NARROW SYSTEMS REVISITED

CHRIS LAMBIE-HANSON

Abstract. We investigate connections between set theoretic compactness principles and cardinal arithmetic, introducing and studying generalized narrow system properties as a way to approach two open questions about two-cardinal tree properties. The first of these questions asks whether the strong tree property at a regular cardinal $\kappa \geq \omega_2$ implies the Singular Cardinals Hypothesis (SCH) above κ. We show here that a certain narrow system property at κ that is closely related to the strong tree property, and holds in all known models thereof, suffices to imply SCH above κ. The second of these questions asks whether the strong tree property can consistently hold simultaneously at all regular cardinals $\kappa \geq \omega_2$. We show here that the analogous question about the generalized narrow system property has a positive answer. We also highlight some connections between generalized narrow system properties and the existence of certain strongly unbounded subadditive colorings.

1. Introduction

One of the oldest problems in set theory is the determination of the values of the *continuum function*, which takes an infinite cardinal κ and outputs 2^κ, i.e., the cardinality of the power set of κ. This function is subject to two easily proven constraints:

- (monotonicity) $\kappa \leq \lambda \Rightarrow 2^\kappa \leq 2^\lambda$;
- (König's theorem) the cofinality of 2^κ is strictly greater than κ.

Shortly after Cohen introduced the technique of forcing in 1963 [1], Easton proved in [4] that, at least when restricted to regular cardinals, i.e., cardinals equal to their own cofinality, these are the only constraints placed on cardinal arithmetic by the axioms of ZFC. More precisely, working in some model V of set theory, given any proper class function F defined on the class of all regular cardinals that satisfies monotonicity and the analogue of König's theorem, there exists some larger model $V[G]$ of ZFC with the same cardinals and cofinalities such that, in $V[G]$, we have $2^\kappa = F(\kappa)$ for every infinite regular cardinal κ.

The study of the continuum function at singular, i.e., nonregular, cardinals, has proven to be much more subtle and difficult. A series of remarkable results due to Silver [23], Galvin and Hajnal [5], and, later, Shelah [22], shows that there are highly nontrivial constraints on the behavior of the continuum function at singular cardinals. These results have placed the Singular Cardinals Hypothesis (SCH) and its variants at the center of much recent research. SCH has a few

2020 *Mathematics Subject Classification.* 03E05, 03E35, 03E55, 03E04.

Key words and phrases. narrow systems, Singular Cardinals Hypothesis, Shelah's Strong Hypothesis, tree property, large cardinals, subadditive colorings.

The author was supported by the Czech Academy of Sciences (RVO 67985840) and the GAČR project 23-04683S. We thank the anonymous referee for a number of helpful corrections and suggestions.

different formulations; in its simplest, which is the one we will adopt in this paper, it asserts that, for every singular cardinal μ, if $2^\kappa < \mu$ for all $\kappa < \mu$, then $2^\mu = \mu^+$. The consistency of the failure of SCH with the axioms of ZFC was first shown by Magidor in [16]; notably, this requires the consistency of the existence of certain *large cardinals*, which goes beyond the consistency of ZFC alone. The exact large cardinal strength necessary to obtain the consistency of the failure of SCH was established by Gitik in [6] to be a measurable cardinal κ with Mitchell order κ^{++}.

Another significant line of research in modern set theory, which, as we will see, has considerable overlap with the study of the continuum function and cardinal arithmetic more broadly, concerns the study of *compactness principles*, i.e., statements asserting that certain mathematical structures necessarily reflect the properties of their small substructures. Compactness principles typically hold at (and sometimes characterize) large cardinals, and much research into them centers on the extent to which these compactness principles can hold at smaller cardinals, and the extent to which these principles can be said to capture the "essence" of the respective large cardinal. To take a classical example, the tree property characterizes weakly compact cardinals among strongly inaccessible cardinals, while Mitchell [20] showed that the tree property at $\aleph_2$ is equiconsistent with the existence of a weakly compact cardinal. A number of questions remain open, though, about the extent to which the tree property can hold at smaller cardinals. The most prominent, due to Magidor, asks whether it is consistent that the tree property holds simultaneously at all regular cardinals greater than or equal to $\aleph_2$.

Generalizations of the tree property, known collectively as *two-cardinal tree properties*, were introduced in the 1970s by Jech [8] and Magidor [15] to provide combinatorial characterizations of strongly compact and supercompact cardinals. Let us now recall some of the important definitions, in their modern formulation (see the end of this section for some notational conventions).

Definition 1.1. Suppose that $\kappa \leq \lambda$ are uncountable cardinals, with κ regular. A (κ, λ)*-tree* is a structure $\mathcal{T} = \langle T_x \mid x \in \mathscr{P}_\kappa\lambda \rangle$ such that

- for all $x \in \mathscr{P}_\kappa\lambda$, T_x is a nonempty collection of subsets of x;
- for all $x \subseteq y \in \mathscr{P}_\kappa\lambda$ and all $t \in T_y$, we have $t \cap x \in T_x$.

A (κ, λ)-tree $\mathcal{T}$ is *thin* if $|T_x| < \kappa$ for all $x \in \mathscr{P}_\kappa\lambda$. A *cofinal branch* through $\mathcal{T}$ is a set $b \subseteq \lambda$ such that $b \cap x \in T_x$ for all $x \in \mathscr{P}_\kappa\lambda$.

The (κ, λ)*-tree property*, denoted $\mathsf{TP}(\kappa, \lambda)$, is the assertion that every thin (κ, λ)-tree has a cofinal branch. The *ineffable* (κ, λ)*-tree property*, denoted $\mathsf{ITP}(\kappa, \lambda)$, is the assertion that, for every thin (κ, λ)-tree $\mathcal{T} = \langle T_x \mid x \in \mathscr{P}_\kappa\lambda \rangle$ and every choice function $d \in \prod_{x \in \mathscr{P}_\kappa\lambda} T_x$, there is a set $b \subseteq \lambda$ such that the set

$$\{x \in \mathscr{P}_\kappa\lambda \mid b \cap x = d(x)\}$$

is stationary in $\mathscr{P}_\kappa\lambda$.

The *strong tree property* at κ, denoted TP_κ, is the assertion that $\mathsf{TP}(\kappa, \lambda)$ holds for all $\lambda \geq \kappa$.[1] The *super tree property* at κ, denoted ITP_κ, is the assertion that $\mathsf{ITP}(\kappa, \lambda)$ holds for all $\lambda \geq \kappa$.

Fact 1.2. Suppose that κ is an inaccessible cardinal.

- (Jech [8]) κ is strongly compact if and only if TP_κ holds.

[1] To head off potential confusion, we note that TP_κ is stronger than the classical tree property at κ, which is typically denoted $\mathsf{TP}(\kappa)$ and is equivalent to $\mathsf{TP}(\kappa, \kappa)$ in our notation.

- (Magidor [15]) κ is supercompact if and only if ITP_κ holds.

The modern study of two-cardinal tree properties at accessible cardinals began in the 2000s, when the relevant definitions (including, e.g., the notion of a *thin* (κ, λ)-tree introduced above) were isolated by Weiß [30]. Since then, they have been the focus of a large amount of research, much of which has been directed toward the study of their influence on cardinal arithmetic. Most notably, results of Viale [29] and Krueger [10] together show that, for a regular cardinal $\kappa \geq \omega_2$, ISP_κ, which is a strengthening of ITP_κ also introduced by Weiß in [30], implies that SCH holds above κ. In [13, Theorem A], Stejskalová and the author show that SCH above κ (and in fact Shelah's Strong Hypothesis (SSH), a strengthening of SCH, above κ) already follows from a significant weakening of ISP_κ that holds if, e.g., κ is strongly compact or if $\kappa = \omega_2$ and we are in an extension by Mitchell forcing starting with a strongly compact cardinal.

We note also the seminal result of Solovay [25] stating that, if κ is a strongly compact cardinal, then SCH holds above κ. Recalling that, among inaccessible cardinals, TP_κ characterizes strongly compact cardinals whereas ITP_κ characterizes supercompact cardinals, this, together with the results mentioned in the previous paragraph, naturally leads to the following question, already asked in, e.g., [7] and [3]:

Question 1.3. *Suppose that $\kappa \geq \omega_2$ is a regular cardinal. Does ITP_κ (or TP_κ) imply SCH above κ?*

The analogue of Magidor's question is also of interest for these two-cardinal tree properties (see [3] for more discussion of this question):

Question 1.4. *Is it consistent that ITP_κ (or TP_κ) holds for all regular cardinals $\kappa \geq \omega_2$?*

Questions 1.3 and 1.4 have a tight connection with one another: by a theorem of Specker [26], if μ is a cardinal and $2^\mu = \mu^+$, then the tree property fails at μ^{++} and therefore, *a fortiori*, $\mathsf{TP}_{\mu^{++}}$ fails. Thus, if μ is a singular strong limit cardinal and SCH holds at μ, then $\mathsf{TP}_{\mu^{++}}$ fails, so a positive answer to Question 1.3 would entail a negative answer to Question 1.4.

Motivated by these question, we prove here some results that we feel hint at a positive answer to Question 1.3 or at least indicate that genuinely new ideas would be needed to establish a negative answer. This work will involve introducing and analyzing generalizations of the classical notion of a *narrow κ-system*, introduced by Magidor and Shelah [18] to facilitate study of the tree property, particularly at successors of singular cardinals.

In this paper, we generalize the notion of narrow system from the setting of cardinals κ to arbitrary directed partial orders Λ and show that these generalized system can play the same role in the study of generalized tree properties that narrow κ-systems play in the study of the classical tree property at κ. We introduce the generalized narrow system properties $\mathsf{NSP}(\Lambda)$, asserting that every narrow Λ-system has a cofinal branch, and study these properties, particularly in relation to their connections to Questions 1.3 and 1.4.

In Section 2, before introducing narrow systems in their full generality, we define a specific type of system, which we call a *concrete $\mathscr{P}_\kappa\lambda$-system*, that is particularly relevant to the study of $\mathsf{TP}(\kappa, \lambda)$. The narrow system property $\mathsf{cNSP}(\mathscr{P}_\kappa\lambda)$ then asserts that every narrow concrete $\mathscr{P}_\kappa\lambda$-system has a cofinal branch, and cNSP_κ denotes the assertion that $\mathsf{cNSP}(\mathscr{P}_\kappa\lambda)$ holds for all $\lambda \geq \kappa$. cNSP_κ holds in all known models of TP_κ and is often used, at least implicitly, in verifications that TP_κ holds in a given model, especially if κ is the successor of a singular cardinal. At the same time, we show that this narrow system property is strong enough to imply instances of SSH:

Theorem A. *Suppose that $\kappa \geq \omega_2$ is a regular cardinal and cNSP_κ holds. Then SSH holds above κ.*

Then, in Section 3, we introduce the notion of narrow Λ-systems and the narrow Λ-system property ($\mathsf{NSP}(\Lambda)$) for an arbitrary directed partial order Λ and, as an illustration of their utility, use them to prove that generalized tree properties hold at successors of singular limits of strongly compact cardinals (Theorem 3.12).

