to the construction of $\langle \Phi^\sigma \mid \sigma \in {}^{<\omega_1}2\rangle$. We will ensure throughout the construction that each Φ^σ is supported on $\vec{e}$. If $\sigma \in {}^{<\omega_1}2$ and $\mathrm{lh}(\sigma)$ is a limit ordinal, then we are obliged to set $\Phi^\sigma := \bigcup_{\beta<\mathrm{lh}(\sigma)} \Phi^{\sigma\restriction\beta}$. Thus, to complete the construction, it suffices to specify how to produce $\Phi^{\sigma^\frown\langle 0\rangle}$ and $\Phi^{\sigma^\frown\langle 1\rangle}$ from Φ_σ. To this end, fix $\gamma < \omega_1$ and $\sigma \in {}^\gamma 2$, and suppose that we have constructed Φ_σ. For readability, we will write $\Phi^{\sigma,\ell}$ in place of $\Phi^{\sigma^\frown\langle\ell\rangle}$ for $\ell < 2$.

Fix $\ell < 2$. To construct $\Phi^{\sigma,\ell}$, it is enough to define $\varphi^{\sigma,\ell}_\gamma$. To begin, since $\gamma < \omega_1$, we can use Proposition 2.7 to find a function $\psi : x_\gamma \to H$ such that $\psi =^* \varphi^\sigma_\alpha$ for all $\alpha < \gamma$. Moreover, since Φ^σ is supported on $\vec{e}$, we can assume that the support of ψ is a subset of $\bigcup\{e_w \mid w \in \mathrm{Fn}^+(\gamma)\}$. If $\gamma \notin \mathrm{Lim}(C)$, then simply let $\varphi^{\sigma,\ell}_\gamma = \psi$. If $\gamma \in \mathrm{Lim}(C)$, then define $\varphi^{\sigma,\ell}_\gamma$ by setting

$$\varphi^{\sigma,\ell}_\gamma(x) = \begin{cases} \ell_H & \text{if } x \in y_\gamma; \\ \psi(x) & \text{if } x \in x_\gamma \setminus y_\gamma. \end{cases}$$

Note that we still have $\varphi^{\sigma,\ell}_\gamma =^* \varphi^\sigma_\alpha$ for all $\alpha < \gamma$, due to the fact that $y_\gamma \cap x_\alpha$ is finite for all such α. We have also satisfied the requirement that each $\Phi^{\sigma,\ell}$ is supported on $\vec{e}$. This completes the construction.

We now prepare for an application of $\mathrm{w}\diamondsuit(\omega_1)$. Let $E := \bigcup\{e_w \mid w \in \mathrm{Fn}^+(\omega_1)\}$, and note that $|E| \leq \aleph_1$ and $y_\gamma \subseteq E$ for all $\gamma \in \mathrm{Lim}(C)$. Let $\langle j_\alpha \mid \alpha < |E|\rangle$ be an injective enumeration of E. For $\gamma < \omega_1$, let $E_\gamma := \{j_\alpha \mid \alpha < \gamma\}$. Let D be the set of $\gamma \in \lim(C)$ such that, for all $w \in \mathrm{Fn}^+(\gamma)$, we have $e_w \subseteq E_\gamma$, and note that D is club in ω_1. Additionally note that, by construction, for each $\gamma \in D$ we have $y_\gamma \subseteq E_\gamma$, and hence every $\tau \in {}^\gamma 2$ induces a function $h_\tau : y_\gamma \to 2$ by setting $h_\tau(j_\alpha) = \tau(\alpha)$ for all $\alpha < \gamma$ such that $j_\alpha \in y_\gamma$.

Now define a function $F : {}^{<\omega_1}2 \to 2$ by setting $F(\tau) := 1$ for all τ such that $\gamma := \mathrm{lh}(\tau) \in D$ and $h_\tau =^* 0$, and setting $F(\tau) := 0$ for all other $\tau \in {}^{<\omega_1}2$. Let $g \in {}^{\omega_1}2$ witness the instance of $\mathrm{w}\diamondsuit(\omega_1)$ associated with F. We claim that $\Phi^g = \bigcup_{\beta<\omega_1} \Phi^{g\restriction\beta} = \langle \varphi_\alpha \mid \alpha < \omega_1\rangle$ is nontrivial. Suppose for the sake of contradiction that ψ trivializes Φ^g. Define a function $f \in {}^{\omega_1}2$ by setting, for all $\alpha < \omega_1$,

$$f(\alpha) = \begin{cases} 0 & \text{if } \psi(j_\alpha) = 0_H \\ 1 & \text{otherwise.} \end{cases}$$

By our choice of g, we can find $\gamma \in D$ such that $g(\gamma) = F(f \restriction \gamma)$.

Let $\ell := g(\gamma)$ and $\tau := f \restriction \gamma$. By construction of Φ^g, we know that $\varphi_\gamma \restriction y_\gamma =^* \ell_H$. By the assumption that ψ trivializes Φ^g, it follows that $\psi \restriction y_\gamma =^* \ell_H$. Our definition of h_τ then implies that $h_\tau =^* \ell$, but then our definition of F implies that $F(\tau) = 1 - \ell$, contradicting the fact that $g(\gamma) = F(f \restriction \gamma)$ and completing the proof. □

We now turn to higher dimensions.

Theorem 3.6. *Suppose that $1 \leq n < \omega$, $\vec{x} = \langle x_\alpha \mid \alpha < \omega_n\rangle$ is an ascending sequence of sets, H is a nonzero abelian group with $|H| \leq 2^{\omega_1}$, and $\mathrm{w}\diamondsuit(S^{k+1}_k)$ holds for all $k < n$. Then there is a nontrivial coherent n-family*

$$\Phi = \left\langle \varphi_b : \bigcap_{i<n} x_{b(i)} \to H \;\middle|\; b \in (\omega_n)^n \right\rangle.$$

Moreover, if $\vec{e} = \langle e_w \mid w \in \mathrm{Fn}^+(\omega_n)\rangle$ and $C \subseteq \omega_n$ witness that $\vec{x}$ is ascending, then we can arrange so that Φ is supported on $\vec{e}$.

Proof. The proof is by induction on n. The case of $n = 1$ is precisely Theorem 3.5, so assume that $n > 1$ and that we have established the theorem for all $1 \leq m < n$.

Let $\langle e_w \mid w \in \mathrm{Fn}^+(\omega_n)\rangle$ and $C \subseteq \omega_n$ witness that $\vec{x}$ is ascending. Let θ be a sufficiently large regular cardinal, and let $\lhd$ be a fixed well-ordering of $H(\theta)$. For $\gamma < \omega_n$, let $E_\gamma := \bigcup\{e_w \mid w \in \mathrm{Fn}^+(\gamma)\}$, and let $\bar{E}_\gamma := E_\gamma \cap x_\gamma$.

Let $\mathcal{S} := \{\sigma \in {}^{<\omega_n}2 \mid \mathrm{supp}(\sigma) \subseteq \mathrm{Lim}(C) \cap S^n_{n-1}\}$, and let $\mathcal{S}^+ := \{g \in {}^{\omega_n}2 \mid \mathrm{supp}(g) \subseteq \mathrm{Lim}(C) \cap S^n_{n-1}\}$. Similarly to the proof of Theorem 3.5, for each $\sigma \in \mathcal{S}$, we will construct a coherent H-valued n-family $\Phi^\sigma = \langle \varphi^\sigma_b \mid b \in \mathrm{lh}(\sigma)^n\rangle$ in such a way that if $\sigma \sqsubseteq \tau \in \mathcal{S}$, then $\Phi^\sigma \sqsubseteq \Phi^\tau$. We will then find some $g \in \mathcal{S}^+$ such that $\Phi^g := \bigcup_{\alpha<\omega_n} \Phi^{g\restriction\alpha}$ is nontrivial.

We now describe the construction of $\langle \Phi^\sigma \mid \sigma \in \mathcal{S}\rangle$, which will be done by recursion on $\mathrm{lh}(\sigma)$. We will ensure along the way that, for all $\gamma < \omega_n$ and all $\sigma \in \mathcal{S} \cap {}^\gamma 2$, the family Φ^σ is supported on $\langle e_w \mid w \in \mathrm{Fn}^+(\gamma)\rangle$. As in the proof of Theorem 3.5, the case in which $\mathrm{lh}(\sigma)$ is a limit ordinal is trivial, so fix a $\sigma \in \mathcal{S}$ and assume that Φ^σ has been constructed. We describe how to construct $\Phi^{\sigma^\frown\ell}$ for $\ell < 2$ if $\mathrm{lh}(\sigma) \in \mathrm{Lim}(C) \cap S^n_{n-1}$ and for $\ell = 0$ otherwise.

Let $\gamma := \mathrm{lh}(\sigma)$. Since $\gamma < \omega_n$, Proposition 2.7 implies that $\Phi^\sigma \restriction\restriction x_\gamma$ is trivial. Let $\Psi^\sigma = \langle \psi^\sigma_a \mid a \in \gamma^{n-1}\rangle$ be an element of $\mathrm{Triv}_H(\Phi^\sigma \restriction\restriction x_\gamma)$. Since $\Phi^\sigma \restriction\restriction x_\gamma$ is supported on $\bar{E}_\gamma$, we can assume that Ψ^σ is also supported on $\bar{E}_\gamma$. If $\gamma \notin \mathrm{Lim}(C) \cap S^n_{n-1}$, then simply let $\Phi^{\sigma^\frown 0} = \Phi^\sigma {}^\frown \langle \Psi^\sigma\rangle$. It is easily verified that this maintains the requirements of the construction; to see that it maintains coherence, recall Remark 2.17.

If $\gamma \in \mathrm{Lim}(C) \cap S^n_{n-1}$, then we work a little bit harder. Let $D_\gamma \subseteq C \cap \gamma$ be a club in γ with $\mathrm{otp}(D_\gamma) = \omega_{n-1}$, and let $\pi_\gamma : \gamma \to \omega_{n-1}$ be a bijection such that $\pi \restriction D_\gamma$ is order-preserving and continuous. In particular, $\bar{D}_\gamma := \pi[D_\gamma]$ is a club in ω_{n-1}. Note that $\{\eta \in \bar{D}_\gamma \mid \pi_\gamma^{-1}[\eta] \subseteq \pi^{-1}(\eta)\}$ is a club in ω_{n-1}. By thinning out D_γ (and hence also $\bar{D}_\gamma$), we can assume that $\pi_\gamma^{-1}[\eta] \subseteq \pi_\gamma^{-1}(\eta)$ for all $\eta \in \bar{D}_\gamma$.