In Section 4, we connect narrow Λ-systems with strongly unbounded subadditive colorings, proving both that instances of $\mathsf{NSP}(\Lambda)$ entail the nonexistence of such functions on $\Lambda^{[2]}$ and, in turn, that the nonexistence of such functions on $(\mathscr{P}_\kappa\lambda)^{[2]}$ can be used in place of cNSP_κ in the hypothesis of Theorem A (cf. Corollaries 4.3 and 4.9, respectively).

The remainder of the paper is devoted to a global consistency result showing that Question 1.4 has a positive answer if the two-cardinal tree properties are replaced by generalized narrow system properties:

Theorem B. *Suppose that there is a proper class of supercompact cardinals. Then there is a (class) forcing extension in which $\mathsf{NSP}(\Lambda)$ holds for every directed partial order Λ.*

Section 5 contains the proof of a technical branch preservation lemma for generalized narrow systems, and then Section 6 applies this lemma to prove Theorem B.

1.1. **Notational conventions.** Unless otherwise noted, we follow standard set theoretic notational conventions and refer the reader to [9] for any undefined notions. On denotes the class of all ordinals. Given an infinite cardinal κ and a set X, $\mathscr{P}(X)$ denotes the power set of X, and $\mathscr{P}_\kappa X$ denotes $\{x \subseteq X \mid |x| < \kappa\}$. If x is a set of ordinals, then the *strong supremum* of x is the ordinal $\mathrm{ssup}(x) := \sup\{\alpha + 1 \mid \alpha \in x\}$, i.e., $\mathrm{ssup}(x)$ is the least ordinal β such that $\alpha < \beta$ for all $\alpha \in x$. Given a partial order Λ, we let $\Lambda^{[2]}$ denote the set of ordered pairs (u, v) from Λ such that $u <_\Lambda v$. Sets of the form $\mathscr{P}_\kappa X$ will be interpreted as partial orders with the order relation given by $\subsetneq$. In particular, $(\mathscr{P}_\kappa X)^{[2]}$ denotes the set of pairs (x, y) of elements of $\mathscr{P}_\kappa X$ with $x \subsetneq y$. The *cofinality* of a partial order Λ, denoted $\mathrm{cf}(\Lambda)$, is the minimal cardinality of a subset $\Lambda_0 \subseteq \Lambda$ such that, for all $u \in \Lambda$, there is $v \in \Lambda_0$ with $u \leq_\Lambda v$.

2. Concrete systems

Before we introduce the general notion of a *(narrow) Λ-system* for an arbitrary directed order Λ, and in order to help motivate the more abstract general definition, we first consider an important special case.

Definition 2.1. Suppose that $\kappa \leq \lambda$ are uncountable cardinals, with κ regular. A *concrete $\mathscr{P}_\kappa\lambda$-system* is a structure $\mathcal{S} = \langle S_x \mid x \in A\rangle$ such that

(1) A is a $\subseteq$-cofinal subset of $\mathscr{P}_\kappa\lambda$;
(2) for all $x \in A$, $\emptyset \neq S_x \subseteq \mathscr{P}(x)$;
(3) for all $x \subseteq y$, both in A, there is $t \in S_y$ such that $t \cap x \in S_x$.

The *width* of $\mathcal{S}$ is defined to be $\text{width}(\mathcal{S}) := \sup\{|S_x| \mid x \in A\}$. We say that $\mathcal{S}$ is a *narrow* concrete $\mathscr{P}_\kappa\lambda$-system if $\text{width}(\mathcal{S})^+ < \kappa$. A *cofinal branch* through $\mathcal{S}$ is a set $b \subseteq \lambda$ such that the set $\{x \in A \mid b \cap x \in S_x\}$ is $\subseteq$-cofinal in $\mathscr{P}_\kappa\lambda$.

Classical narrow systems, with levels indexed by ordinals, were introduced by Magidor and Shelah in [18] as a central tool in the study of the tree property, particularly at successors of singular cardinals. Indeed, all known verifications of the tree property at the successor of a singular cardinal μ at least implicitly go through the following two steps:

(1) Show that every μ^+-tree $\mathcal{T}$ has a narrow subsystem $\mathcal{S}$ of height μ^+.
(2) Show that every narrow system of height μ^+ has a cofinal branch; in particular, $\mathcal{S}$ has a cofinal branch, which gives rise to a cofinal branch through $\mathcal{T}$.

One of the motivating observations for this paper is that narrow concrete $\mathscr{P}_\kappa\lambda$-systems play an analogous role for (κ, λ)-trees. For example, by an analogue of the two-step argument outlined above, we can show that the two-cardinal tree property TP_κ holds if κ is the successor of a singular limit of strongly compact cardinals. Since a more general version of this statement is true, we postpone its proof until after we introduce the more general definition of "narrow system"; it follows as a special case of Theorem 3.12 below.

We now turn to showing that the existence of cofinal branches through certain narrow concrete systems implies instances of SCH (and SSH). To state the results concisely, we introduce the following terminology.

Definition 2.2. Let κ be a regular uncountable cardinal. For a cardinal $\lambda \geq \kappa$, we say that the *concrete narrow $\mathscr{P}_\kappa\lambda$-system property* holds (denoted $\mathsf{cNSP}(\mathscr{P}_\kappa\lambda)$) if every narrow concrete $\mathscr{P}_\kappa\lambda$-system has a cofinal branch. We say that cNSP_κ holds if $\mathsf{cNSP}(\mathscr{P}_\kappa\lambda)$ holds for all $\lambda \geq \kappa$.

Remark 2.3. It is worth taking the time to compare Definitions 2.1 and 2.2 with Definition 1.1, as the definitions of narrow concrete $\mathscr{P}_\kappa\lambda$-systems and thin (κ, λ)-trees are quite similar. The two salient differences are:

- The definition of narrow concrete $\mathscr{P}_\kappa\lambda$-system is more restrictive with regards to the size of each level, requiring $\text{width}(\mathcal{S})^+ < \kappa$, whereas a thin (κ, λ)-tree $\mathcal{T}$ is only required to satisfy $|T_x| < \kappa$ for all $x \in \mathscr{P}_\kappa\lambda$.
- On the other hand, the definition of thin (κ, λ)-tree is more restrictive with regards to the *coherence properties* of the structure, requiring that, for all $x \subseteq y$ and *all* $t \in T_y$, we have $t \cap x \in T_x$, whereas the analogous requirement in the definition of narrow concrete $\mathscr{P}_\kappa\lambda$-system only requires the existence of one such t.

Therefore, it is not immediately evident whether either TP_κ or cNSP_κ implies the other, though we shall see that, in general, cNSP_κ is easier to arrange than, and does not imply, TP_κ (cf. Remark 6.3 below). The question of whether TP_κ implies cNSP_κ remains open and very much of interest.

Our verifications of SSH will go through the machinery of *covering matrices* introduced by Viale in his proof that SCH follows from the Proper Forcing Axiom [27].

Definition 2.4. Let $\theta < \lambda$ be regular cardinals. A *θ-covering matrix for λ* is a matrix $\mathcal{D} = \langle D(i,\beta) \mid i < \theta,\ \beta < \lambda\rangle$ such that:

(1) for all $\beta < \lambda$, $\langle D(i,\beta) \mid i < \theta\rangle$ is a $\subseteq$-increasing sequence and $\bigcup_{i<\theta} D(i,\beta) = \beta$;
(2) for all $\beta < \gamma < \lambda$ and $i < \theta$, there is $j < \theta$ such that $D(i,\beta) \subseteq D(j,\gamma)$.

We will especially be interested in covering matrices satisfying certain additional properties.

Definition 2.5. Suppose that $\theta < \lambda$ are regular cardinals and $\mathcal{D}$ is a θ-covering matrix for λ.

(1) $\mathcal{D}$ is *transitive* if, for all $\beta < \gamma < \lambda$ and all $i < \theta$, if $\beta \in D(i,\gamma)$, then $D(i,\beta) \subseteq D(i,\gamma)$.
(2) $\mathcal{D}$ is *uniform* if, for every limit ordinal $\beta < \lambda$, there is $i < \theta$ such that $D(i,\beta)$ contains a club in β.
(3) $\mathsf{CP}(\mathcal{D})$ holds if there is an unbounded $A \subseteq \lambda$ such that $[A]^\theta$ is covered by $\mathcal{D}$, i.e., for all $X \in [A]^\theta$, there are $\beta < \lambda$ and $i < \theta$ for which $X \subseteq D(i,\beta)$.

The following fact can readily be deduced from the proof of [21, Lemma 2.4] (cf. also [13, Lemma 4.4])

Fact 2.6. Suppose that μ is a singular cardinal, $\theta = \mathrm{cf}(\mu)$, and $\langle \mu_i \mid i < \theta\rangle$ is an increasing sequence of regular cardinals that is cofinal in μ. Then there exists a uniform, transitive, θ-covering matrix $\mathcal{D} = \langle D(i,\beta) \mid i < \theta,\ \beta < \mu^+\rangle$ for μ^+ such that $|D(i,\beta)| < \mu_i$ for all $i < \theta$ and $\beta < \mu^+$.

The following theorem is proven in [13] (it was previously known in the case in which μ is strong limit (cf. [28, Lemma 6])).

Theorem 2.7 ([13, Lemma 4.7])**.** *Suppose that μ is a singular cardinal, $\theta = \mathrm{cf}(\mu)$, and $\mathcal{D}$ is a uniform, transitive θ-covering matrix for μ^+. Then, for every $x \in \mathscr{P}_\mu \mu^+$, there is $\gamma_x < \mu^+$ such that, for all $\beta \in [\gamma_x, \mu^+)$, there is $i < \theta$ such that, for all $j \in [i,\theta)$, we have $x \cap D(j,\beta) = x \cap D(j,\gamma_x)$.*

We will also need to recall some basic information about Shelah's Strong Hypothesis. SSH is the assertion that $\mathrm{pp}(\mu) = \mu^+$ for every singular cardinal μ, where $\mathrm{pp}(\mu)$ denotes the *pseudopower* of μ. For a cardinal κ, we say that SSH holds *above* κ if $\mathrm{pp}(\mu) = \mu^+$ for every singular cardinal $\mu > \kappa$. For our purposes, we will not need to recall the definition of $\mathrm{pp}(\mu)$; the following facts will suffice:

Fact 2.8. In what follows, if $\vec{\mu} = \langle \mu_i \mid i < \theta\rangle$ is a sequence of regular cardinals, then $\prod \vec{\mu}$ denotes the set of functions f such that $\mathrm{dom}(f) = \theta$ and $f(i) < \mu_i$ for all $i < \theta$. Given $f, g \in \prod \vec{\mu}$, we say that $f <^* g$ if there is $i < \theta$ such that $f(j) < g(j)$ for all $j \in [i,\theta)$. The second and third facts below are both implicit in [22]; the cited references provide more explicit explanations.

(1) [22, §2, Claim 2.4] If μ is a singular cardinal of uncountable cofinality and $\{\nu < \mu \mid \mathrm{pp}(\nu) = \nu^+\}$ is stationary in μ, then $\mathrm{pp}(\mu) = \mu^+$.
(2) [19, Observation 4.4] Suppose that μ is a singular cardinal and $\mathrm{pp}(\mu) > \mu^+$. Then there is an increasing sequence of regular cardinals $\vec{\mu} = \langle \mu_i \mid i < \mathrm{cf}(\mu)\rangle$ converging to μ such that $\mathrm{cf}(\prod \vec{\mu}, <^*) > \mu^+$.

(3) [13, Proposition 4.18] Let κ be an infinite cardinal such that SSH holds above κ. Then SCH holds above κ.

The connection between covering matrices and SSH comes via the following result.

Theorem 2.9. [13, Theorem 4.19] *Suppose that μ is a singular cardinal, $\theta = \mathrm{cf}(\mu)$, and $\vec{\mu} = \langle \mu_i \mid i < \theta \rangle$ is an increasing sequence of regular cardinals converging to μ. Suppose moreover that $\mathcal{D} = \langle D(i, \beta) \mid i < \theta,\ \beta < \mu^+ \rangle$ is a θ-covering matrix for μ^+ such that*

(1) for all $i < \theta$ and $\beta < \mu^+$, we have $|D(i, \beta)| < \mu_i$;
(2) $\mathsf{CP}(\mathcal{D})$ holds.

Then $\mathrm{cf}(\prod \vec{\mu}, <^*) = \mu^+$.

We are now ready for the main result of this section, which will then yield Theorem A.

Theorem 2.10. *Suppose that μ is a singular cardinal, $\theta = \mathrm{cf}(\mu)$, and there is a regular cardinal $\kappa \in [\theta^{++}, \mu)$ such that $\mathsf{cNSP}(\mathscr{P}_\kappa \mu^+)$ holds. Then $\mathsf{CP}(\mathcal{D})$ holds for every uniform, transitive θ-covering matrix $\mathcal{D}$ for μ^+.*

Proof. Fix a uniform, transitive θ-covering matrix $\mathcal{D} = \langle D(i, \beta) \mid i < \theta,\ \beta < \mu^+ \rangle$ for μ^+, and let $A := \{x \in \mathscr{P}_\kappa \mu^+ \mid \mathrm{cf}(\mathrm{ssup}(x)) > \theta\}$. Since $\kappa > \theta^+$ is a regular cardinal, A is cofinal in $\mathscr{P}_\kappa \mu^+$; note that, for $x \in A$, we have $\sup(x) = \mathrm{ssup}(x)$. For each $x \in A$, let $\gamma_x < \mu^+$ be the least ordinal satisfying the conclusion of Theorem 2.7; namely, for all $\beta \in [\gamma_x, \mu^+)$ and all sufficiently large $j < \theta$, we have $x \cap D(j, \beta) = x \cap D(j, \gamma_x)$. Note that we must have $\gamma_x \geq \sup(x)$ and, if $x \subseteq y$ are both in A, then $\gamma_x \leq \gamma_y$.

For each $x \in A$, let

$$S_x := \{x \cap D(i, \gamma_x) \mid i < \theta \text{ and} \sup(x \cap D(i, \gamma_x)) = \sup(x)\}.$$

Since $\mathrm{cf}(\sup(x)) > \theta$ and $x = \bigcup_{i<\theta}(x \cap D(i, \gamma_x))$, it must be the case that $x \cap D(i, \gamma_x) \in S_x$ for all sufficiently large $i < \theta$.

We claim that $\mathcal{S} = \langle S_x \mid x \in A \rangle$ is a concrete $\mathscr{P}_\kappa \mu^+$-system. We have already verified clauses (1) and (2) of Definition 2.1. To verify clause (3), fix $x \subseteq y$, both in A. By construction, for all sufficiently large $i < \theta$, we have $y \cap D(i, \gamma_y) \in S_y$ and

$$(y \cap D(i, \gamma_y)) \cap x = D(i, \gamma_y) \cap x = D(i, \gamma_x) \cap x \in S_x,$$

where the second equality holds by the choice of γ_x. Therefore, we have found $t \in S_y$ for which $t \cap x \in S_x$, as desired.

Moreover, we have $|S_x| \leq \theta$ for all $x \in A$, so $\mathcal{S}$ is a *narrow* concrete $\mathscr{P}_\kappa \mu^+$-system. We can therefore apply $\mathsf{cNSP}(\mathscr{P}_\kappa \mu^+)$ to find a cofinal branch b through $\mathcal{S}$.

Claim 2.11. *b is unbounded in μ^+.*

Proof. Fix $\alpha < \mu^+$; we will show that $b \setminus \alpha$ is nonempty. Find $x \in A$ such that $\alpha \in x$ and $b \cap x \in S_x$. By the definition of S_x, it follows that $\sup(b \cap x) = \sup(x) > \alpha$. □

We will therefore be done if we show that $[b]^\theta$ is covered by $\mathcal{D}$, as then b will witness $\mathsf{CP}(\mathcal{D})$. To this end, fix $z \in [b]^\theta$. Since b is a cofinal branch through $\mathcal{S}$, we can find $x \in A$ such that $z \subseteq x$ and $b \cap x \in S_x$. Then $z \subseteq b \cap x$, and there is $i < \theta$ such that $b \cap x = x \cap D(i, \gamma_x)$; therefore, $z \subseteq D(i, \gamma_x)$, as desired. □

We are now ready to prove Theorem A, asserting that, for a regular cardinal $\kappa \geq \omega_2$, cNSP_κ implies SSH above κ.

Proof of Theorem A. By Fact 2.8(1), to establish SSH above κ, it suffices to show that $\mathrm{pp}(\mu) = \mu^+$ for every singular cardinal $\mu > \kappa$ of countable cofinality. Fix such a μ. Next, by Fact 2.8(2), to establish $\mathrm{pp}(\mu) = \mu^+$, it suffices to prove that $\mathrm{cf}(\prod \vec{\mu}, <^*) = \mu^+$ for every increasing sequence of regular cardinals $\vec{\mu} = \langle \mu_i \mid i < \omega \rangle$ converging to μ. Fix such a sequence $\vec{\mu}$. By Fact 2.6, there is a uniform, transitive ω-covering matrix $\mathcal{D} = \langle D(i, \beta) \mid i < \omega,\ \beta < \mu^+ \rangle$ for μ^+ such that $|D(i, \beta)| < \mu_i$ for all $i < \omega$ and $\beta < \mu^+$. By Theorem 2.10 and the assumption that cNSP_κ holds, we know that $\mathsf{CP}(\mathcal{D})$ holds, and then, by Theorem 2.9, we have $\mathrm{cf}(\prod \vec{\mu}, <^*) = \mu^+$, as desired. □

3. General systems

We now move to the more general setting of systems indexed by arbitrary directed partial orders. Given a partial order $(\Lambda, \leq_\Lambda)$, we will sometimes abuse notation and use the symbol Λ to denote the partial order. If a partial order is denoted by Λ, it should be understood that its order relation is denoted by $\leq_\Lambda$. The strict portion of $\leq_\Lambda$ will be denoted by $<_\Lambda$. Given $u \in \Lambda$, let $u^\uparrow$ denote $\{v \in \Lambda \mid u <_\Lambda v\}$. Since all of the questions considered here become trivial when addressing systems indexed by partial orders with maximal elements, we will always assume that we are working with partial orders that do not have maximal elements, even when this assumption is not explicitly stated.

Definition 3.1. Suppose that Λ is a partial order and κ is an infinite cardinal. We say that Λ is *κ-directed* if every element of $\mathscr{P}_\kappa \Lambda$ has an upper bound, i.e., for every $x \in \mathscr{P}_\kappa \Lambda$, there is $v \in \Lambda$ such that $u \leq_\Lambda v$ for all $u \in x$. We say that Λ is *directed* if it is $\aleph_0$-directed; equivalently, for all $u, v \in \Lambda$, there is $w \in \Lambda$ such that $u, v \leq_\Lambda w$.

Definition 3.2. Suppose that Λ is a directed partial order. The *directedness* of Λ, denoted d_Λ, is the largest cardinal κ such that Λ is κ-directed. It is readily verified that this is well-defined and that d_Λ is a regular cardinal for every directed partial order Λ.

Definition 3.3. Let R be a binary relation on a set X. For $x, y \in X$, we will typically write $x <_R y$ to denote $(x, y) \in R$ and $x \leq_R y$ to denote the statement

$$(x, y) \in R \text{ or } x = y.$$

Two elements x and y of X are said to be *R-comparable* if either $x \leq_R y$ or $y \leq_R x$. Otherwise, x and y are *R-incomparable*.

Definition 3.4. Let Λ be a directed partial order. A *Λ-system* is a structure

$$\mathcal{S} = \langle \langle S_u \mid u \in \Lambda \rangle, \mathcal{R} \rangle$$

satisfying the following conditions.

(1) $\langle S_u \mid u \in \Lambda \rangle$ is a sequence of pairwise disjoint nonempty sets. We will sometimes refer to $\bigcup_{u \in \Lambda} S_u$ as the *underlying set* of $\mathcal{S}$, and we will sometimes simply denote it by S. For each $x \in S$, let $\ell(x)$ denote the unique $u \in \Lambda$ such that $x \in S_u$.

(2) $\mathcal{R}$ is a nonempty set of binary, transitive relations on S.

(3) For all $x, y \in S$ and $R \in \mathcal{R}$, if $x <_R y$, then $\ell(x) <_\Lambda \ell(y)$.
(4) For all $x, y, z \in S$ and $R \in \mathcal{R}$, if $x, y <_R z$ and $\ell(x) \leq_\Lambda \ell(y)$, then $x \leq_R y$.
(5) For all $(u, v) \in \Lambda^{[2]}$, there are $x \in S_u$, $y \in S_v$, and $R \in \mathcal{R}$ such that $x <_R y$.

If $\mathcal{S}$ is a Λ-system, then we define $\mathrm{width}(\mathcal{S})$ to be $\max\{\sup\{|S_u| \mid u \in \Lambda\}, |\mathcal{R}|\}$. We say that $\mathcal{S}$ is a *narrow Λ-system* if $\mathrm{width}(\mathcal{S})^+ < d_\Lambda$.

Definition 3.5. If $\mathcal{S} = \langle\langle S_u \mid u \in \Lambda\rangle, \mathcal{R}\rangle$ is a Λ-system, $x, y \in S$, and $R \in \mathcal{R}$, then we say that x and y are *R-compatible*, denoted $x \parallel_R y$, if there is $z \in S$ such that $x, y \leq_R z$. We say that x and y are *R-incompatible*, denoted $x \perp_R y$, if there is no such z. Note that, if $x \parallel_R y$ and $\ell(x) \leq_\Lambda \ell(y)$, then Clause 4 of Definition 3.4 implies that $x \leq_R y$.

Given $R \in \mathcal{R}$, a *branch through R in $\mathcal{S}$* is a set $b \subseteq S$ such that, for all $x, y \in b$, we have $x \parallel_R y$ (note that this implies that $|b \cap S_u| \leq 1$ for all $u \in \Lambda$). We will sometimes say that b is a *branch in $\mathcal{S}$* to mean that there is $R \in \mathcal{R}$ such that b is a branch through R in $\mathcal{S}$. A branch b is said to be *cofinal* if $\{u \in \Lambda \mid b \cap S_u \neq \emptyset\}$ is cofinal in Λ.