For all $\eta < \omega_{n-1}$, let $y^\gamma_\eta := x_\gamma \cap x_{\pi_\gamma^{-1}(\eta)}$. Given $w \in \mathrm{Fn}^+(\omega_{n-1})$, let $\hat{w}^\gamma \in \mathrm{Fn}^+(\gamma)$ be defined by letting $\mathrm{dom}(\hat{w}^\gamma) := \pi_\gamma^{-1}[\mathrm{dom}(w)]$ and, for all $\alpha \in \mathrm{dom}(\hat{w}^\gamma)$, setting $\hat{w}^\gamma(\alpha) = w(\pi_\gamma(\alpha))$. For such w, let $\bar{e}^\gamma_w := x_\gamma \cap e_{\hat{w}^\gamma}$.

Claim 3.7. *$\vec{y} = \langle y^\gamma_\eta \mid \eta < \omega_{n-1}\rangle$ is an ascending sequence of sets, as witnessed by $\langle \bar{e}^\gamma_w \mid w \in \mathrm{Fn}^+(\omega_{n-1})\rangle$ and $\bar{D}_\gamma$.*

Proof. We must verify conditions (2)(a–c) of Definition 3.3. To see (a), fix $w \in \mathrm{Fn}^+(\omega_{n-1})$, and note that

$$d^{\vec{y}}_w = d^{\vec{x}}_{\hat{w}^\gamma} \cap x_\gamma = d^{\vec{x}}_{\hat{w}^\gamma \cup \{(\gamma,1)\}}.$$

By the fact that $\gamma \in C$, we know that

$$(\vec{y} \text{ respects } w) \Leftrightarrow (\vec{x} \text{ respects } \hat{w}^\gamma \cup \{(\gamma,1)\}) \Leftrightarrow (\vec{x} \text{ respects } \hat{w}^\gamma).$$

Thus, if $\vec{y}$ respects w, the fact that $\vec{x}$ is ascending implies that

$$\bar{e}^\gamma_w = x_\gamma \cap e_{\hat{w}^\gamma} \in [d^{\vec{y}}_w]^{\aleph_0},$$

so (a) is satisfied.

For item (b), fix $\eta \in \bar{D}_\gamma$ and $a \in [\eta]^{<\omega}$. Let $\hat{a}^\gamma := \pi_\gamma^{-1}[a]$ and $\beta = \pi_\gamma^{-1}(\eta)$. Since $\eta \in \bar{D}_\gamma$, we have $\hat{a}^\gamma \subseteq \beta$ and $\beta \in C$, and hence $x_\beta \not\subseteq^* \bigcup_{\alpha \in \hat{a}^\gamma} x_\alpha$. In particular, if we define $\hat{w} \in \mathrm{Fn}^+(\gamma)$ by letting $\mathrm{dom}(\hat{w}) = \hat{a}^\gamma \cup \{\beta\}$, $\hat{w}(\beta) = 1$, and $\hat{w} \restriction \hat{a}^\gamma = 0$, then $\vec{x}$ respects $\hat{w}$. Since $\gamma \in C$, $\vec{x}$ respects $\hat{w} \cup \{(\gamma, 1)\}$. But

$$d^{\vec{x}}_{\hat{w}\cup\{(\gamma,1)\}} = (x_\gamma \cap x_\beta) \setminus \bigcup_{\alpha \in \hat{a}^\gamma} x_\alpha = y_\eta \setminus \bigcup_{\xi \in a} y_\xi.$$

Since this set is infinite, we have $y_\eta \not\subseteq^* \bigcup_{\xi \in a} y_\xi$, as desired.

For (c), fix $a \in [\bar{D}_\gamma]^{<\omega}$ and $w \in \mathrm{Fn}^+(\min(a))$ such that $\vec{y}$ respects w. Let $\hat{a}^\gamma := \pi_\gamma^{-1}[a]$. Note that $\hat{w}^\gamma \in \mathrm{Fn}^+(\min(\hat{a}^\gamma))$, $\hat{a}^\gamma \in [C]^{<\omega}$, and $\vec{x}$ respects $\hat{w}^\gamma$. Therefore, since $\gamma \in C$, we know that $e_{\hat{w}^\gamma} \cap x_\gamma \cap \bigcap_{\alpha \in \hat{a}^\gamma} x_\alpha$ is infinite. But $e_{\hat{w}^\gamma} \cap x_\gamma \cap \bigcap_{\alpha \in \hat{a}^\gamma} x_\alpha = \bar{e}^\gamma_w \cap \bigcap_{\eta \in a} y_\eta$, so this instance of (c) is satisfied. □

By the inductive hypothesis, we can find a nontrivial coherent $(n-1)$-family

$$\bar{T}^\gamma = \left\langle \bar{\tau}^\gamma_a : \bigcap_{i<n-1} y_{a(i)} \to H \;\middle|\; a \in (\omega_{n-1})^{n-1} \right\rangle$$

that is supported on $\bar{E}_\gamma$. Through re-indexing via π_γ, this yields a nontrivial coherent $(n-1)$-family

$$T^\gamma = \left\langle \tau^\gamma_a : x_\gamma \cap \bigcap_{i<n-1} x_{a(i)} \to H \;\middle|\; a \in \gamma^{n-1} \right\rangle$$

that is also supported on $\bar{E}_\gamma$. Then Proposition 2.15 implies that $(\Psi^\sigma + T^\gamma) \in \mathrm{Triv}_H(\Phi^\sigma \restriction\restriction x_\gamma)$ and $\Psi^\sigma \not\approx_{\Phi^\sigma \restriction\restriction x_\gamma, H} (\Psi^\sigma + T^\gamma)$. Finally, set $\Phi^{\sigma,0} := \Phi^\sigma {}^\frown \langle \Psi^\sigma \rangle$, and set $\Phi^{\sigma,1} := \Phi^\sigma {}^\frown \langle \Psi^\sigma + T^\gamma \rangle$. It is routine to verify that these definitions maintain coherence and that the resulting families $\Phi^{\sigma,\ell}$ are supported on $\langle e_w \mid w \in \mathrm{Fn}^+(\gamma + 1) \rangle$. This completes the construction of $\langle \Phi^\sigma \mid \sigma \in \mathcal{S} \rangle$.

Let $E = E_{\omega_n} := \bigcup \{e_w \mid w \in \mathrm{Fn}^+(\omega_n)\}$. Note that the sequence $\langle E_\gamma \mid \gamma \leq \omega_n \rangle$ is $\subseteq$-increasing and continuous, and that $|E_\gamma| < \omega_n$ for all $\gamma < \omega_n$.

For all $\gamma \leq \omega_n$, let $\mathcal{P}^\gamma$ denote the set of all alternating $(n-1)$-families of the form

$$\Psi = \left\langle \psi_a : E_\gamma \cap \bigcap_{i<n-1} x_{a(i)} \to H \;\middle|\; a \in \gamma^{n-1} \right\rangle.$$

If $\gamma < \delta \leq \omega_n$, $\Psi \in \mathcal{P}^\gamma$, and $\Psi' \in \mathcal{P}^\delta$, then we say that Ψ' *extends* Ψ, written $\Psi' \trianglerighteq \Psi$, if, for all $a \in \gamma^{n-1}$, we have $\psi_a = \psi'_a \restriction (E_\gamma \cap \bigcap_{i<n-1} x_{a(i)})$.

Claim 3.8. *For all $\gamma < \omega_n$ and all $\Psi \in \mathcal{P}^\gamma$, there is at most one $\sigma \in \mathcal{S} \cap {}^\gamma 2$ such that $\Psi \in \mathrm{Triv}_H(\Phi^\sigma \restriction\restriction E_\gamma)$.*

Proof. Suppose for the sake of contradiction that there are $\gamma < \omega_n$, $\Psi \in \mathcal{P}_\gamma$, and distinct $\sigma_0, \sigma_1 \in \mathcal{S} \cap {}^\gamma 2$ such that $\Psi \in \mathrm{Triv}_H(\Phi^{\sigma_\ell} \restriction\restriction E_\gamma)$ for $\ell < 2$. Let $\beta < \gamma$ be the least ordinal at which σ_0 and σ_1 disagree; without loss of generality, assume that $\sigma_\ell(\beta) = \ell$ for $\ell < 2$. By the definition of $\mathcal{S}$, we must have $\beta \in \mathrm{Lim}(C) \cap S^n_{n-1}$. Let $\tau = \sigma_0 \restriction \beta = \sigma_1 \restriction \beta$. In the construction, we fixed a $\Psi^\tau \in \mathrm{Triv}_H(\Phi^\tau \restriction\restriction x_\beta)$ and a nontrivial coherent $(n-1)$-family T^β, both supported on $\bar{E}_\beta$, such that $\Phi^{\tau,0} = \Phi^\tau {}^\frown \langle \Psi^\tau \rangle$ and $\Phi^{\tau,1} = \Phi^\tau {}^\frown \langle \Psi^\tau + T^\beta \rangle$. Then our assumptions imply that, for all $a \in \beta^{n-1}$, we have

$$\begin{aligned}(-1)^{n+1}(\psi^\tau_a) \restriction E_\gamma &= \varphi^{\sigma_0}_{a^\frown\langle\beta\rangle} \restriction E_\gamma \\ &=^* (-1)^{n-1}\psi_a \restriction E_\gamma + \sum_{i<n-1}(-1)^i \psi_{a^i\frown\langle\beta\rangle} \restriction E_\gamma \quad =^* \varphi^{\sigma_1}_{a^\frown\langle\beta\rangle} \restriction E_\gamma \\ &= (-1)^{n+1}(\psi^\tau_a + \tau^\beta_a) \restriction E_\gamma.\end{aligned}$$

In particular, from the first and last terms of the above equation we see that $\tau^\beta_a \restriction E_\gamma =^* 0$, contradicting the fact that T^β is a nontrivial coherent $(n-1)$-family supported on $\bar{E}_\beta \subseteq E_\gamma$. □

Fix a coding function G with domain ${}^{\leq\omega_n}2$ and a club $D \subseteq \mathrm{Lim}(C)$ with the following properties:

- for all $\gamma \in D \cup \{\omega_n\}$, $G \restriction {}^\gamma 2$ is a surjection from ${}^\gamma 2$ to $\mathcal{P}^\gamma$ (this is where we use the assumption that $|H| \leq 2^{\omega_1}$);
- for all $\gamma < \delta$, both in $D \cup \{\omega_n\}$, and all $\sigma \in {}^\gamma 2$ and $\sigma' \in {}^\delta 2$, if $\sigma' \sqsupseteq \sigma$, then $G(\sigma') \trianglerighteq G(\sigma)$.