Remark 3.6. The concrete $\mathscr{P}_\kappa\lambda$-systems of Section 2 are indeed special cases of Definition 3.4: suppose that $\mathcal{S} = \langle S_x \mid x \in A\rangle$ is a concrete $\mathscr{P}_\kappa\lambda$-system. Then there is a natural way to view $\mathcal{S}$ as an $(A, \subseteq)$-system in the sense of Definition 3.4. Namely, for each $x \in A$, let $S'_x := \{x\} \times S_x$, and define a binary relation R on $\bigcup_{x \in A} S'_x$ by letting $(x, t) <_R (y, s)$ iff $x \subsetneq y$ and $s \cap x = t$. Then $\mathcal{S}' := \langle\langle S'_x \mid x \in A\rangle, \{R\}\rangle$ is readily verified to be an $(A, \subseteq)$-system in the sense of Definition 3.4, and cofinal branches through $\mathcal{S}$ in the sense of Definition 2.1 naturally correspond to cofinal branches through $\mathcal{S}'$ in the sense of Definition 3.4.

Definition 3.7. Let Λ be a directed partial order. We say that the *Λ-narrow system property* (denoted $\mathsf{NSP}(\Lambda)$) holds if every narrow Λ-system has a cofinal branch.

For notational simplicity, we often prefer to work with systems having only one relation. The following proposition shows that, in the context of questions about the existence of narrow Λ-systems without cofinal branches, this involves no loss of generality.

Proposition 3.8. *Suppose that Λ is a directed partial order and*

$$\mathcal{S} = \langle\langle S_u \mid u \in \Lambda\rangle, \mathcal{R}\rangle$$

is a Λ-system. Then there is a Λ-system $\mathcal{S}' = \langle\langle S'_u \mid u \in \Lambda\rangle, \mathcal{R}'\rangle$ such that

- $|\mathcal{R}'| = 1$*;*
- *if* $\mathrm{width}(\mathcal{S})$ *is finite, then so is* $\mathrm{width}(\mathcal{S}')$*;*
- *if* $\mathrm{width}(\mathcal{S})$ *is infinite, then* $\mathrm{width}(\mathcal{S}') = \mathrm{width}(\mathcal{S})$*;*
- *$\mathcal{S}'$ has a cofinal branch if and only if $\mathcal{S}$ has a cofinal branch.*

Proof. For each $u \in \Lambda$, let $S'_u := S_u \times \mathcal{R}$, and let $\mathcal{R}'$ consist of a single binary relation $<'$ defined as follows: for all $x_0, x_1 \in S$ and $R_0, R_1 \in \mathcal{R}$, let $(x_0, R_0) <' (x_1, R_1)$ if and only if $R_0 = R_1$ and $x_0 <_{R_0} x_1$. It is readily verified that $\mathcal{S}'$ thus defined is a Λ-system $\mathrm{width}(\mathcal{S}')$ is as required. If $R \in \mathcal{R}$ and $b \subseteq S$ is a cofinal branch through R in $\mathcal{S}$, then $b' := \{(x, R) \mid x \in b\}$ is a cofinal branch in $\mathcal{S}'$. Conversely, if d' is a cofinal branch in $\mathcal{S}'$, then there must be a single $R \in \mathcal{R}$ such that every element of d' is of the form (x, R) for some $x \in S$. Then $d := \{x \in S \mid (x, R) \in d'\}$ is a cofinal branch through R in $\mathcal{S}$. □

The following basic proposition is reminiscent of König's Infinity Lemma, asserting that every infinite finitely-branching tree has an infinite branch.

Proposition 3.9. *Suppose that Λ is a directed partial order and $\mathcal{S}$ is a Λ-system with finite width. Then $\mathcal{S}$ has a cofinal branch.*

Proof. By Proposition 3.8, we can assume that $\mathcal{S}$ has a single relation, which we will denote by R. Since width$(\mathcal{S})$ is finite, we can fix an $n < \omega$ such that $|S_u| \leq n$ for all $u \in \Lambda$. Enumerate each S_u as $\langle x_{u,k} \mid k < n \rangle$, with repetitions if necessary (i.e., if $|S_u| < n$). Since Λ is directed, $\mathcal{F} := \{u^\uparrow \mid u \in \Lambda\}$ is a filter over Λ. Let $\mathcal{U}$ be an ultrafilter over Λ extending $\mathcal{F}$.

Temporarily fix $u \in \Lambda$. Since $\mathcal{S}$ is a Λ-system, it follows that, for every $v \in u^\uparrow$, we can find (not necessarily unique) $j(u,v), k(u,v) < n$ such that $x_{u,j(u,v)} <_R x_{v,k(u,v)}$. Since $\mathcal{U}$ is an ultrafilter extending $\mathcal{F}$, we can then find fixed numbers $j(u), k(u) < n$ such that the set

$$X_u := \{v \in u^\uparrow \mid (j(u,v), k(u,v)) = (j(u), k(u))\}$$

is in $\mathcal{U}$.[2] We can then find fixed numbers $j^*, k^* < n$ such that the set

$$Y := \{u \in \Lambda \mid (j(u), k(u)) = (j^*, k^*)\}$$

is in $\mathcal{U}$. In particular, Y is cofinal in Λ. Let $b := \{x_{u,j^*} \mid u \in Y\}$. Since Y is cofinal in Λ, in order to show that b is a cofinal branch in $\mathcal{S}$ it suffices to show that, for all $u_0, u_1 \in Y$, we have $x_{u_0,j^*} \parallel_R x_{u_1,j^*}$. To this end, fix such u_0, u_1. Since $X_{u_0}, X_{u_1} \in \mathcal{U}$, we can fix $v \in X_{u_0} \cap X_{u_1}$. Then $x_{u_0,j^*}, x_{u_1,j^*} <_R x_{v,k^*}$, so $x_{u_0,j^*} \parallel_R x_{u_1,j^*}$, as desired. $\square$

An analogous result holds at strongly compact cardinals:

Proposition 3.10. *Suppose that κ is a strongly compact cardinal, Λ is a directed partial order with $d_\Lambda \geq \kappa$, and $\mathcal{S}$ is a Λ-system such that* width$(\mathcal{S}) < \kappa$. *Then $\mathcal{S}$ has a cofinal branch.*

Proof. The proof is essentially the same as that of Proposition 3.9 and is thus mostly left to the reader. We remark only that, due to the fact that $d_\Lambda \geq \kappa$, the filter $\mathcal{F} := \{u^\uparrow \mid u \in \Lambda\}$ is κ-complete and, since κ is strongly compact, it can be extended to a κ-complete ultrafilter $\mathcal{U}$ over Λ. The rest of the proof is precisely as in Proposition 3.9. $\square$

As mentioned already, classical narrow systems were introduced by Magidor and Shelah in the context of the study of the tree property at successors of singular cardinals; their first application came in the proof that, if μ is a singular limit of strongly compact cardinals, then the tree property holds at μ^+ [18, Theorem 3.1]. To help get a feel for the utility of narrow Λ-systems, we present here the analogous result in the more general setting. We first need to recall the notion of a κ-Λ-tree for an arbitrary directed partial order Λ.

Definition 3.11 ([14])**.** Let Λ be a directed partial order. A Λ-*tree* is a structure $\mathcal{T} = (\langle T_u \mid u \in \Lambda \rangle, <_\mathcal{T})$ such that the following conditions all hold.

(i) $\langle T_u \mid u \in \Lambda \rangle$ is a sequence of nonempty, pairwise disjoint sets.
(ii) $<_\mathcal{T}$ is a strict partial order on $\bigcup_{u \in \Lambda} T_u$.

[2] We are using the implicit assumption that Λ has no maximal element to ensure that we can find such $j(u)$ and $k(u)$.

(iii) For all $u, v \in \Lambda$, all $s \in T_u$, and all $t \in T_v$, if $s <_{\mathcal{T}} t$, then $u <_\Lambda v$.
(iv) $<_{\mathcal{T}}$ is *tree-like*, i.e., for all $u <_\Lambda v <_\Lambda w$, all $r \in T_u$, all $s \in T_v$ and all $t \in T_w$, if $r, s <_{\mathcal{T}} t$, then $r <_{\mathcal{T}} s$.
(v) For all $u \leq_\Lambda v$ in Λ and all $t \in T_v$, there is a unique $s \in T_u$, denoted $t \restriction u$, such that $s \leq_{\mathcal{T}} t$.

For a cardinal κ, we say that $\mathcal{T}$ is a κ-Λ-tree if, in addition to the above requirements, we have $|T_u| < \kappa$ for all $u \in \Lambda$. If $\mathcal{T}$ is a Λ-tree, then a *cofinal branch* through $\mathcal{T}$ is a function $b \in \prod_{u \in \Lambda} T_u$ such that, for all $u <_\Lambda v$ in Λ, we have $b(u) <_{\mathcal{T}} b(v)$. The (κ, Λ)*-tree property*, denoted $\mathsf{TP}_\kappa(\Lambda)$, is the assertion that every κ-Λ-tree has a cofinal branch. We let $\mathsf{TP}(\Lambda)$ denote $\mathsf{TP}_{d_\Lambda}(\Lambda)$.

Theorem 3.12. *Suppose that μ is a singular limit of strongly compact cardinals and Λ is a μ^+-directed partial order. Then $\mathsf{TP}_{\mu^+}(\Lambda)$ holds.*

Proof. Let $\theta := \mathrm{cf}(\mu)$, and let $\langle \mu_i \mid i < \theta \rangle$ be an increasing sequence of strongly compact cardinals, converging to μ, with $\mu_0 > \theta$. Let $\mathcal{T} = \langle \langle T_u \mid u \in \Lambda \rangle, <_{\mathcal{T}} \rangle$ be a Λ-tree with $|T_u| \leq \mu$ for all $u \in \Lambda$. We first show that $\mathcal{T}$ has a narrow subsystem indexed by a cofinal subset of Λ.

Claim 3.13. *There is a cofinal $\Gamma \subseteq \Lambda$ and, for each $u \in \Gamma$, a nonempty $S_u \subseteq T_u$ such that $\mathcal{S} := \langle \langle S_u \mid u \in \Gamma \rangle, \{<_{\mathcal{S}}\} \rangle$ is a narrow Γ-system, where $<_{\mathcal{S}}$ is the restriction of $<_{\mathcal{T}}$ to $\bigcup_{u \in \Gamma} S_u$.*

Proof. For all $u \in \Lambda$, enumerate T_u as $\langle t^u_\eta \mid \eta < \mu \rangle$ (with repetitions, if necessary). Fix an elementary embedding $j : V \to M$ witnessing that μ_0 is $|\Lambda|$-strongly compact. In particular, we have

- $\mathrm{crit}(j) = \mu_0$;
- $j(\mu_0) > |\Lambda|$;
- there is $W \in M$ such that $W \subseteq j(\Lambda)$, $|W|^M < j(\mu_0)$, and $j``\Lambda \subseteq W$.