As long as all intervals between successive elements of D have size ω_{n-1}, it is routine to construct such a function G, using the fact that $|\mathcal{P}_\gamma| \leq 2^{\omega_{n-1}}$ for each $\gamma < \omega_n$.

Now define a function $F : {}^{<\omega_n}2 \to 2$ as follows. If $\gamma \notin D \cap S^n_{n-1}$, simply let $F \restriction {}^\gamma 2 = 0$. Suppose now that $\gamma \in D \cap S^n_{n-1}$ and $\rho \in {}^\gamma 2$. If there does not exist $\sigma \in \mathcal{S} \cap {}^\gamma 2$ such that $G(\rho) \in \mathrm{Triv}_H(\Phi^\sigma \restriction\restriction E_\gamma)$, then let $F(\rho) = 0$. If there is $\sigma \in \mathcal{S} \cap {}^\gamma 2$ such that $G(\rho) \in \mathrm{Triv}_H(\Phi^\sigma \restriction\restriction E_\gamma)$, then, by Claim 3.8, there is a unique such σ; denote this by $\sigma(\rho)$. By Proposition 2.18 and the construction of $\Phi^{\sigma(\rho),\ell}$, there is at most one $\ell < 2$ such that $G(\rho)$ extends to an element of $\mathrm{Triv}_H(\Phi^{\sigma(\rho),\ell})$. More precisely, if $G(\rho) = \Psi$, then there is at most one $\ell < 2$ for which there exists a $\Psi' \in \mathrm{Triv}_H(\Phi^{\sigma(\rho),\ell})$ such that, for all $a \in \gamma^{n-1}$, we have $\psi'_a \restriction E_\gamma = \psi_a$. Then choose $F(\rho) \in 2$ so that $G(\rho)$ does *not* extend to an element of $\mathrm{Triv}_H(\Phi^{\sigma(\rho),\ell})$.

Let $g \in {}^{\omega_n}2$ witness the instance of $\mathrm{w}\diamondsuit(S^n_{n-1})$ associated with F. Since this only depends on the values g takes on S^n_{n-1}, or any relatively club subset thereof, we can assume that $g \in \mathcal{S}$. We claim that $\Phi^g = \langle \varphi_u \mid u \in (\omega_n)^n \rangle$ is nontrivial. Suppose for the sake of contradiction that $\Psi = \langle \psi_a \mid a \in (\omega_n)^{n-1} \rangle$ trivializes Φ^g. By the construction of G, we can find a function $f \in {}^{\omega_n}2$ such that, for all $\gamma \in D \cup \{\omega_n\}$, we have

$$G(f \restriction \gamma) = \left\langle \psi_a \restriction E_\gamma \cap \bigcap_{i<n-1} a_{v(i)} \;\middle|\; a \in \gamma^{n-1} \right\rangle.$$

Moreover, for all $\gamma \in D$, it follows that $G(f \restriction \gamma)$ trivializes $\Phi^{g\restriction\gamma} \restriction\restriction E_\gamma$.

By our choice of g, we can find $\gamma \in D \cap S^n_{n-1}$ such that $g(\gamma) = F(f \restriction \gamma)$. By our construction of F, we then know that $\sigma(f \restriction \gamma) = g \restriction \gamma$ and that $G(f \restriction \gamma)$ does not extend to an element of $\mathrm{Triv}_H(\Phi^{g\restriction(\gamma+1)})$. Therefore, *a fortiori*, $G(f \restriction \gamma)$ does not extend to an element of $\mathrm{Triv}_H(\Phi^g)$. But this is contradicted by the fact that Ψ extends $G(f \restriction \gamma)$ and is an element of $\mathrm{Triv}_H(\Phi^g)$. This contradiction shows that Φ^g is indeed nontrivial, thus completing the proof of the theorem. □

We can get rid of the weak diamond assumptions in the previous theorem at the cost of increasing the size of the group into which we are mapping. For a group H and an ordinal β, we write $H^{(\beta)}$ to denote the direct sum of β-many copies of H; concretely, this is the group consisting of all finitely-supported functions $f : \beta \to H$. If $\alpha < \beta$, then we consider $H^{(\alpha)}$ as a subgroup of $H^{(\beta)}$ in the obvious way. If we have fixed a particular nonzero element $1_H \in H$, then, for $\eta < \beta$, we let 1_η denote the element of $H^{(\beta)}$ whose support is precisely $\{\eta\}$ and takes value 1_H at η.

Theorem 3.9. *Suppose that $1 \leq n < \omega$, H is a nonzero abelian group, and $\vec{x} = \langle x_\alpha \mid \alpha < \omega_n \rangle$ is an ascending sequence of sets. Then there is a nontrivial coherent n-family*

$$\Phi = \left\langle \varphi_b : \bigcap_{i<n} x_{b(i)} \to H^{(\omega_n)} \;\middle|\; b \in (\omega_n)^n \right\rangle.$$

Moreover, if $\vec{e} = \langle e_w \mid w \in \mathrm{Fn}^+(\omega_n) \rangle$ and $C \subseteq \omega_n$ witness that $\vec{x}$ is ascending, then we can arrange so that Φ is supported on $\vec{e}$.

Proof. Fix for the remainder of the proof an arbitrary nonzero element of H, and denote it by 1_H. The proof is by induction on n. Suppose first that $n = 1$, and fix $\vec{e}$ and C witnessing that $\vec{x}$ is ascending. We will construct a nontrivial coherent 1-family $\Phi = \langle \varphi_\beta : x_\beta \to H^{(\omega_1)} \mid \beta < \omega_1 \rangle$ that is supported on $\vec{e}$ by recursion on β, maintaining the recursive hypothesis that, for all $\beta < \omega_1$, φ_β maps into $H^{(\beta+1)}$ (i.e., for all $z \in x_\beta$, the support of $\varphi_\beta(z)$ is a subset of $\beta + 1$).

Suppose that $\beta < \omega_1$ and we have constructed $\Phi^\beta = \langle \varphi_\alpha \mid \alpha < \beta \rangle$. As before, let $E_\beta = \bigcup \{e_w \mid w \in \mathrm{Fn}^+(\beta)\}$ and $\bar{E}_\beta = E_\beta \cap x_\beta$. Since $\beta < \omega_1$, Proposition 2.7 implies that $\Phi_\beta \restriction\restriction x_\beta$ is trivial. Fix a trivialization $\psi^\beta : x_\beta \to H^{(\omega_1)}$. By our recursive assumption, $\Phi_\beta \restriction\restriction x_\beta$ is supported on $\bar{E}_\beta$ and, for each $\alpha < \beta$, φ_α maps into $H^{(\beta)}$. Therefore, we can assume that ψ^β is also supported on $\bar{E}_\beta$ and maps into $2^{(\beta)}$. If $\beta \notin \mathrm{Lim}(C)$, then simply let $\varphi_\beta = \psi^\beta$; this readily maintains the recursive hypotheses.

Suppose now that $\beta \in \mathrm{Lim}(C)$. Precisely as in the proof of Theorem 3.5, fix a set $y_\beta \in [\bar{E}_\beta]^\omega$ such that, for all $\alpha < \beta$, $y_\beta \cap x_\alpha$ is finite. Now define φ_β by setting, for all $z \in x_\beta$,

$$\varphi_\beta(z) = \begin{cases} \psi^\beta(z) & \text{if } z \notin y_\beta \\ 1_\beta & \text{if } z \in y_\beta. \end{cases}$$

Since $y_\beta \cap x_\alpha$ is finite for all $\alpha < \beta$, this maintains the coherence of Φ. It is easily verified that all other recursive requirements are satisfied by this definition. This completes the construction of Φ.

We claim that Φ is nontrivial. Suppose for the sake of contradiction that $\psi : \bigcup \{x_\beta \mid \beta < \omega_1\} \to H^{(\omega_1)}$ trivializes Φ. Note that $\langle E_\beta \mid \beta < \omega_1 \rangle$ is a continuous $\subseteq$-increasing sequence of countable sets. Therefore, since, for each $z \in \mathrm{dom}(\psi)$, $\mathrm{supp}(\psi(z))$ is a finite subset of ω_1, we can find $\beta \in \mathrm{Lim}(C)$ such that $\psi \restriction E_\beta$ maps into $H^{(\beta)}$. However, when we constructed φ_β, we found an infinite subset $y_\beta \subseteq \bar{E}_\beta$ and ensured that $\beta \in \mathrm{supp}(\varphi_\beta(z))$ for all $z \in y_\beta$. This implies that $\psi \restriction x_\beta \neq^* \varphi_\beta$, contradicting the assumption that ψ trivializes Φ.

Suppose now that $1 < n < \omega$ and we have established the theorem for all $1 \leq m < n$. Again fix $\vec{e}$ and C witnessing that $\vec{x}$ is ascending. For notational convenience, we will index our construction of a nontrivial coherent n-family by $[\omega_n]^n$, whose elements we may think of as strictly increasing sequences of length n, rather than by the full $(\omega_n)^n$. Recall that this involves no loss of generality by Remark 2.2. We will construct a nontrivial coherent n-family $\Phi = \langle \varphi_b : \bigcap_{i<n} x_{b(i)} \to H^{(\omega_n)} \mid b \in [\omega_n]^n \rangle$ by recursion on $\max(b)$. We will ensure that Φ is supported on $\vec{e}$ and will maintain the recursive hypothesis that, for all $b \in [\omega_n]^n$, φ_b maps into $H^{(\max(b)+\omega_{n-1})}$.