Let $j(\mathcal{T}) = \mathcal{T}' = \langle T'_v \mid v \in j(\Lambda) \rangle$. Since $|W|^M < j(\mu_0) < j(\mu^+)$ and $j(\Lambda)$ is $j(\mu^+)$-directed in M, we can find $z \in j(\Lambda)$ such that $w <_{j(\Lambda)} z$ for all $w \in W$; in particular, $j(u) <_{j(\Lambda)} z$ for all $u \in \Lambda$. Choose an arbitrary $t \in T'_z$. For each $u \in \Lambda$, enumerate $T'_{j(u)}$ as $\langle (t')^{j(u)}_\eta \mid \eta < j(\mu) \rangle$. For each $u \in \Lambda$, there is $i_u < \theta$ and $\eta_u < j(\mu_{i_u})$ such that $(t')^{j(u)}_{\eta_u} <_{j(\mathcal{T})} t$. Since Λ is μ^+-directed, we can find a fixed $i < \theta$ and a cofinal $\Gamma \subseteq \Lambda$ such that $i_u = i$ for all $u \in \Gamma$. Then, for all $u <_\Lambda v$, both in Γ, we have $(t')^{j(u)}_{\eta_u}, (t')^{j(v)}_{\eta_v} <_{j(\mathcal{T})} t$, and hence $(t')^{j(u)}_{\eta_u} <_{j(\mathcal{T})} (t')^{j(v)}_{\eta_v}$. In particular,

$$M \models \exists \eta, \xi < j(\mu_i) \left[(t')^{j(u)}_\eta <_{j(\mathcal{T})} (t')^{j(v)}_\xi \right],$$

as witnessed by $\eta = \eta_u$ and $\xi = \xi_v$. By elementarity, we have

$$V \models \exists \eta, \xi < \mu_i \left[t^u_\eta <_{\mathcal{T}} t^v_\xi \right].$$

It is now readily verified that, if we let $S_u := \{ t^u_\eta \mid \eta < \mu_i \}$ for all $u \in \Gamma$, then $\mathcal{S}$ as in the statement of the claim is indeed a narrow Γ-system: clauses (1)–(4) of Definition 3.4 are immediate, and clause (5) follows from the elementarity argument in the previous paragraph. □

Let $\mathcal{S}$ be as given by the claim, and let $i < \theta$ be such that $\mathrm{width}(\mathcal{S}) < \mu_i$. Then we can apply Proposition 3.10 with μ_i and Γ in place of κ and Λ, respectively, to conclude that $\mathcal{S}$ has a cofinal branch, $b \subseteq S$. This readily gives rise to a cofinal branch $b' \in \prod_{u \in \Lambda} T_u$ through $\mathcal{T}$: for each $u \in \Lambda$, find $v \in \Gamma$ such that $u \leq_\Lambda v$ and $b \cap S_v \neq \emptyset$. Let s be the unique element of $b \cap S_v$, and then let $b'(u) := s \restriction u$. □

4. Subadditive colorings

In this brief section, we highlight a connection between the narrow system properties introduced in the previous section and the existence of certain strongly unbounded subadditive colorings on arbitrary directed partial orders. Such colorings on *ordinals* have been extensively studied and have proven to be useful in a variety of contexts (cf. [12]). Here we generalize the notion to arbitrary directed orders, show that instances of the narrow system property imply the nonexistence of certain strongly unbounded subadditive colorings, and then show that the nonexistence of certain strongly unbounded subadditive colorings can replace the narrow system property hypothesis in the statement of Theorem A.

Definition 4.1. Suppose that Λ is a directed partial order and θ is an infinite regular cardinal. Let $c : \Lambda^{[2]} \to \theta$ be a function.

1. We say that c is *subadditive* if, for all triples $u <_\Lambda v <_\Lambda w$ from Λ, we have
   (a) $c(u,w) \leq \max\{c(u,v), c(v,w)\}$; and
   (b) $c(u,v) \leq \max\{c(u,w), c(v,w)\}$.
2. We say that c is *strongly unbounded* if, for every cofinal subset $\Gamma \subseteq \Lambda$, $c``\Gamma^{[2]}$ is unbounded in θ.

Proposition 4.2. *Suppose that Λ is a directed partial order, θ is an infinite regular cardinal, and $c : \Lambda^{[2]} \to \theta$ is a strongly unbounded subadditive function. Then there is a Λ-system with width θ and no cofinal branch.*

Proof. We will define a Λ-system $\mathcal{S} = \langle \langle S_u \mid u \in \Lambda \rangle, \mathcal{R} \rangle$. First, for each $u \in \Lambda$, let $S_u := \{u\} \times \theta$, and let $\mathcal{R} = \{R\}$ consist of a single relation defined as follows: for $(u,v) \in \Lambda^{[2]}$ and $i, j < \theta$, set $(u,i) <_R (v,j)$ if and only if $i = j$ and $c(u,v) \leq i$. The fact that $\mathcal{S}$ is a Λ-system follows from the subadditivity of c, and it is evident that $\mathrm{width}(\mathcal{S}) = \theta$. Now suppose for the sake of contradiction that $\mathcal{S}$ has a cofinal branch, b. Then b is necessarily of the form $\{(u,i) \mid u \in \Gamma\}$ for some fixed $i < \theta$ and some cofinal $\Gamma \subseteq \Lambda$. But then $c``\Gamma^{[2]} \subseteq i+1$, contradicting the fact that c is strongly unbounded. □

Corollary 4.3. *Suppose that Λ is a directed partial order and $\mathsf{NSP}(\Lambda)$ holds. Then, for every infinite regular cardinal θ with $\theta^+ < d_\Lambda$, there does not exist a strongly unbounded subadditive coloring $c : \Lambda^{[2]} \to \theta$.* □

We now show that the nonexistence of strongly unbounded subadditive colorings from $(\mathscr{P}_\kappa \mu^+)^{[2]}$ to $\mathrm{cf}(\mu)$ can be used in place of $\mathsf{cNSP}(\mathscr{P}_\kappa \mu^+)$ to yield the conclusion of Theorem 2.10.

Theorem 4.4. *Suppose that $\kappa < \mu$ are infinite cardinals such that*

- $\mathrm{cf}(\mu) < \kappa$; *and*
- *there does not exist a strongly unbounded subadditive coloring*

$$c : (\mathscr{P}_\kappa \mu^+)^{[2]} \to \mathrm{cf}(\mu).$$

Then $\mathsf{CP}(\mathcal{D})$ holds for every uniform, transitive $\mathrm{cf}(\mu)$-covering matrix for μ^+.

Proof. Let $\theta := \mathrm{cf}(\mu)$, and let $\mathcal{D} = \langle D(i,\beta) \mid i < \theta,\ \beta < \mu^+ \rangle$ be a uniform, transitive θ-covering matrix for μ^+. By Theorem 2.7, $\mathcal{D}$ has the property that, for every $x \in \mathscr{P}_\kappa \mu^+$, there is $\gamma_x < \mu^+$ such that, for all $\beta \in [\gamma_x, \mu^+)$, there is $i < \theta$ such that, for all $j \in [i, \theta)$, we have $x \cap D(j,\beta) = x \cap D(j,\gamma_x)$. For each $x \in \mathscr{P}_\kappa \mu^+$ and each $j < \theta$, let $x_j := x \cap D(j,\gamma_x)$. Note that $x = \bigcup_{j<\theta} x_j$.

Claim 4.5. *For all $(x, y) \in (\mathscr{P}_\kappa \mu^+)^{[2]}$, there is $i < \theta$ such that, for all $j \in [i, \theta)$, we have $x_j = y_j \cap x$.*

Proof. Fix $(x, y) \in (\mathscr{P}_\kappa \mu^+)^{[2]}$, and let $\gamma := \max\{\gamma_x, \gamma_y\}$. Then, by the definition of x_j and y_j and the choice of γ_x and γ_y, there is $i < \theta$ such that, for all $j \in [i, \theta)$, we have $x_j = x \cap D(j, \gamma)$ and $y_j = y \cap D(j, \gamma)$. But then, for all $j \in [i, \theta)$, we have $y_j \cap x = D(j, \gamma) \cap x = x_j$, as desired. □

Now define a function $c : (\mathscr{P}_\kappa \mu^+)^{[2]} \to \theta$ by letting $c(x, y)$ be the least $i < \theta$ as in Claim 4.5 for all $(x, y) \in (\mathscr{P}_\kappa \mu^+)^{[2]}$.

Claim 4.6. *c is subadditive.*

Proof. Fix $x \subsetneq y \subsetneq z$ in $\mathscr{P}_\kappa \mu^+$, and fix $j < \theta$. First, if $j \geq \max\{c(x, y), c(y, z)\}$, then $z_j \cap y = y_j$ and $y_j \cap x = x_j$. It follows that $z_j \cap x = x_j$, and from this we can conclude that $c(x, z) \leq \max\{c(x, y), c(y, z)\}$.

Second, if $j \geq \max\{c(x, z), c(y, z)\}$, then $z_j \cap y = y_j$ and $z_j \cap x = x_j$. It then follows that $y_j \cap x = (z_j \cap y) \cap x = z_j \cap x = x_j$, and again we can conclude that $c(x, y) \leq \max\{c(x, z), c(y, z)\}$. Therefore, c is subadditive. □

By assumption, c cannot be strongly unbounded. Therefore, there is a $\subseteq$-cofinal $X \subseteq \mathscr{P}_\kappa \mu^+$ and an $i < \theta$ such that $c(x, y) \leq i$ for all $x \subsetneq y$ in X.

Claim 4.7. *For all $j \in [i, \theta)$ and all $x, y \in X$, we have $x_j \cap y = y_j \cap x$.*

Proof. Fix such j, x, and y, and find $z \in X$ such that $x \cup y \subseteq z$. Then $x_j = z_j \cap x$ and $y_j = z_j \cap y$, so $x_j \cap y = (z_j \cap x) \cap y = (z_j \cap y) \cap x = y_j \cap x$. □

For all $j \in [i, \theta)$, let $A_j = \bigcup_{x \in X} x_j$. It follows immediately from the previous claim that, for all $x \in X$, we have $A_j \cap x = x_j$.

Claim 4.8. *There is $j \in [i, \theta)$ such that A_j is unbounded in μ^+.*

Proof. If not, then, for every $j \in [i, \theta)$, there would be $\beta_j < \mu^+$ such that $A_j \subseteq \beta_j$. Let $\beta := \sup\{\beta_j \mid j \in [i, \theta)\} < \mu^+$, and find $x \in X$ such that $\beta \in x$. Then, for all large enough $j < \theta$, we must have $\beta \in x_j$ and hence $\beta \in A_j$, contradicting the fact that $A_j \subseteq \beta_j \subseteq \beta$. □

Fix $j \in [i, \theta)$ such that A_j is unbounded in μ^+. We claim that A_j witnesses $\mathsf{CP}(\mathcal{D})$. To this end, fix $w \in [A_j]^\theta$. Let $x \in X$ be such that $w \subseteq x$. Then $w \subseteq A_j \cap x = x_j \subseteq D(j, \gamma_x)$, so $[A_j]^\theta$ is indeed covered by $\mathcal{D}$, as desired. □

Corollary 4.9. *Suppose that $\kappa \geq \omega_2$ is a regular cardinal and, for every singular cardinal $\mu > \kappa$ of countable cofinality, there does not exist a strongly unbounded subadditive coloring $c : (\mathscr{P}_\kappa \mu^+)^{[2]} \to \omega$. Then SSH holds above κ.*

Proof. By Theorem 4.4, the hypothesis implies that, for every singular cardinal $\mu > \kappa$ of countable cofinality and every uniform, transitive ω-covering matrix $\mathcal{D}$ for μ^+, we have $\mathsf{CP}(\mathcal{D})$. Then SSH above κ follows exactly as in proof of Theorem A at the end of Section 2. □

5. A preservation lemma

The remainder of the paper is dedicated to the proof of Theorem B, our global consistency result. In this section, we prove a technical preservation lemma indicating that if a sufficiently closed forcing adds a rich set of branches to a narrow Λ-system, then that system necessarily has a cofinal branch in the ground model. The lemma is a generalization of [11, Lemma 4.3] (which itself is a slight improvement on a previous result of Sinapova [24, Theorem 14]) from the context of classical (ordinal-indexed) narrow systems to the context of narrow Λ-systems for arbitrary directed orders Λ. We first need a preliminary definition.