Fix $\gamma < \omega_n$ and suppose that we have specified $\Phi^\gamma = \langle \varphi_b \mid b \in [\gamma]^n \rangle$. We describe how to specify $\varphi_{a \cup \{\gamma\}}$ for $a \in [\gamma]^{n-1}$. As before, let $E_\gamma := \bigcup \{e_w \mid w \in \mathrm{Fn}^+(\gamma)\}$, and let $\bar{E}_\gamma := E_\gamma \cap x_\gamma$. Since $\mathrm{cf}(\gamma) < \omega_n$, Proposition 2.7 implies that $\Phi^\gamma \restriction\restriction x_\gamma$ is trivial. Fix $\Psi^\gamma = \langle \psi^\gamma_a : x_\gamma \cap \bigcap_{i<n-1} x_{a(i)} \to H^{(\omega_{n+1})} \mid a \in [\gamma]^{n-1} \rangle$ in $\mathrm{Triv}_{H^{(\omega_n)}}(\Phi^\gamma \restriction\restriction x_\gamma)$. Since $\Phi^\gamma \restriction\restriction x_\gamma$ is supported on $\bar{E}_\gamma$ and, for each $b \in [\gamma]^n$, φ_b maps into $H^{(\gamma+\omega_{n-1})}$, we can assume that Ψ^γ is also supported on $\bar{E}_\gamma$ and, for each $a \in [\gamma]^{n-1}$, ψ^γ_a maps into $H^{(\gamma+\omega_{n-1})}$. Moreover, if $\mathrm{cf}(\gamma) = \omega_{n-1}$, then we in fact know that, for each $b \in [\gamma]^n$, φ_b maps into $H^{(\gamma)}$, so we can require that ψ^γ_a does for each $a \in [\gamma]^{n-1}$ as well.

If $\gamma \notin \mathrm{Lim}(C) \cap S^n_{n-1}$, then simply let $\varphi_{a\cup\{\gamma\}} = (-1)^{n+1}\psi^\gamma_a$ for all $a \in [\gamma]^{n-1}$. Suppose now that $\gamma \in \mathrm{Lim}(C) \cap S^n_{n-1}$. Recall in this case that ψ^γ_a maps into $2^{(\gamma)}$ for each $a \in [\gamma]^{n-1}$. First, exactly as in the proof of Theorem 3.6, the inductive hypothesis implies that we can find a nontrivial coherent $(n-1)$-family

$$T^\gamma = \left\langle \tau^\gamma_a : x_\gamma \cap \bigcap_{i<n-1} x_{a(i)} \to H^{(\omega_{n-1})} \;\middle|\; a \in [\gamma]^{n-1} \right\rangle$$

that is supported on $\bar{E}_\gamma$. The idea now is to add a "shifted" version of T^γ to Ψ^γ. More precisely, for each $a \in [\gamma]^{n-1}$, define a function

$$\mathrm{sh}_\gamma(\tau^\gamma_a) : x_\gamma \cap \bigcap_{i<n-1} x_{a(i)} \to H^{(\gamma+\omega_{n-1}))}$$

by setting, for all $z \in x_\gamma \cap \bigcap_{i<n-1} x_{a(i)}$ and all $\eta < \gamma + \omega_{n-1}$,

$$\mathrm{sh}_\gamma(\tau^\gamma_a)(z)(\eta) = \begin{cases} 0 & \text{if } \eta < \gamma \\ \tau^\gamma_a(z)(\xi) & \text{if } \eta = \gamma + \xi. \end{cases}$$

Then let $\varphi_{a\cup\{\gamma\}} = (-1)^{n+1}(\psi^\gamma_a + \mathrm{sh}_\gamma(\tau^\gamma_a)))$ for all $a \in [\gamma]^{n-1}$. It is easily verified, using the coherence of T^γ and the fact that Ψ^γ trivializes $\Phi^\gamma \restriction\restriction x_\gamma$, that this maintains coherence and all of the other requirements of the recursive construction.

We claim that the family Φ thus constructed is nontrivial. For the sake of contradiction, suppose that it is trivial, and fix a $\Psi = \langle \psi_a \mid a \in [\omega_n]^{n-1} \rangle$ in $\mathrm{Triv}_{H^{(\omega_n)}}(\Phi)$. Find $\gamma \in \mathrm{Lim}(C) \cap S^n_{n-1}$ such that, for all $a \in [\gamma]^n$, ψ_a maps into $H^{(\gamma)}$.

The argument now differs slightly depending on whether $n = 2$ or $n > 2$. Suppose first that $n = 2$. Define a function $\rho : x_\gamma \to H^{(\omega_{n-1})}$ by setting, for all $z \in x_\gamma$ and all $\xi < \omega_{n-1}$, $\rho(z)(\xi) = -\psi_\gamma(z)(\gamma + \xi)$. By our choice of γ and the construction of Φ, we know that, for all $\alpha < \gamma$, we have

- $\psi_\gamma - \psi_\alpha =^* \varphi_{\alpha\gamma}$;
- ψ_α maps into $H^{(\gamma)}$;
- for all $z \in x_\alpha \cap x_\gamma$ and all $\xi < \omega_{n-1}$, $\varphi_{\alpha\gamma}(z)(\gamma + \xi) = -\tau^\gamma_\alpha(z)(\xi)$.

Putting this all together, we see that $\rho \restriction (x_\alpha \cap x_\gamma) =^* \tau^\gamma_\alpha$ for all $\alpha < \gamma$, and hence ρ witnesses that T^γ is trivial, contradicting our choice of T^γ.

Suppose next that $n > 2$. For each $d \in [\gamma]^{n-2}$, define a function $\rho_d : x_\gamma \cap \bigcap_{i<n-2} x_{d(i)} \to H^{(\omega_{n-1})}$ by setting, for all $z \in x_\gamma \cap \bigcap_{i<n-2} x_{d(i)}$ and all $\xi < \omega_{n-1}$, $\rho_d(z)(\xi) = (-1)^{n+1}\psi_{d\cup\{\gamma\}}(z)(\gamma + \xi)$. By our choice of Φ, we know that, for all $a \in [\gamma]^{n-1}$, we have

- $(-1)^{n-1}\psi_a + \sum_{i<n-1}(-1)^i \psi_{a^i \cup \{\gamma\}} =^* \varphi_{a\cup\{\gamma\}}$;

- ψ_a maps into $H^{(\gamma)}$;
- for all $z \in x_\gamma \cap \bigcap_{i<n-1} x_{a(i)}$ and all $\xi < \omega_{n-1}$,
$$\varphi_{a\cup\{\gamma\}}(z)(\gamma+\xi) = (-1)^{n+1}\tau^\gamma_a(z)(\xi).$$

Putting this together, we see that, for all $a \in [\gamma]^{n-1}$, we have
$$\sum_{i<n-1}(-1)^i \rho_{a^i} =^* \tau^\gamma_a,$$
and hence $\langle \rho_d \mid d \in [\gamma]^{n-2}\rangle$ witnesses that T^γ is trivial, contradicting our choice of T^γ and completing the proof of the theorem. □

4. Ideals and their characteristics

In this section, we isolate some properties of ideals $\mathcal{I}$ that ensure that they carry $\subseteq^*$-cofinal ascending sequences; this will then yield a proof of Theorem A.

The following cardinal characteristics associated with ideals were introduced in [11] in the context of tall ideals on ω. We reproduce the definition here in a more general context.

Definition 4.1. Suppose that $\mathcal{I}$ is an ideal on a set X properly extending the ideal of all finite subsets of X with no $\subseteq^*$-maximal set.

- $\mathrm{cof}^*(\mathcal{I}) = \min\{|\mathcal{Y}| \mid \mathcal{Y} \subseteq \mathcal{I} \wedge (\forall x \in \mathcal{I})(\exists y \in \mathcal{Y})(x \subseteq^* y)\}$.
- $\mathrm{non}^*(\mathcal{I}) = \min\{|\mathcal{E}| \mid \mathcal{E} \subseteq \mathcal{I} \cap [X]^\omega \wedge (\forall x \in \mathcal{I})(\exists e \in \mathcal{E})(|e \cap x| < \aleph_0)\}$.[4]

The next lemma demonstrates the relevance of these characteristics to our results by showing that, if $\mathrm{cof}^*(\mathcal{I}) = \mathrm{non}^*(\mathcal{I})$, then $\mathcal{I}$ carries an ascending $\subseteq^*$-cofinal sequence.

Lemma 4.2. *Suppose that $\mathcal{I}$ is an ideal on a set X properly extending the ideal of all finite subsets of X with no $\subseteq^*$-maximal set, and suppose that $\mathrm{cof}^*(\mathcal{I}) = \mathrm{non}^*(\mathcal{I}) = \kappa$. Then there is an ascending sequence $\vec{x} = \langle x_\alpha \mid \alpha < \kappa\rangle$ of elements of $\mathcal{I}$ that is cofinal in $(\mathcal{I}, \subseteq^*)$.*

Proof. Let $\langle z_\alpha \mid \alpha < \kappa\rangle$ enumerate a $\subseteq^*$-cofinal subset of $\mathcal{I}$. We will simultaneously construct $\vec{x} = \langle x_\alpha \mid \alpha < \kappa\rangle$ and $\vec{e} = \langle e_w \mid w \in \mathrm{Fn}^+(\kappa)\rangle$ by recursion on α and $\max(\mathrm{dom}(w))$ such that each x_α is in $\mathcal{I}$ and such that $\langle e_w \mid w \in \mathrm{Fn}^+(\kappa)\rangle$ and κ witness that $\vec{x}$ is ascending. We will ensure that, for all $\alpha < \kappa$, we have $z_\alpha \subseteq x_\alpha$, which will in turn ensure that $\vec{x}$ is $\subseteq^*$-cofinal in $\mathcal{I}$. To this end, suppose that $\beta < \kappa$ and we have constructed $\vec{x} \restriction \beta$ and $\vec{e} \restriction \mathrm{Fn}^+(\beta)$ in such a way that $\vec{e} \restriction \mathrm{Fn}^+(\beta)$ and β witness that $\vec{x} \restriction \beta$ is ascending. We describe how to choose x_β and $\langle e_w \mid w \in \mathrm{Fn}^+(\beta+1) \wedge \beta \in \mathrm{dom}(w)\rangle$.