Definition 5.1. Suppose that Λ is a directed partial order and

$$\mathcal{S} = \langle \langle S_u \mid u \in \Lambda \rangle, \mathcal{R} \rangle$$

is a Λ-system with $\mathrm{width}(\mathcal{S}) = \theta$. Then a *full set of branches in* $\mathcal{S}$ is a set $\{b_i \mid i < \theta\}$ such that

- for all $i < \theta$, b_i is a branch in $\mathcal{S}$;
- for all $u \in \Lambda$, there is $i < \theta$ such that $b_i \cap S_u \neq \emptyset$.

Proposition 5.2. *Suppose that* Λ *is a directed partial order,*

$$\mathcal{S} = \langle \langle S_u \mid u \in \Lambda \rangle, \mathcal{R} \rangle$$

is a Λ*-system with* $\mathrm{width}(\mathcal{S}) = \theta < d_\Lambda$*, and* $\{b_i \mid i < \theta\}$ *is a full set of branches in* $\mathcal{S}$*. Then there is* $i < \theta$ *such that* b_i *is a cofinal branch in* $\mathcal{S}$*.*

Proof. Suppose not. Then, for every $i < \theta$, there is $u_i \in \Lambda$ such that $b_i \cap S_v = \emptyset$ for all $v \in u_i^{\uparrow}$. Since $d_\Lambda > \theta$, we can find $u^* \in \Lambda$ such that $u_i \leq_\Lambda u^*$ for all $i < \theta$. Since Λ has no maximal element, $(u^*)^{\uparrow} \neq \emptyset$. However, for all $v \in (u^*)^{\uparrow}$ and all $i < \theta$, we have $b_i \cap S_v = \emptyset$, contradicting the fact that $\{b_i \mid i < \theta\}$ is a full set of branches. □

We are now ready for the main preservation lemma.

Lemma 5.3. *Suppose that* Λ *is a directed partial order,* $\mathcal{S}$ *is a narrow* Λ*-system,* $\theta = \mathrm{width}(S)$*,* $\mathbb{P}$ *is a* θ^+*-closed forcing poset, and*

$$\Vdash_{\mathbb{P}} \text{“}\textit{there is a full set of branches in } \mathcal{S}\text{”}.$$

Then, in V*, there is a cofinal branch in* $\mathcal{S}$*.*

Proof. Suppose for the sake of contradiction that there is no cofinal branch in $\mathcal{S}$. By assumption, we can fix $\mathbb{P}$-names $\{\dot{b}_i \mid i < \theta\}$ such that

$$\Vdash_{\mathbb{P}} \text{“}\{\dot{b}_i \mid i < \theta\} \text{ is a full set of branches in } \mathcal{S}\text{”}.$$

Using the θ^+-closure of $\mathbb{P}$, construct a decreasing sequence $\langle p_i \mid i < \theta \rangle$ of conditions in $\mathbb{P}$ such that, for each $i < \theta$:

- there is $R_i \in \mathcal{R}$ such that $p_i \Vdash_{\mathbb{P}}$ “$\dot{b}_i$ is a branch through R_i”;
- p_i decides the truth value of the statement “$\dot{b}_i$ is a cofinal branch in $\mathcal{S}$”;
- if $p_i \Vdash_{\mathbb{P}}$ “$\dot{b}_i$ is not a cofinal branch”, then there is $u_i \in \Lambda$ such that $p_i \Vdash_{\mathbb{P}}$ “$\forall v \in u_i^{\uparrow}\ (\dot{b}_i \cap S_v = \emptyset)$”.

Again using the θ^+-closure of $\mathbb{P}$, let p^* be a lower bound for $\langle p_i \mid i < \theta \rangle$. Let

$$A := \{i < \theta \mid p_i \Vdash_{\mathbb{P}} \text{“}\dot{b}_i \text{ is a cofinal branch”}\}$$

and, using the fact that $d_\Lambda > \theta$, find a $u^* \in \Lambda$ such that $u_i \leq_\Lambda u^*$ for all $i \in \theta \setminus A$. Note that, by Proposition 5.2, it must be the case that $A \neq \emptyset$.

Claim 5.4. *Suppose that $p \leq_{\mathbb{P}} p^*$ and $i \in A$. Then there are $q_0, q_1 \leq_{\mathbb{P}} p$ and $x_0, x_1 \in S$ such that*

(1) for $\varepsilon < 2$, $q_\varepsilon \Vdash_{\mathbb{P}}$ "$x_\varepsilon \in \dot{b}_i$";
(2) $x_0 \perp_{R_i} x_1$.

Proof. Suppose not, and let p and i form a counterexample. Let

$$b := \{x \in S \mid \exists q \leq_{\mathbb{P}} p \ (q \Vdash_{\mathbb{P}} \text{“}x \in \dot{b}_i\text{”})\}.$$

We claim that b is a cofinal branch through R_i in $\mathcal{S}$. Since $i \in A$ and $p \leq_{\mathbb{P}} p_i$, it is immediate that $\{u \in \Lambda \mid b \cap S_u \neq \emptyset\}$ is cofinal in Λ. Also, for all $x, y \in b$, our assumption that p and i form a counterexample to the claim implies that $x \parallel_{R_i} y$. Thus, b is a cofinal branch in $\mathcal{S}$, contradicting our assumption that no such branch exists. □

Claim 5.5. *Suppose that $p_0, p_1 \leq_{\mathbb{P}} p^*$ and $i \in A$. Then there are $q_0 \leq_{\mathbb{P}} p_0$, $q_1 \leq_{\mathbb{P}} p_1$, and $x_0, x_1 \in S$ such that*

(1) for $\varepsilon < 2$, $q_\varepsilon \Vdash_{\mathbb{P}}$ "$x_\varepsilon \in \dot{b}_i$";
(2) $x_0 \perp_{R_i} x_1$.

Proof. First apply Claim 5.4 to obtain $q_{0,0}, q_{0,1} \leq_{\mathbb{P}} p_0$ and $x_{0,0}, x_{0,1} \in S$ such that $q_{0,\varepsilon} \Vdash_{\mathbb{P}}$ "$x_{0,\varepsilon} \in \dot{b}_i$" for $\varepsilon < 2$ and $x_{0,0} \perp_{R_i} x_{0,1}$. Then find $q_1 \leq_{\mathbb{P}} p_1$ and $x_1 \in S$ such that $\ell(x_{0,0}), \ell(x_{0,1}) \leq_\Lambda \ell(x_1)$ and $q_1 \Vdash_{\mathbb{P}}$ "$x_1 \in \dot{b}_i$". It cannot be the case that x_1 is R_i-compatible with both $x_{0,0}$ and $x_{0,1}$, as otherwise x_1 would witness that $x_{0,0}$ and $x_{0,1}$ are R_i-compatible. Therefore, we can fix $\varepsilon < 2$ such that $x_{0,\varepsilon} \perp_{R_i} x_1$. Let $q_0 := q_{0,\varepsilon}$ and $x_0 := x_{0,\varepsilon}$. Then q_0, q_1, x_0, and x_1 are as desired. □

Claim 5.6. *Suppose that $p \leq p^*$. Then there are $q_0, q_1 \leq_{\mathbb{P}} p$ and $\{x^i_\varepsilon \mid i \in A,\ \varepsilon < 2\} \subseteq S$ such that*

(1) for every $i \in A$ and $\varepsilon < 2$, we have $q_\varepsilon \Vdash_{\mathbb{P}}$ "$x^i_\varepsilon \in \dot{b}_i$";
(2) for every $i \in A$, we have $x^i_0 \perp_{R_i} x^i_1$.

Proof. We recursively build two decreasing sequences $\langle q_{0,i} \mid i < \theta\rangle$ and $\langle q_{1,i} \mid i < \theta\rangle$ from $\mathbb{P}$, together with elements $\{x^i_\varepsilon \mid i \in A,\ \varepsilon < 2\}$ as follows.

First, let $q_{0,0} = q_{1,0} = p$. If $j < \theta$ is a limit ordinal, $\varepsilon < 2$, and we have defined $\langle q_{\varepsilon,i} \mid i < j\rangle$, then let $q_{\varepsilon,j}$ be any lower bound for $\langle q_{\varepsilon,i} \mid i < j\rangle$. If $i \in \theta \setminus A$, $\varepsilon < 2$, and $q_{\varepsilon,i}$ has been defined, then simply let $q_{\varepsilon,i+1} = q_{\varepsilon,i}$. Finally, suppose that $i \in A$ and we have defined $\langle q_{0,j} \mid j \leq i\rangle$ and $\langle q_{1,j} \mid j \leq i\rangle$. Then apply Claim 5.5 to $q_{0,i}$, $q_{1,i}$, and i to obtain $q_{0,i+1} \leq_{\mathbb{P}} q_{0,i}$, $q_{1,i+1} \leq_{\mathbb{P}} q_{1,i}$, and $x^i_0, x^i_1 \in S$ such that

- for $\varepsilon < 2$, $q_{\varepsilon,i+1} \Vdash_{\mathbb{P}}$ "$x^i_\varepsilon \in \dot{b}_i$";
- $x^i_0 \perp_{R_i} x^i_1$.

At the end of the construction, for each $\varepsilon < 2$, let q_ε be a lower bound for $\langle q_{\varepsilon,i} \mid i < \theta\rangle$. Then q_0, q_1, and $\{x^i_\varepsilon \mid i \in A,\ \varepsilon < 2\}$ are as desired. □

Now use Claim 5.6 and the closure of $\mathbb{P}$ to recursively build a tree of conditions $\{p_\sigma \mid \sigma \in {}^{<\theta}2\}$ and elements $\{x^{\sigma,i}_\varepsilon \mid \sigma \in {}^{<\theta}2,\ i \in A,\ \varepsilon < 2\}$ of S as follows. We will maintain the hypothesis that, for all $\tau, \sigma \in {}^{<\theta}2$, if τ is an initial segment of σ, then $p_\sigma \leq_{\mathbb{P}} p_\tau$.