First, let $W_\beta = \{w \in \mathrm{Fn}^+(\beta) \mid \vec{x} \restriction \beta \text{ respects } w\}$, and let
$$\mathcal{E}_\beta = \{e_w \cap \bigcap_{\alpha\in a} x_\alpha \mid w \in W_\beta,\ a \in [\beta]^{<\omega}, \text{ and } \mathrm{dom}(w) < a\}.$$

By the inductive hypothesis, we know that $\mathcal{E}_\beta \subseteq [X]^\omega$. Moreover, since $\beta < \kappa = \mathrm{non}^*(\mathcal{I})$, we can find $x_{\beta,0} \in \mathcal{I}$ such that $e \cap x_{\beta,0}$ is infinite for all $e \in \mathcal{E}_\beta$. Since $\beta < \kappa = \mathrm{cof}^*(\mathcal{I})$, we can find $x_{\beta,1} \in \mathcal{I}$ such that, for all $a \in [\beta]^{<\omega}$, we have $x_{\beta,1} \not\subseteq^* \bigcup_{\alpha\in a} x_\alpha$.

[4] In [11] the set $\mathcal{E}$ is only assumed to be a subset of $[X]^\omega$, not $\mathcal{I} \cap [X]^\omega$. In the context of tall ideals on ω this does not change the definition; in the general context, the definition we give here seems the more useful one, at least for our purposes.

Let $x_\beta = x_{\beta,0} \cup x_{\beta,1} \cup z_\beta$. Our choice of $x_{\beta,0}$ ensures that this maintains clause (2)(c) of Definition 3.3, and our choice of $x_{\beta,1}$ ensures that this maintains clause (2)(b) of the same definition. Now, for every $w \in \mathrm{Fn}^+(\beta+1)$ such that $\beta \in \mathrm{dom}(w)$ and $\vec{x} \restriction (\beta+1)$ respects w, let e_w be an arbitrary element of $[d_w^{\vec{x}\restriction(\beta+1)}]^{\aleph_0}$. This completes the inductive step of the construction and hence the proof of the lemma. □

An ideal of particular interest to us is the ideal $\emptyset \times \mathrm{Fin}$, which can be concretely defined as an ideal on $\omega \times \omega$ in the following way. First, for all functions $f : \omega \to \omega$, recall that $I_f = \{(k,n) \in \omega \times \omega \mid n < f(k)\}$. Then $\emptyset \times \mathrm{Fin}$ is the set

$$\{x \subseteq \omega \times \omega \mid (\exists f \in {}^\omega\omega)(x \subseteq I_f)\}.$$

Recall that $\mathfrak{d}$ denotes the *dominating number*, i.e., the least size of a cofinal subset of $({}^\omega\omega, \leq^*)$.

Proposition 4.3. $\mathfrak{d} = \mathrm{cof}^*(\emptyset \times \mathrm{Fin}) = \mathrm{non}^*(\emptyset \times \mathrm{Fin})$.

Proof. The fact that $\mathfrak{d} = \mathrm{cof}^*(\emptyset \times \mathrm{Fin})$ follows immediately from the definitions. Let us show that $\mathfrak{d} = \mathrm{non}^*(\emptyset \times \mathrm{Fin})$. First, suppose that $\mathcal{F} \subseteq {}^\omega\omega$ has size $\mathfrak{d}$ and is cofinal in $({}^\omega\omega, \leq^*)$. For each $f \in {}^\omega\omega$, let I_f^+ denote the graph of f, i.e., the set $\{(k, f(k)) \mid k < \omega\}$. Let $\mathcal{E} = \{I_f^+ \mid f \in \mathcal{F}\}$. We claim that $\mathcal{E}$ witnesses that $\mathrm{non}^*(\emptyset \times \mathrm{Fin}) \leq \mathfrak{d})$. To see this, fix an arbitrary $x \in \emptyset \times \mathrm{Fin}$. Then there is $g \in {}^\omega\omega$ such that $x \subseteq I_g$. Then $I_g^+ \in \mathcal{E}$ and $I_g \cap I_g^+ = \emptyset$. Thus, $\mathrm{non}^*(\emptyset \times \mathrm{Fin}) \leq \mathfrak{d}$.

For the other inequality, suppose that $\mathcal{E}$ is a family of infinite elements of $\emptyset \times \mathrm{Fin}$ with $|\mathcal{E}| < \mathfrak{d}$. We must find $x \in \emptyset \times \mathrm{Fin}$ such that $|e \cap x| = \aleph_0$ for all $e \in \mathcal{E}$. For each $e \in \mathcal{E}$, fix $f_e \in {}^\omega\omega$ such that $e \subseteq I_{f_e}$. Let $a_e = \{k < \omega \mid (\exists m < \omega)((k,m) \in e)\}$. Since e is an infinite element of $\emptyset \times \mathrm{Fin}$, a_e must be infinite. Moreover, a_e partitions ω into pairwise disjoint intervals $\langle J_n^e \mid n < \omega\rangle$, where $J_0^e = [0, a_e(0)]$ and, for all $n < \omega$, $J_{n+1}^e = (a_e(n), a_e(n+1)]$. Define a function $g_e \in {}^\omega\omega$ as follows. For all $k < \omega$, let $n < \omega$ be such that $k \in J_n^e$, and then let $g_e(k) = f_e(a_e(n))$.

Using the fact that $|\mathcal{E}| < \mathfrak{d}$, find $h \in {}^\omega\omega$ such that, for all $e \in \mathcal{E}$, we have $h \not\leq^* g_e$. By increasing h if necessary, we may assume that it is weakly increasing, i.e., $h(k_0) \leq h(k_1)$ for all $k_0 < k_1 < \omega$. For each $e \in \mathcal{E}$, let $b_e := \{k < \omega \mid g_e(k) < h(k)\}$, and let $c_e := \{n < \omega \mid b_e \cap J_n^e\} \neq \emptyset$. Since each J_n^e is finite, we have $c_e \in [\omega]^\omega$. Moreover, for all $n \in c_e$, we can fix $k \in b_e \cap J_n^e$ and conclude that

$$f_e(a_e(n)) = g_e(k) < h(k) \leq h(a_e(n)),$$

and hence there is $m \leq h(a_e(n))$ such that $(a_e(n), m) \in e$. In particular, letting $x = I_h$, it follows that $x \in \emptyset \times \mathrm{Fin}$ and $|e \cap x| = \aleph_0$ for all $e \in \mathcal{E}$. □

Putting the results of this section together with Theorems 3.6 and 3.9 immediately yields the following corollary.

Corollary 4.4. *Suppose that* $1 \leq n < \omega$, $\mathcal{I}$ *is an ideal on a set* X *properly extending the ideal of all finite subsets of* X *with no* $\subseteq^*$*-maximal set. Suppose also that* $\mathrm{cof}^*(\mathcal{I}) = \mathrm{non}^*(\mathcal{I}) = \omega_n$. *Then*

(1) $\lim^n \mathbf{A}_{\mathcal{I}}[\mathbb{Z}^{(\omega_n)}] \neq 0$*;*

(2) if $\mathrm{w}\diamondsuit(S_k^{k+1})$ *holds for all* $k < n$*, then* $\lim^n \mathbf{A}_{\mathcal{I}} \neq 0$.

Setting $\mathcal{I} = \emptyset \times \mathrm{Fin}$ and invoking Proposition 4.3 and Fact 2.22 then yields the following important special case, which is Theorem A from the introduction.

Corollary 4.5. *Suppose that $1 \leq n < \omega$ and $\mathfrak{d} = \omega_n$. Then*

(1) $\lim^n \mathbf{A}[\mathbb{Z}^{(\omega_n)}] \neq 0$*;*
(2) if $\mathrm{w}\diamondsuit(S^{k+1}_k)$ *holds for all* $k < n$*, then* $\lim^n \mathbf{A} \neq 0$*.*

In particular, if $\lim^n \mathbf{A}[H] = 0$ *for all* $1 \leq n < \omega$ *and all abelian groups* H*, then* $2^{\aleph_0} > \aleph_\omega$*.*

We end this section by noting one other important special case. Given a regular uncountable cardinal κ, let $\mathcal{I}_\kappa$ denote the ideal $[\kappa]^{<\kappa}$. It is easily verified that $\mathrm{cof}^*(\mathcal{I}_\kappa) = \mathrm{non}^*(\mathcal{I}_\kappa) = \kappa$. Our framework therefore immediately yields the following corollary.

Corollary 4.6. *Suppose that $1 \leq n < \omega$. Then*

(1) $\lim^n \mathbf{A}_{\mathcal{I}_\kappa}[\mathbb{Z}^{(\omega_n)}] \neq 0$*;*
(2) if $\mathrm{w}\diamondsuit(S^{k+1}_k)$ *holds for all* $k < n$*, then* $\lim^n \mathbf{A}_{\mathcal{I}_\kappa} \neq 0$*.*

This corollary is not new, as the inverse systems $\mathbf{A}_{\mathcal{I}_\kappa}[H]$ have been somewhat extensively studied, although under different names (for instance, in [4] they are denoted $\mathbf{C}(\kappa, H)$). The first derived limits of these systems are also familiar to set theorists, although in a different guise: the assertion "$\lim^1 \mathbf{A}_{\mathcal{I}_\kappa}[\mathbb{Z}] \neq 0$" is equivalent to the existence of a coherent κ-Aronszajn subtree of ${}^{<\kappa}\omega$.

Clause (1) of Corollary 4.6 is implicit in [17]; for a more set-theoretic proof, see [3]. Clause (2) has, to the best of our knowledge, not explicitly appeared anywhere, but it follows from the techniques in [19]. We include the corollary here to indicate that these systems can be incorporated into the general framework we develop in this paper.

5. Simultaneous nonvanishing from squares and weak diamonds

In this final section, we prove Theorem B from the introduction. We first recall the following definition.

Definition 5.1. Suppose that λ is a regular uncountable cardinal and $S \subseteq \lambda$ is stationary. The principle $\square(\lambda, S)$ asserts the existence of a sequence $\vec{C} = \langle C_\alpha \mid \alpha < \lambda \rangle$ such that:

(1) for every limit ordinal $\alpha < \lambda$, C_α is a club in α;
(2) for every limit ordinal $\alpha < \beta$, if $\alpha \in \mathrm{Lim}(C_\beta)$, then $C_\beta \cap \alpha = C_\alpha$;
(3) for every limit ordinal $\alpha < \lambda$, $C_\alpha \cap S = \emptyset$.

Such a sequence is called a $\square(\lambda, S)$-sequence.

Remark 5.2. Condition (3) implies that there exists no club $C \subseteq \lambda$ such that for all $\alpha \in \mathrm{Lim}(\lambda)$, $C \cap \alpha = C_\alpha$, i.e., there is no *thread* through $\vec{C}$. For a proof, if it were not so, let $\gamma \in C \cap S$ and $\alpha \in \mathrm{Lim}(\lambda) \setminus (\gamma + 1)$. Then $\gamma \in C \cap \alpha = C_\alpha$, a contradiction. In particular, if $S \subseteq \lambda$ is stationary, then a $\square(\lambda, S)$-sequence is *a fortiori* a $\square(\lambda)$-sequence.

If δ is an ordinal, then a δ*-chain* in $({}^\omega\omega, \leq^*)$ is a sequence $\vec{f} = \langle f_\alpha \mid \alpha < \delta \rangle$ such that, for all $\alpha < \beta < \delta$, we have $f_\alpha <^* f_\beta$. If H is an abelian group and $\vec{f}$ is a δ-chain in $({}^\omega\omega, \leq^*)$, then when we speak of an *H-valued n-family indexed over $\vec{f}$*, we mean an n-family of the form

$$\left\langle \varphi_a : \bigcap_{\alpha \in b} I_{f_\alpha} \to H \;\middle|\; a \in [\delta]^n \right\rangle.$$

Lemma 5.3. *Suppose that $\kappa < \lambda$ are uncountable regular cardinals, n is a positive integer, and H is an abelian group with $|H| \leq 2^\kappa$. Suppose moreover that*

(1) for every κ-chain in $({}^\omega\omega, \leq^*)$ there exists a nontrivial coherent H-valued n-family indexed over that chain;

(2) there exists a stationary $S \subseteq S^\lambda_\kappa$ such that both $\mathrm{w}\diamondsuit(S)$ and $\square(\lambda, S)$ hold.

Then for every λ-chain in $({}^\omega\omega, \leq^*)$ there exists a nontrivial coherent H-valued $(n+1)$-family indexed over that chain.

Proof. Fix a λ-chain $\vec{f} = \langle f_\alpha \mid \alpha < \lambda\rangle$ in $({}^\omega\omega, \leq^*)$ and a $\square(\lambda, S)$-sequence $\langle C_\alpha \mid \alpha < \lambda\rangle$. Let $\mathcal{S}$ be the set of $\sigma \in {}^{<\lambda}2$ such that $\mathrm{supp}(\sigma) \subseteq S$, and let $\mathcal{S}^+$ be the set of $g \in {}^\lambda 2$ such that $\mathrm{supp}(g) \subseteq S$. For all $\sigma \in \mathcal{S}$, we will construct a coherent (and trivial) $(n+1)$-family $\Phi^\sigma = \langle \varphi^\sigma_b : \bigcap_{\alpha \in b} I_{f_\alpha} \to H \mid b \in [\mathrm{lh}(\sigma)]^{n+1}\rangle$. We will arrange so that, if $\sigma, \tau \in \mathcal{S}$ and $\sigma \sqsubseteq \tau$, then $\Phi^\sigma \sqsubseteq \Phi^\tau$. At the end of the proof, we will find $g \in \mathcal{S}$ such that $\Phi^g = \bigcup_{\alpha<\lambda} \Phi^{g\restriction\alpha}$ is nontrivial. The construction is by recursion on $\mathrm{lh}(\sigma)$, and we will use $\vec{C}$ to ensure coherence by maintaining the following recursion hypothesis for each $\sigma \in \mathcal{S}$:

$$(\dagger) \qquad \forall\gamma \in \mathrm{Lim}(\mathrm{lh}(\sigma)) \setminus S \ \forall\beta \in \mathrm{Lim}(C_\gamma) \ \forall a \in [\beta]^n \left[\varphi^\sigma_{a\cup\{\beta\}} =^* \varphi^\sigma_{a\cup\{\gamma\}}\right]$$

We now turn to the construction. First, if $\sigma \in \mathcal{S}$, $\mathrm{lh}(s) = \delta \in \mathrm{Lim}(\lambda)$, and $\Phi^{\sigma\restriction\alpha}$ has been constructed for all $\alpha < \delta$, then let $\Phi^\sigma = \bigcup_{\alpha<\delta} \Phi^{\sigma\restriction\alpha}$.

It remains to describe the successor step of the construction. Thus, suppose that $\sigma \in \mathcal{S}$, $\mathrm{lh}(\sigma) = \delta$, and we have constructed Φ^σ. We describe how to construct $\Phi^{\sigma,\ell}$ for $\ell < 2$ if $\delta \in S$ and $\ell = 0$ otherwise. We distinguish between a number of cases.

Case 1: δ is a successor ordinal. In this case there is nothing new to check with respect to $(\dagger)$, so any coherent extension $\Phi^{\sigma,0}$ of Φ^σ will do. Such an extension exists by Theorem 2.7 and Remark 2.17.

Case 2: $\delta \in \mathrm{Lim}(\lambda) \setminus S$ and $\delta > \sup(\mathrm{Lim}(C_\delta))$. Let $\gamma = \sup(\mathrm{Lim}(C_\delta))$. In the present case, $\gamma = \max(\mathrm{Lim}(C_\delta))$ and $\mathrm{otp}(C_\delta \setminus \gamma) = \omega$, thus witnessing that $\mathrm{cf}(\delta) = \omega$. Hence again by Theorem 2.7 there exists $\Psi^\sigma = \langle \psi^\sigma_a \mid a \in [\delta]^n\rangle$ that trivializes Φ^σ.

In order to construct $\Phi^{\sigma,0}$, we need to define $\varphi^{\sigma,0}_{a\cup\{\delta\}}$ for all $a \in [\delta]^n$. Fix such an a if $\gamma \in a$, then set $\varphi^{\sigma,0}_{a\cup\{\delta\}} = 0$. If $\gamma \notin a$, then let $h = |a \setminus \gamma|$, and define $\varphi^{s,0}_{a\cup\{\delta\}} : \bigcap_{i<n} I_{f_{\alpha_i}} \cap I_{f_\delta} \to H$ by setting

$$\varphi^{s,0}_{a\cup\{\delta\}}(x) = \begin{cases} (-1)^h \varphi^\sigma_{a\cup\{\gamma\}}(x) & \text{if } x \in I_{f_\gamma}; \\ (-1)^n \psi^\sigma_a(x) & \text{if } x \notin I_{f_\gamma}. \end{cases}$$

We now verify that this choice of $\Phi^{\sigma,0}$ satisfies the requirements of the construction. For readability, we will omit the superscripts σ and 0 during this verification. To verify condition $(\dagger)$, we must show that, for all $\beta \in \mathrm{Lim}(C_\delta)$ and all $a \in [\beta]^n$, we have $\varphi_{a\cup\{\beta\}} =^* \varphi_{a\cup\{\delta\}}$. Fix such β and a. By the construction of $\Phi^{s,0}$, we know that $\varphi_{a\cup\{\gamma\}} =^* \varphi_{a\cup\{\delta\}}$. By the properties of $\vec{C}$, we know that either $\beta = \gamma$ or $\beta \in \mathrm{Lim}(C_\gamma)$. In the latter case, the fact that Φ^σ satisfies $(\dagger)$ implies that $\varphi_{a\cup\{\beta\}} =^* \varphi_{a\cup\{\gamma\}}$. Since $I_{f_\beta} \subseteq^* I_{f_\gamma} \subseteq^* I_{f_\delta}$, this in turn implies that $\varphi_{a\cup\{\beta\}} =^* \varphi_{a\cup\{\delta\}}$, as desired.

We next verify coherence. It suffices to show that, for all $b \in [\delta]^{n+1}$, we have $(-1)^{n+1}\varphi_b + \sum_{i<n+1}(-1)^i \varphi_{b^i\cup\{\delta\}} =^* 0$. Fix such a b. We distinguish between two

cases. First suppose that $\gamma \in b$ (say $\gamma = b(k)$, so $|b \setminus \gamma| = n - k$). Then

$$\begin{aligned}(-1)^{n+1}\varphi_b + \sum_{i<n+1}(-1)^i\varphi_{b^i\cup\{\delta\}} &= (-1)^{n+1}\varphi_b + (-1)^k\varphi_{b^k\cup\{\delta\}} \\ &= (-1)^{n+1}\varphi_b + (-1)^k(-1)^{n-k}\varphi_{b^k\cup\{\gamma\}} \\ &= (-1)^{n+1}\varphi_b + (-1)^n\varphi_b = 0.\end{aligned}$$

Suppose on the other hand that $\gamma \notin b$, and let $k = |b \cap \gamma|$ (so $|b \setminus \gamma| = n+1-k$). Then, when restricting to I_{f_γ}, we have

$$\begin{aligned}&(-1)^{n+1}\varphi_b + \sum_{i<n+1}(-1)^i\varphi_{b^i\cup\{\delta\}} \\ =&(-1)^{n+1}\varphi_b + \sum_{i<k}(-1)^i(-1)^{n+1-k}\varphi_{b^i\cup\{\gamma\}} + \sum_{i=k}^{n}(-1)^i(-1)^{n-k}\varphi_{b^i\cup\{\gamma\}} \\ =&(-1)^{n+1-k}\Big((-1)^k\varphi_b + \sum_{i<k}(-1)^i\varphi_{b^i\cup\{\gamma\}} + \sum_{i=k}^{n}(-1)^{i+1}\varphi_{b^i\cup\{\gamma\}}\Big) \\ =&(-1)^{n+1-k}\sum_{i<n+2}(-1)^i\varphi_{(b\cup\{\gamma\})^i} =^* 0.\end{aligned}$$

Outside of I_{f_γ}, we have

$$\begin{aligned}(-1)^{n+1}\varphi_b + \sum_{i<n+1}(-1)^i\varphi_{b^i\cup\{\delta\}} &= (-1)^{n+1}\varphi_b + \sum_{i<n+1}(-1)^i(-1)^n\psi_{b^i} \\ &=^* (-1)^{n+1}\varphi_b + (-1)^n\varphi_b = 0.\end{aligned}$$

Thus, we have verified coherence.