Let $p_\emptyset := p^*$. If $\eta < \theta$ is a limit ordinal, $\sigma \in {}^\eta 2$, and $p_{\sigma \restriction \xi}$ has been defined for every $\xi < \eta$, then let p_σ be any lower bound for $\langle p_{\sigma\restriction\xi} \mid \xi < \eta\rangle$. If $\sigma \in {}^{<\theta}2$ and p_σ has been defined, then apply Claim 5.6 to find $p_{\sigma ^\frown \langle 0 \rangle}, p_{\sigma ^\frown \langle 1 \rangle} \leq p_\sigma$, and $\{x^{\sigma,i}_\varepsilon \mid i \in A,\ \varepsilon < 2\} \subseteq S$ such that

(1) for every $i \in A$ and $\varepsilon < 2$, we have $p_{\sigma ^\frown \langle \varepsilon \rangle} \Vdash_{\mathbb{P}}$ "$x^{\sigma,i}_\varepsilon \in \dot{b}_i$";
(2) for every $i \in A$, we have $x^{\sigma,i}_0 \perp_{R_i} x^{\sigma,i}_1$.

For each $f \in {}^\theta 2$, let p_f be a lower bound for $\langle p_{f\restriction\eta} \mid \eta < \theta\rangle$. Choose $B \subseteq {}^\theta 2$ with $|B| = \theta^+$, and use the fact that $d_\Lambda > \theta^+$ to find $v \in \Lambda$ such that $\ell(x^{f\restriction\eta,i}_\varepsilon) <_\Lambda v$ for all $f \in B$, $\eta < \theta$, $i \in A$, and $\varepsilon < 2$. We can also assume that $u^* <_\Lambda v$.

For each $f \in B$, use the fact that $\{\dot{b}_i \mid i < \theta\}$ is forced to be a full set of branches in $\mathcal{S}$ to find a $q_f \leq_{\mathbb{P}} p_f$, an $i_f < \theta$, and an $x_f \in S_v$ such that $q_f \Vdash_{\mathbb{P}}$ "$x_f \in \dot{b}_{i_f}$". Since $u^* <_\Lambda v$ and each q_f extends p^*, it must be the case that $i_f \in A$ for all $f \in B$. Since $|B| = \theta^+ > \mathrm{width}(\mathcal{S})$, we can find distinct $f, g \in B$, $i \in A$, and $x \in S_v$ such that $i_f = i_g = i$ and $x_f = x_g = x$. Let $\eta^* < \theta$ be the least η such that $f(\eta) \neq g(\eta)$, and let $\sigma := f \restriction \eta^* = g \restriction \eta^*$. Without loss of generality, assume that $f(\eta^*) = 0$ and $g(\eta^*) = 1$. Then $q_f \leq_{\mathbb{P}} q_{\sigma ^\frown \langle 0 \rangle}$, and therefore $q_f \Vdash_{\mathbb{P}}$ "$x^{\sigma,i}_0 \in \dot{b}_i$". Similarly, $q_g \Vdash_{\mathbb{P}}$ "$x^{\sigma,i}_1 \in \dot{b}_i$". Since both q_f and q_g extend p_i and force x to be in $\dot{b}_i$, and since $\ell(x^{\sigma,i}_0), \ell(x^{\sigma,i}_1) <_\Lambda v = \ell(x)$, it must be the case that $x^{\sigma,i}_0 <_{R_i} x$ and $x^{\sigma,i}_1 <_{R_i} x$, contradicting the fact that $x^{\sigma,i}_0 \perp_{R_i} x^{\sigma,i}_1$. □

6. A global consistency result

We are finally ready to prove our consistency result. For organizational reasons, it will be helpful to have the following definition.

Definition 6.1. For every infinite regular cardinal κ, we say that κ has the *strong narrow system property*, denoted SNSP_κ, if, for every directed partial order Λ with $d_\Lambda \geq \kappa$, every Λ-system $\mathcal{S}$ with $\mathrm{width}(\mathcal{S})^+ < \kappa$ has a cofinal branch.

Theorem 6.2. *Let $\mu < \kappa$ be regular uncountable cardinals, with κ supercompact, and let $\mathbb{P} := \mathrm{Coll}(\mu, {<}\kappa)$. Then, in $V^{\mathbb{P}}$, SNSP_κ holds and moreover is indestructible under κ-directed closed set forcing.*

Proof. Let G be $\mathbb{P}$-generic over V. Since trivial forcing is κ-directed closed, it suffices to prove that, if $\mathbb{Q}$ is a κ-directed closed set forcing in $V[G]$ and H is $\mathbb{Q}$-generic over $V[G]$, then SNSP_κ holds in $V[G][H]$.

To this end, fix a κ-directed closed $\mathbb{Q} \in V[G]$ and a $\mathbb{Q}$-generic filter H over $V[G]$. In $V[G][H]$, let Λ be a κ-directed partial order, and let $\mathcal{S} = \langle\langle S_u \mid u \in \Lambda\rangle, \mathcal{R}\rangle$ be a Λ-system with $\mathrm{width}(\mathcal{S}) < \mu$. For concreteness, assume that the underlying sets of both $\mathbb{Q}$ and Λ are ordinals. We will show that, in $V[G][H]$, there is a cofinal branch in $\mathcal{S}$. By Proposition 3.8, we can assume that $\mathcal{S}$ has a single relation, which we will denote by R.

In V, let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for $\mathbb{Q}$, and let $\dot{\Lambda}$ be a $\mathbb{P} * \dot{\mathbb{Q}}$-name for Λ. Fix a cardinal $\delta > \kappa$ such that $|\mathscr{P}(\mathbb{P} * \dot{\mathbb{Q}})| < \delta$ and $\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}}$ "$|\dot{\Lambda}| < \delta$", and let $j : V \to M$ be an elementary embedding witnessing that κ is δ-supercompact, i.e., $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \delta$, and ${}^\delta M \subseteq M$. We have $j(\mathbb{P}) = \mathrm{Coll}(\mu, {<}j(\kappa))$ so, by [17, Lemma 3] (cf. also [2, Fact 6.11]), the natural complete embedding ι of $\mathbb{P}$ into $j(\mathbb{P})$ can be extended to a complete embedding ι' of $\mathbb{P} * \dot{\mathbb{Q}}$ into $j(\mathbb{P})$ in such a way that the quotient forcing $j(\mathbb{P})/\iota'[\mathbb{P} * \dot{\mathbb{Q}}]$ is μ-closed. Let $\dot{\mathbb{R}}$ be a $\mathbb{P} * \dot{\mathbb{Q}}$-name for this quotient forcing. We then have $j(\mathbb{P}) \cong \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$, and $\dot{\mathbb{R}}$ is forced by $\mathbb{P} * \dot{\mathbb{Q}}$ to be μ-closed.

Let $\mathbb{R}$ be the realization of $\dot{\mathbb{R}}$ in $V[G][H]$, and let K be an $\mathbb{R}$-generic filter over $V[G][H]$. Since, for all $p \in \mathbb{P}$, $j(p) = p$, which is naturally identified with $(p, 1_{\dot{\mathbb{Q}}}, 1_{\dot{\mathbb{R}}})$ in $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$, we have $j``G \subseteq G * H * K$, so, in $V[G][H][K]$, we can extend j to $j : V[G] \to M[G][H][K]$.

By the closure of M, we know that $j``H \in M[G][H][K]$. Moreover, in that model, $j``H$ is a directed subset of $j(\mathbb{Q})$ with $|j``H| < \delta < j(\kappa)$, and $j(\mathbb{Q})$ is $j(\kappa)$-directed closed. We can therefore find $q^* \in j(\mathbb{Q})$ such that $q^* \leq_{j(\mathbb{Q})} j(q)$ for all $q \in H$. Let $H^+$ be $j(\mathbb{Q})$-generic over $V[G][H][K]$ with $q^* \in H^+$. Then $j``H \subseteq H^+$, so, in $V[G][H][K][H^+]$, we can extend j one last time to $j : V[G][H] \to M[G][H][K][H^+]$.

Again by the closure of M, we have $j``\Lambda \in M[G][H][K][H^+]$. Moreover, in that model, $j``\Lambda$ is a subset of $j(\Lambda)$ with $|j``\Lambda| < \delta < j(\kappa)$, and $j(\Lambda)$ is $j(\kappa)$-directed. We can therefore find $v^* \in j(\Lambda)$ such that $j(u) <_{j(\Lambda)} v^*$ for all $u \in \Lambda$. Let $\theta := \mathrm{width}(\mathcal{S})$. Since $\theta < \mu$, we have $\theta = j(\theta) = \mathrm{width}(j(\mathcal{S}))$. Write $j(\mathcal{S})$ as $\langle \langle S'_v \mid v \in j(\Lambda) \rangle, \{j(R)\} \rangle$. Enumerate S'_{v^*} as $\langle y_i \mid i < \theta \rangle$, with repetitions if $|S'_{v^*}| < \theta$. For each $i < \theta$, let $b_i := \{x \in S \mid j(x) <_{j(R)} y_i\}$.

We claim that $\{b_i \mid i < \theta\}$ is a full set of branches in $\mathcal{S}$. Let us first verify that each b_i is a branch in $\mathcal{S}$. To this end, fix $i < \theta$ and $x, y \in b_i$. Then, in $j(\mathcal{S})$, we have $j(x), j(y) <_{j(R)} y_i$, and hence $j(x) \parallel_{j(R)} j(y)$. By elementarity, we have $x \parallel_R y$, as desired.

We next verify that, for all $u \in \Lambda$, there is $i < \theta$ such that $b_i \cap S_u \neq \emptyset$. To this end, fix $u \in \Lambda$. Since $j(u) <_{j(\Lambda)} v^*$, clause 5 of Definition 3.4 implies that there are $i < \theta$ and $w \in S'_{j(u)}$ such that $w <_{j(R)} y_i$. Since $|S_u| \leq \theta < \kappa$, we have $S'_{j(u)} = j``S_u$, so we can find $x \in S_u$ such that $j(x) = w$. Then $x \in b_i \cap S_u$.

We have thus shown that $\{b_i \mid i < \theta\} \in V[G][H][K][H^+]$ is a full set of branches in $\mathcal{S}$. Therefore, since $\theta^+ \leq \mu$, we can apply Lemma 5.3 in $V[G][H]$ to $\mathcal{S}$ and the μ-closed poset $\mathbb{R} * j(\mathbb{Q})$ to conclude that, in $V[G][H]$, there is a cofinal branch in $\mathcal{S}$, thus completing the proof of the theorem. ☐

Remark 6.3. Theorem 6.2 provides a way of verifying our earlier claim from Remark 2.3 that, in general, cNSP_κ does not imply TP_κ. For example, if κ is supercompact and $\mathbb{P} = \mathrm{Coll}(\omega_1, {<}\kappa)$, then, in $V^{\mathbb{P}}$, we have $\kappa = \aleph_2$, and Theorem 6.2 implies that cNSP_κ holds. On the other hand, CH holds in $V^{\mathbb{P}}$, so there exists an $\aleph_2$-Aronszajn tree, and hence even $\mathsf{TP}(\kappa, \kappa)$ fails.

Note that the assertion "SNSP_κ holds for every infinite regular cardinal κ" is equivalent to the assertion "$\mathsf{NSP}(\Lambda)$ holds for every directed partial order Λ". The following therefore yields Theorem B.