Case 3: $\delta \in \mathrm{Lim}(\lambda) \setminus S$ **and** $\sup(\mathrm{Lim}(C_\delta)) = \delta$**.** For $a \in [\delta]^n$, let $\gamma(a) = \min(\mathrm{Lim}(C_\delta) \setminus (\max(a)+1))$. Then, for all $x \in I_{f_\delta} \cap \bigcap_{\alpha\in a} I_{f_\alpha}$, set

$$\varphi^{\sigma,0}_{a\cup\{\delta\}} = \begin{cases}\varphi^{\sigma}_{a\cup\{\gamma(a)\}}(x) & \text{if } x \in I_{f_{\gamma(a)}} \\ 0 & \text{otherwise.}\end{cases}$$

Note that the "otherwise" case of the above definition only occurs for finitely many $x \in \mathrm{dom}(\varphi^{\sigma,0}_{a\cup\{\delta\}})$.

We verify condition (†), again omitting superscripts for readability. For all $\beta \in \mathrm{Lim}(C_\delta)$ and $a \in [\beta]^n$ we have $\varphi_{a\cup\{\delta\}} =^* \varphi_{a\cup\{\gamma(a)\}} =^* \varphi_{a\cup\{\beta\}}$, where the second equality holds by the induction assumption since either $\gamma(a) = \beta$ or $\gamma(a) \in \mathrm{Lim}(C_\beta) = \mathrm{Lim}(C_\delta) \cap \beta$. Since moreover $I_{f_{\gamma(a)}} \subseteq^* I_{f_\beta} \subseteq^* I_{f_\delta}$, we have $\varphi_{a\cup\{\delta\}} =^* \varphi_{a\cup\{\beta\}}$, as desired.

We now verify coherence. Let $b \in [\delta]^{n+1}$, and let $\gamma = \gamma(b^0)$. Then

$$\begin{aligned}(-1)^{n+1}\varphi_b + \sum_{i<n+1}(-1)^i\varphi_{b^i\cup\{\delta\}} &=^* (-1)^{n+1}\varphi_b + \sum_{i<n+1}(-1)^i\varphi_{b^i\cup\{\gamma\}} \\ &=^* \sum_{i<n+2}(-1)^i\varphi_{(b\cup\{\gamma\})^i} =^* 0,\end{aligned}$$

where the first equality (mod finite) uses the fact that $\gamma(b^i) = \gamma$ for all $i < n$ and $\varphi_{b^n\cup\{\gamma\}} =^* \varphi_{b^n\cup\{\gamma(b^n)\}}$ by (†), and the last equality holds because Φ^σ satisfies coherence.

Case 4: $\delta \in S$. In this case there is nothing to check with respect to (†), so we only need to produce two coherent families $\Phi^{\sigma,0}$ and $\Phi^{\sigma,1}$. First observe that $\sup(\mathrm{Lim}(C_\delta)) = \delta$, since $\delta \in S \subseteq S^\lambda_\kappa$. We can thus define $\Phi^{\sigma,0}$ exactly as in Case 3.

By the hypothesis of the theorem, every κ-chain in $({}^\omega\omega, \leq^*)$ carries a nontrivial coherent H-valued n-family. By Proposition 2.5 and Remark 2.6, the same is true for every chain whose length has cofinality κ. In particular, we can fix a nontrivial coherent H-valued n-family

$$T^\delta = \langle \tau^\delta_a : \bigcap_{\alpha \in a} I_{f_\alpha} \to H \mid a \in [\delta]^n \rangle.$$

Note that, since $I_{f_\alpha} \subseteq^* I_{f_\delta}$ for all $\alpha < \delta$, the restriction $T^\delta \restriction\restriction I_{f_\delta}$ remains nontrivial. Then define $\Phi^{\sigma,1}$ by setting $\varphi^{\sigma,1}_{a\cup\{\delta\}} = \varphi^{\sigma,0}_{a\cup\{\delta\}} + \tau^\delta_a$ for all $a \in [\delta]^n$. The coherence of $\Phi^{\sigma,1}$ then follows from the coherence of $\Phi^{\sigma,0}$ and T^δ.

This completes the construction. We now prove that there is $g \in \mathcal{S}^+$ such that $\Phi^g = \bigcup_{\beta<\lambda} \Phi^{g\restriction\beta}$ is nontrivial. We first record a couple of simple claims.

Claim 5.4. *Suppose that $\delta \in S$ and $\sigma \in {}^\delta 2$. Then no trivialization of Φ^σ extends both to a trivialization of $\Phi^{\sigma,0}$ and a trivialization of $\Phi^{\sigma,1}$.*

Proof. This follows from the construction of $\Phi^{\sigma,0}$ and $\Phi^{\sigma,1}$ and from Propositions 2.15 and 2.18. □

Claim 5.5. *Let Υ be a n-family over $\vec{f} \restriction \beta$ for some $\beta < \lambda$. Then there exists at most one $\sigma \in {}^\beta 2$ such that Υ trivializes Φ^σ.*

Proof. This is proven in exactly the same way as Claim 3.8. □

Now we use a coding similar to that used in the proof of Theorem 3.6. More precisely, let $\mathcal{P}^\gamma$ be the set of alternating H-valued n-families indexed over $\vec{f} \restriction \gamma$. Then let G be a function with domain ${}^{\leq\lambda}2$ such that for every $\gamma \in S^\lambda_\kappa \cup \{\lambda\}$ the map $G \restriction {}^\gamma 2$ is a surjection onto $\mathcal{P}^\gamma$, and such that if $\rho \sqsubseteq \rho' \in \mathrm{dom}(G)$, then $G(\rho) \sqsubseteq G(\rho')$. Such a G exists because the intervals between successive elements of S^λ_κ have size κ, and $|H| \leq 2^\kappa$. We also write Υ^ρ for $G(\rho)$.

Finally, we define $F : {}^{<\lambda}2 \to 2$ as follows. If $\mathrm{lh}(\rho) \in S$ and Υ^ρ trivializes Φ^σ for some σ with $\mathrm{lh}(\sigma) = \mathrm{lh}(\rho)$ (such a σ is unique by Claim 5.5) then we let $F(\rho) = \ell \in \{0, 1\}$ be such that Υ^ρ does not extend to a trivialization of $\Phi^{\sigma,\ell}$ (such an ℓ exists by Claim 5.4). Otherwise, let $F(\rho) \in \{0, 1\}$ be arbitrary.

By $\mathrm{w}\diamondsuit(S)$ there exists $g : \lambda \to 2$ such that for all $b : \lambda \to 2$ there exists $\alpha \in S$ such that $g(\alpha) = F(b \restriction \alpha)$. Since this only depends on the values that g takes on S, we can assume that $g \in \mathcal{S}^+$. Now suppose for the sake of contradiction that some Υ trivializes $\Phi^g = \bigcup_{\alpha<\lambda} \Phi^{g\restriction\alpha}$. Find b such that $\Upsilon = \Upsilon^b$, and find $\alpha \in S$ such that $g(\alpha) = F(b \restriction \alpha)$. Then $\Upsilon^b \restriction (\vec{f} \restriction \alpha) = \Upsilon^{b\restriction\alpha}$ trivializes $\Phi^g \restriction (\vec{f} \restriction \alpha) = \Phi^{g\restriction\alpha}$. By our choice of F, g, and α, then, no extension of $\Upsilon^{b\restriction\alpha}$ can trivialize $\Phi^{g\restriction(\alpha+1)}$ and hence, *a fortiori*, no extension of $\Upsilon^{b\restriction\alpha}$ can trivialize Φ^g, contradicting the fact that Υ extends $\Upsilon^{b\restriction\alpha}$ and trivializes Φ^g and completing the proof of the theorem. □

We can now show the consistency of the simultaneous nonvanishing of $\lim^k \mathbf{A}$ for many values of k. We first show that for all n it is consistent that $\bigwedge_{2\leq k\leq n} \lim^k \mathbf{A} \neq 0$. This can be realized in the kind of model considered in [19]. Recall that, for a regular cardinal κ and a set I, the poset $\mathrm{Fn}(I, 2, \kappa)$ consists of all partial functions $p : I \to 2$ such that $|p| < \kappa$, ordered by reverse inclusion. Recall also that Hechler forcing $\mathbb{H}$ consists of conditions of the form $p = (s_p, f_p)$ such that s_p is a finite partial function from ω to ω and $f_p \in {}^\omega\omega$. If $p, q \in \mathbb{H}$, then $q \leq p$ if and only if $s_q \supseteq s_p$, $f_q \geq f_p$ and, for all $n \in \mathrm{dom}(s_q) \setminus \mathrm{dom}(s_p)$, we have $s_q(n) \geq f_p(n)$.

Lemma 5.6. *Suppose that $1 \leq n < \omega$. Then there exists a model of* ZFC *in which*

(1) $\mathfrak{b} = \mathfrak{d} = \omega_n$;
(2) for all $k < n$, $\mathrm{w}\diamondsuit(S^{k+1}_k)$ holds;
(3) for all $k < n$, there exists a stationary set $S \subseteq S^n_k$ such that $\square(\omega_n, S) + \mathrm{w}\diamondsuit(S)$ holds.

Proof. Our model will be the following forcing extension of L:

$$V' = L^{\mathbb{H}_{\omega_n} \times \mathbb{C}_1 \times \cdots \times \mathbb{C}_n},$$

where $\mathbb{H}_{\omega_n}$ is the finite support iteration of Hechler forcings of length ω_n and $\mathbb{C}_k = \mathrm{Fn}(\omega_{n+k}, 2, \omega_k)$ for all $1 \leq k \leq n$, i.e., $\mathbb{C}_k$ is the forcing to add ω_{n+k}-many Cohen subsets to ω_k.