Theorem 6.4. *Suppose that there is a proper class of supercompact cardinals. Then there is a class forcing extension in which SNSP_κ holds for every infinite regular cardinal κ.*

Proof. Let $\langle \kappa_\eta \mid \eta \in \mathrm{On} \rangle$ be an increasing, continuous sequence of cardinals such that

- $\kappa_0 = \aleph_0$;
- if η is a limit ordinal (including 0), then $\kappa_{\eta+1} = \kappa_\eta^+$;
- if η is a successor ordinal, then $\kappa_{\eta+1}$ is supercompact.

We may assume that κ_η is singular for every nonzero limit ordinal η; if not, then simply truncate the universe below κ_η for the least nonzero limit ordinal η such that κ_η is regular (and hence strongly inaccessible).

We now force with a class-length iteration of Lévy collapses to turn each κ_η into $\aleph_\eta$. More formally, recursively define posets $\langle \mathbb{P}_\eta \mid \eta \in \mathrm{On} \rangle$ as follows:

- $\mathbb{P}_0$ and $\mathbb{P}_1$ are trivial forcing;
- if η is a successor ordinal, then $\mathbb{P}_{\eta+1} = \mathbb{P}_\eta * \dot{\mathrm{Coll}}(\kappa_\eta, {<}\kappa_{\eta+1})$;
- if η is a nonzero limit ordinal, then $\mathbb{P}_\eta$ is the inverse (i.e., full-support) limit of $\langle \mathbb{P}_\xi \mid \xi < \eta \rangle$ and $\mathbb{P}_{\eta+1} = \mathbb{P}_\eta$.

For ordinals $\xi < \eta$, let $\dot{\mathbb{P}}_{\xi\eta}$ be a $\mathbb{P}_\xi$-name for the quotient $\mathbb{P}_\eta/\mathbb{P}_\xi$. Then $\dot{\mathbb{P}}_{\xi\eta}$ is a name for a full-support iteration of Lévy collapses, each of which is forced to be κ_ξ-directed closed. It follows that $\dot{\mathbb{P}}_{\xi\eta}$ is forced to be κ_ξ-closed. In particular, $(H(\kappa_\xi))^{V^{\mathbb{P}_\xi}} = (H(\kappa_\xi))^{V^{\mathbb{P}_\eta}}$, so $V^{\mathbb{P}} := \bigcup_{\eta \in \mathrm{On}} V^{\mathbb{P}_\eta}$ is a model of ZFC. Also, standard arguments show that, in $V^{\mathbb{P}}$, we have $\kappa_\eta = \aleph_\eta$ for all $\eta \in \mathrm{On}$.

We claim that SNSP_κ holds in $V^{\mathbb{P}}$ for every infinite regular cardinal κ. Note that the infinite regular cardinals in $V^{\mathbb{P}}$ are precisely the cardinals κ_η for which η is either 0 or a successor ordinal. If $\eta \leq 1$, then every system $\mathcal{S}$ such that $(\mathrm{width}(\mathcal{S}))^+ < \kappa_\eta$ has finite width. It therefore follows from Proposition 3.9 that $\mathsf{SNSP}_{\aleph_0}$ and $\mathsf{SNSP}_{\aleph_1}$ are true in ZFC.

We next note that, if η is a nonzero limit ordinal and $\mathcal{S}$ is a system such that $(\mathrm{width}(\mathcal{S}))^+ < \kappa_{\eta+1}$, then there must be a $\xi < \eta$ such that $(\mathrm{width}(\mathcal{S}))^+ < \kappa_\xi$. Therefore, $\mathsf{SNSP}_{\kappa_{\eta+1}}$ will follow from the conjunction of $\mathsf{SNSP}_{\kappa_\xi}$ for all $\xi < \eta$.

We are therefore left with the task of verifying $\mathsf{SNSP}_{\kappa_{\eta+1}}$ for all successor ordinals η. To this end, fix a successor ordinal η and fix in $V^{\mathbb{P}}$ a directed partial order Λ with $d_\Lambda \geq \kappa_{\eta+1}$ and a Λ-system $\mathcal{S}$ with $\mathrm{width}(\mathcal{S})^+ < \kappa_{\eta+1}$.

In V^{P_η}, we have $\kappa_\eta = \aleph_\eta$ and, since $|\mathbb{P}_\eta| < \kappa_{\eta+1}$, we know that $\kappa_{\eta+1}$ is still supercompact. Moreover, $\mathbb{P}_{\eta,\eta+1} = \mathrm{Coll}(\kappa_\eta, {<}\kappa_{\eta+1})$ so, by Theorem 6.2, $\mathsf{SNSP}_{\kappa_{\eta+1}}$ holds in $V^{\mathbb{P}_{\eta+1}}$ and every $\kappa_{\eta+1}$-directed closed set forcing extension thereof. Let $\zeta > \eta + 1$ be large enough so that Λ and $\mathcal{S}$ are in $V^{\mathbb{P}_\zeta}$. In $V^{\mathbb{P}_{\eta+1}}$, $\mathbb{P}_{\eta+1,\zeta}$ is $\kappa_{\eta+1}$-directed closed, so $\mathsf{SNSP}_{\kappa_{\eta+1}}$ holds in $V^{\mathbb{P}_\zeta}$. In particular, $\mathcal{S}$ has a cofinal branch in $V^{\mathbb{P}_\zeta}$ and hence also in $V^{\mathbb{P}}$. $\square$

We close the paper with what we feel are the most prominent remaining open questions.

Question 6.5. *Suppose that $\kappa \geq \omega_2$ is a regular cardinal. Does TP_κ imply cNSP_κ? More specifically, if $\lambda \geq \kappa$ is a cardinal, does $\mathsf{TP}(\kappa, \lambda)$ imply $\mathsf{cNSP}(\mathscr{P}_\kappa \lambda)$? More generally, suppose that Λ is a directed partial order with $d_\Lambda \geq \aleph_2$. Does $\mathsf{TP}(\Lambda)$ imply $\mathsf{NSP}(\Lambda)$?*

By Theorem A, a positive answer to the first part of Question 6.5 would entail a positive answer to Question 1.3 and therefore a negative answer to Question 1.4. On the other hand, a negative answer to Question 6.5 would seem to require genuinely new ideas, since the known methods to verify $\mathsf{TP}(\kappa, \lambda)$ in practice inevitably yield $\mathsf{cNSP}(\mathscr{P}_\kappa \lambda)$, as well.

References

[1] Paul Cohen. The independence of the continuum hypothesis. *Proc. Nat. Acad. Sci. U.S.A.*, 50:1143–1148, 1963.

[2] James Cummings, Matthew Foreman, and Menachem Magidor. Squares, scales and stationary reflection. *J. Math. Log.*, 1(1):35–98, 2001.

[3] James Cummings, Yair Hayut, Menachem Magidor, Itay Neeman, Dima Sinapova, and Spencer Unger. The ineffable tree property and failure of the singular cardinals hypothesis. *Trans. Amer. Math. Soc.*, 373(8):5937–5955, 2020.
[4] William B. Easton. Powers of regular cardinals. *Ann. Math. Logic*, 1:139–178, 1970.
[5] Fred Galvin and András Hajnal. Inequalities for cardinal powers. *Ann. of Math. (2)*, 101:491–498, 1975.
[6] Moti Gitik. The strength of the failure of the singular cardinal hypothesis. *Ann. Pure Appl. Logic*, 51(3):215–240, 1991.
[7] Sherwood Hachtman and Dima Sinapova. ITP, ISP, and SCH. *J. Symb. Log.*, 84(2):713–725, 2019.
[8] Thomas Jech. Some combinatorial problems concerning uncountable cardinals. *Ann. Math. Logic*, 5:165–198, 1972/73.
[9] Thomas Jech. *Set theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, millennium edition, 2003.
[10] John Krueger. Guessing models imply the singular cardinal hypothesis. *Proc. Amer. Math. Soc.*, 147(12):5427–5434, 2019.
[11] Chris Lambie-Hanson. Squares and narrow systems. *J. Symb. Log.*, 82(3):834–859, 2017.
[12] Chris Lambie-Hanson and Assaf Rinot. Knaster and friends III: Subadditive colorings. *J. Symbolic Logic*, 2023. To appear.
[13] Chris Lambie-Hanson and Šárka Stejskalová. Guessing models, trees, and cardinal arithmetic. *Isr. J. Math.*, 2024. To appear.
[14] Chris Lambie-Hanson and Šárka Stejskalová. Strong tree properties, Kurepa trees, and guessing models. *Monatsh. Math.*, 203(1):111–148, 2024.
[15] Menachem Magidor. Combinatorial characterization of supercompact cardinals. *Proc. Amer. Math. Soc.*, 42:279–285, 1974.
[16] Menachem Magidor. On the singular cardinals problem. I. *Israel J. Math.*, 28(1-2):1–31, 1977.
[17] Menachem Magidor. Reflecting stationary sets. *J. Symbolic Logic*, 47(4):755–771 (1983), 1982.
[18] Menachem Magidor and Saharon Shelah. The tree property at successors of singular cardinals. *Arch. Math. Logic*, 35(5-6):385–404, 1996.
[19] Pierre Matet. Meeting numbers and pseudopowers. *MLQ Math. Log. Q.*, 67(1):59–76, 2021.
[20] William Mitchell. Aronszajn trees and the independence of the transfer property. *Ann. Math. Logic*, 5:21–46, 1972/73.
[21] Assaf Sharon and Matteo Viale. Some consequences of reflection on the approachability ideal. *Trans. Amer. Math. Soc.*, 362(8):4201–4212, 2010.
[22] Saharon Shelah. *Cardinal arithmetic*, volume 29 of *Oxford Logic Guides*. The Clarendon Press, Oxford University Press, New York, 1994. Oxford Science Publications.
[23] Jack Silver. On the singular cardinals problem. In *Proceedings of the International Congress of Mathematicians (Vancouver, B.C., 1974), Vol. 1*, pages 265–268. Canad. Math. Congr., Montreal, QC, 1975.
[24] Dima Sinapova. The tree property at $\aleph_{\omega+1}$. *J. Symbolic Logic*, 77(1):279–290, 2012.
[25] Robert M. Solovay. Strongly compact cardinals and the GCH. In *Proceedings of the Tarski Symposium (Proc. Sympos. Pure Math., Vol. XXV, Univ. California, Berkeley, Calif., 1971)*, pages 365–372. Amer. Math. Soc., Providence, R.I., 1974.
[26] E. Specker. Sur un problème de Sikorski. *Colloq. Math.*, 2:9–12, 1949.
[27] Matteo Viale. The proper forcing axiom and the singular cardinal hypothesis. *J. Symbolic Logic*, 71(2):473–479, 2006.
[28] Matteo Viale. A family of covering properties. *Math. Res. Lett.*, 15(2):221–238, 2008.
[29] Matteo Viale. Guessing models and generalized Laver diamond. *Ann. Pure Appl. Logic*, 163(11):1660–1678, 2012.
[30] Christoph Weiß. The combinatorial essence of supercompactness. *Ann. Pure Appl. Logic*, 163(11):1710–1717, 2012.

(Lambie-Hanson) Institute of Mathematics, Czech Academy of Sciences, Žitná 25, Prague 1, 115 67, Czech Republic
Email address: `lambiehanson@math.cas.cz`
URL: `https://users.math.cas.cz/~lambiehanson/`