It follows from [19, Theorem 4.5] that V' has all of the same cardinals and cofinalities as L and satisfies $\mathfrak{b} = \mathfrak{d} = \omega_n$ as well as $\mathrm{w}\diamondsuit(S^{k+1}_k)$ for all $k < n$. To verify item (3) in the statement of the lemma, fix $k < n$, and temporarily work in L. By [13], we can find a stationary subset $S \subseteq S^n_k$ and a $\square(\omega_n, S)$-sequence $\vec{C}$. Let $\mathbb{P} = \mathbb{H}_{\omega_n} \times \mathbb{C}_1 \times \cdots \times \mathbb{C}_{n-1}$. Then $\mathbb{P}$ is an ω_n-cc poset of size ω_{2n-1}, $\mathbb{C}_n$ is ω_n-closed, and $V' = L^{\mathbb{P} \times \mathbb{C}_n}$. In particular, S remains stationary in V'. Since all of the defining properties of $\square(\omega_n, S)$ are upwards absolute from L to V', $\vec{C}$ remains a $\square(\omega_n, S)$-sequence in V'. Moreover, [19, Lemma 4.3] implies that $\mathrm{w}\diamondsuit(S)$ holds in V', so $S \subseteq S^n_k$ is as desired. □

Lemma 5.7. *Assume that*

(1) for all $k < n$, $\mathrm{w}\diamondsuit(S^{k+1}_k)$ holds;
(2) for all $1 \leq k < n$, there exists a stationary set $S \subseteq S^n_k$ such that $\square(\omega_n, S) + \mathrm{w}\diamondsuit(S)$ holds.

Then every ω_n-chain in $({}^\omega\omega, \leq^*)$ carries a nontrivial coherent H-valued k-family, for all $2 \leq k \leq n$ and all countable abelian groups H.

Proof. Fix a countable abelian group H. In ZFC, every ω_1-chain in $({}^\omega\omega, \leq^*)$ carries a nontrivial coherent H-valued 1-family (cf. [19, Proposition 3.4]). Using [19, Lemma 3.6], one can prove by induction that, for every $1 \leq k \leq n$, every ω_k-chain carries a nontrivial coherent H-valued k-family. Finally, suppose that $2 \leq k \leq n$, and pick $S \subseteq S^{\omega_n}_{\omega_{k-1}}$ such that $\square(\omega_n, S) + \mathrm{w}\diamondsuit(S)$ holds. Then Lemma 5.3 implies that every ω_n-chain carries a nontrivial coherent H-valued k-family, as desired. □

The following theorem is now immediate, yielding clause (1) of Theorem B from the introduction.

Theorem 5.8. *Fix $2 \leq n < \omega$. Relative to the consistency of* ZFC*, it is consistent that $\mathfrak{b} = \mathfrak{d} = \omega_n$ and $\bigwedge_{2 \leq k \leq n} \lim^k \boldsymbol{A} \neq 0$ holds.*

Proof. In any model witnessing the conclusion of Lemma 5.6, there exists an ω_n-chain that is cofinal in $({}^\omega\omega, \leq^*)$. Lemma 5.7 then implies that, for all $2 \leq k \leq n$, this ω_n-chain carries a nontrivial coherent $\mathbb{Z}$-valued k-family. Then Proposition 2.4, Remark 2.6, and Fact 2.21 imply that $\lim^k \mathbf{A} \neq 0$ for all $2 \leq k \leq n$. □

We now show that it is consistent that $\lim^n \mathbf{A}$ does not vanish for any $n \geq 2$. By Goblot's Theorem (cf. Proposition 2.7), this requires the dominating number to be at least $\aleph_{\omega+1}$. For technical reasons, we were unable to obtain this simultaneous nonvanishing with $\mathfrak{d} = \aleph_{\omega+1}$, but we were able to achieve it with $\mathfrak{d} = \aleph_{\omega+2}$.

Lemma 5.9. *There exists a model of* ZFC *in which*

(1) $\mathfrak{b} = \mathfrak{d} = \omega_{\omega+2}$*;*
(2) for all $k < \omega$*,* $\mathrm{w}\diamondsuit(S^{k+1}_k)$ *holds;*
(3) for all $k < \omega$*, there exists a stationary set* $S \subseteq S^{\omega_{\omega+2}}_{\omega_k}$ *such that* $\square(\omega_{\omega+2}, S) + \mathrm{w}\diamondsuit(S)$ *holds.*

Proof. Our model will be the following forcing extension of L:

$$V' = L^{\mathbb{H} \times \prod_{1 \leq n < \omega} \mathbb{C}_n \times \mathbb{C}_\omega}$$

satisfies the properties in question, where $\mathbb{H}$ is the finite-support iteration of Hechler forcings of length $\omega_{\omega+2}$, $\mathbb{C}_n = \mathrm{Fn}(\omega_{\omega+n+2}, 2, \omega_n)$ for all $1 \leq n < \omega$, and $\mathbb{C}_\omega = \mathrm{Fn}(\omega_{\omega\cdot 2+2}, 2, \omega_{\omega+2})$.

We first verify clause (1) in the statement of the theorem. Let G be a generic filter for $\mathbb{H} \times \prod_{1\leq n<\omega} \mathbb{C}_n \times \mathbb{C}_\omega$. By standard arguments, forcing over L with this product preserves all cardinalities and cofinalities. We can write G as the product $G = G_0 \times G'$ of generic filters on $\mathbb{H}$ and $\prod_{1\leq n<\omega} \mathbb{C}_n \times \mathbb{C}_\omega$, respectively. Note that $\prod_{1\leq n<\omega} \mathbb{C}_n \times \mathbb{C}_\omega$ is σ-closed, so that in particular it does not add any reals, preserves CH and $\mathbb{H}$ is absolute with respect to it. Therefore, forcing with $\mathbb{H}$ over $L[G']$ forces that $\mathfrak{b} = \mathfrak{d} = \omega_{\omega+2}$, as desired.

We now verify clause (2). Let us further decompose G' as a product $\prod_{1\leq n<\omega} G_n \times G_\omega$, where G_n is generic for $\mathbb{C}_n$ for all $1 \leq n \leq \omega$. Fix some $k < \omega$. Then $\prod_{k+2\leq n<\omega} \mathbb{C}_n \times \mathbb{C}_\omega$ is ω_{k+2}-closed, so that $W = L[\prod_{k+2\leq n<\omega} G_n \times G_\omega]$ satisfies $2^{\omega_k} = \omega_{k+1}$. Now apply [19, Lemma 4.3] in W with $\mathbb{P} = \mathbb{H} \times \prod_{j\leq k} \mathbb{C}_k$ and $\mathbb{Q} = \mathbb{C}_{k+1}$ to conclude that $\mathrm{w}\diamondsuit(S)$ holds for every stationary in $S \subseteq \omega_{k+1}$ with $S \in W$; in particular, for $S = S^{k+1}_k$. More precisely, in W, $\mathbb{P}$ is ω_{k+1}-cc and has size $\omega_{\omega+k+2}$, $\mathbb{Q} = \mathrm{Fn}(\omega_{\omega+k+3}, 2, \omega_{k+1})$ and, in W, we have $(\omega_{\omega+k+2})^{\omega_k} = \omega_{\omega+k+2}$, so [19, Lemma 4.3] implies that $\mathrm{w}\diamondsuit(S^{k+1}_k)$ holds in the extension of W by $\mathbb{P} \times \mathbb{Q}$, i.e., in V'.

We finally verify clause (3). Fix $k < \omega$, and temporarily work in L. By [13], we can find a stationary subset $S \subseteq S^{\omega_{\omega+2}}_{\omega_k}$ and a $\square(\omega_{\omega+2}, S)$-sequence $\vec{C}$. Since $\mathbb{H} \times \prod_{1\leq n<\omega} \mathbb{C}_n$ is $\omega_{\omega+2}$-cc and $\mathbb{C}_\omega$ is $\omega_{\omega+2}$-closed, S remains stationary in V', so $\vec{C}$ remains a $\square(\omega_{\omega+2}, S)$-sequence in V'. Another application of [19, Lemma 4.3], this time with $\mathbb{P} = \mathbb{H} \times \prod_{1\leq n<\omega} \mathbb{C}_n$ and $\mathbb{Q} = \mathbb{C}_\omega$, implies that $\mathrm{w}\diamondsuit(S)$ holds in V'. □

Lemma 5.10. *Assume that* $\lambda \geq \omega_{\omega+1}$ *is a regular cardinal and*

(1) for all $1 \leq k < \omega$*,* $\mathrm{w}\diamondsuit(S^{k+1}_k)$ *holds;*
(2) for all $1 \leq k < \omega$*, there is a stationary set* $S \subseteq S^\lambda_{\omega_k}$ *such that* $\square(\lambda, S) + \mathrm{w}\diamondsuit(S)$ *holds.*

Then every λ*-chain in* $({}^\omega\omega, \leq^*)$ *carries a nontrivial coherent* H*-valued* k*-family for all* $2 \leq k < \omega$ *and all countable abelian groups* H*.*

Proof. Fix a countable abelian group H. As above, we know that every ω_1-chain in $({}^\omega\omega, leq^*)$ carries a nontrivial coherent H-valued 1-family. Using [19, Lemma 3.6], one can prove by induction that, for all $1 \leq k < \omega$, every ω_k-chain carries a nontrivial coherent H-valued k-family. Finally, fix $2 \leq k < \omega$ and fix a stationary

$S \subseteq S^\lambda_{\omega_{k-1}}$ such that $\square(\lambda, S) + \mathrm{w}\diamondsuit(S)$ holds. Then Lemma 5.3 implies that every λ-chain carries a nontrivial coherent H-valued k-family. □

These yield the following result, which is clause (2) of Theorem B from the introduction.

Theorem 5.11. *Relative to the consistency of* ZFC*, it is consistent that* $\mathfrak{b} = \mathfrak{d} = \omega_{\omega+2}$ *and* $\bigwedge_{2 \leq k < \omega} \lim^k \mathbf{A} \neq 0$.

Proof. This follows immediately from Lemmas 5.9 and 5.10, together with the arguments from the proof of Theorem 5.8. □

It is noted in [5] that if one adds $\aleph_2$-many Hechler reals over a model of CH, then $\lim^1 \mathbf{A} = 0$ in the resulting forcing extension. The same argument easily adapts to forcing extensions resulting from adding λ-many Hechler reals to a model of GCH, where $\lambda \geq \aleph_2$ is a regular cardinal. Therefore, the models we construct to witness Theorems 5.8 and 5.11 will satisfy $\lim^1 \mathbf{A} = 0$. The following question therefore remains open.

Question 5.12. *Is* $\bigwedge_{1 \leq k < \omega} \lim^k \mathbf{A} \neq 0$ *consistent with* ZFC*?*

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