# KUREPA TREES, CONTINUOUS IMAGES, AND PERFECT SET PROPERTIES

CHRIS LAMBIE-HANSON AND ŠÁRKA STEJSKALOVÁ

ABSTRACT. Building upon work of Lücke and Schlicht, we study (higher) Kurepa trees through the lens of higher descriptive set theory, focusing in particular on various perfect set properties and representations of sets of branches through trees as continuous images of function spaces. Answering a question of Lücke and Schlicht, we prove that it is consistent with CH that there exist  $\omega_2$ -Kurepa trees and yet, for every  $\omega_2$ -Kurepa tree  $T \subseteq {}^{<\omega_2}\omega_2$ , the set  $[T] \subseteq {}^{\omega_2}\omega_2$  of cofinal branches through T is not a continuous image of  ${}^{\omega_2}\omega_2$ . We also produce models indicating that the existence of Kurepa trees is not necessary to produce closed subsets of  ${}^{\omega_1}\omega_1$  failing to satisfy strong perfect set properties, and prove a number of consistency results regarding *full* and *superthin* trees.

### 1. INTRODUCTION

Two fundamental features of closed subsets of the classical Baire space  ${}^\omega\omega$  are the following:

- (1) every nonempty closed subset of  ${}^{\omega}\omega$  is a continuous image (and, in fact, a retract) of  ${}^{\omega}\omega$ ;
- (2) every closed subset of  ${}^{\omega}\omega$  has the perfect set property.

When passing to the higher Baire spaces of the form  $\kappa \kappa$  for regular uncountable cardinals  $\kappa$ , both of these properties at least consistently fail. Regarding property (1), for an arbitrary regular uncountable cardinal  $\kappa$ , it is proven in [7, Proposition 1.4] that there exists a nonempty closed subset of  $\kappa \kappa$  that is not a retract of  $\kappa \kappa$ . If we assume moreover that  $\kappa^{<\kappa} = \kappa$ , which is a standard assumption in the study of higher Baire spaces, then it is proven in [7, Theorem 1.5] that there exists a nonempty closed subset of  $\kappa \kappa$  that is not a continuous image of  $\kappa \kappa$ .

When discussing higher analogues of the perfect set property, we need to fix an appropriate notion of perfectness. Unlike in the classical Baire space, where there is a single, unambiguous notion of "perfectness", in the higher Baire space  $\kappa \kappa$  there exist various degrees of perfectness, leading to a corresponding spectrum of perfect set properties. These will be discussed in more detail below in Section 3, but for now let us consider the strongest and arguably most natural version. We say that a  $\kappa$ -perfect subset of  $\kappa \kappa$  is the set of all cofinal branches through some cofinally-splitting ( $<\kappa$ )-closed subtree of  $<\kappa \kappa$  (see Section 2 for a precise definition). A set

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 $X \subseteq {}^{\kappa}\kappa$  has the  $\kappa$ -perfect set property, denoted  $\mathsf{PSP}_{\kappa}(X)$ , if either  $|X| \leq \kappa$  or X contains a  $\kappa$ -perfect subset.

As noted in [6, §7], if  $\kappa$  is a regular uncountable cardinal and every closed subset of  ${}^{\kappa}\kappa$  has the perfect set property, then  $\kappa^+$  is inaccessible in L. On the other hand, by an argument of Schlicht (cf. [6, §9]), if  $\kappa$  is a regular uncountable cardinal,  $\nu > \kappa$ is inaccessible, and G is generic for the Lévy collapse  $\operatorname{Coll}(\kappa, < \nu)$ , then, in V[G] $\mathsf{PSP}_{\kappa}(X)$  holds for every closed  $X \subseteq {}^{\kappa}\kappa$  (in fact, for every  $\Sigma_1^1 X \subseteq {}^{\kappa}\kappa$ . Thus, the  $\kappa$ -perfect set property for closed subsets of  ${}^{\kappa}\kappa$  for some regular uncountable  $\kappa$  is equiconsistent with the existence of an inaccessible cardinal.

A starring role in the story of the preceding two paragraphs is played by *Kurepa* trees. Indeed, suppose that  $\kappa$  is a regular uncountable cardinal that is not strong limit, e.g., a successor cardinal. If  $T \subseteq {}^{<\kappa}\kappa$  is a  $\kappa$ -Kurepa tree<sup>1</sup> then the set [T]of all cofinal branches through T is a closed subset of  ${}^{\kappa}\kappa$  that fails to satisfy the  $\kappa$ -perfect set property (see [6] for a proof of this; the idea of using Kurepa trees as a counterexample for the  $\kappa$ -perfect set property goes back at least to [17] and [9]). The fact that  $\mathsf{PSP}_{\kappa}(X)$  holding for all closed  $X \subseteq {}^{\kappa}\kappa$  implies that  $\kappa^+$  is inaccessible in L then follows from an argument of Solovay (cf. [5, §4]).

It is thus natural to also consider Kurepa trees in the context of item (1) above, leading to the following general question: for a fixed regular uncountable cardinal  $\kappa$ , given a  $\kappa$ -Kurepa tree T, under what circumstances is [T] a continuous image of, or even a retract of, the space  $\kappa \kappa$ . This and related questions were considered by Lücke and Schlicht in [8], where they obtained a number of interesting results, including the following:

- If there is  $\mu < \kappa$  such that  $\mu^{\omega} \ge \kappa$ , and  $T \subseteq {}^{<\kappa}\kappa$  is a  $\kappa$ -Kurepa tree with  $|[T]| > \kappa^{<\kappa}$ , then [T] is not a continuous image of  ${}^{\kappa}\kappa$ . In particular, assuming instances of GCH, the question has an easy negative answer if  $\kappa$  is a successor of a cardinal of countable cofinality, e.g.,  $\omega_1$ .
- If V = L, then, for every regular uncountable cardinal  $\kappa$  that is not the successor of a cardinal of countable cofinality, there exists a  $\kappa$ -Kurepa tree  $T \subseteq {}^{<\kappa}\kappa$  such that [T] is a retract of  ${}^{\kappa}\kappa$ .
- If  $\mu < \kappa$  are regular uncountable cardinals, with  $\kappa$  inaccessible, then there is a forcing extension in which (1)  $\kappa = \mu^+$ , (2) there exists a  $\kappa$ -Kurepa tree  $T \subseteq {}^{<\kappa}\kappa$  such that [T] is a retract of  ${}^{\kappa}\kappa$ ; and (3) there exists a  $\kappa$ -Kurepa tree  $S \subseteq T$  such that [S] is not a continuous image of  ${}^{\kappa}\kappa$ .
- If there is a  $\kappa$ -Kurepa tree, then there is a  $\kappa$ -Kurepa tree  $T \subseteq {}^{<\kappa}\kappa$  such that [T] is not a retract of  ${}^{\kappa}\kappa$ .

The work of Lücke and Schlicht left open a number of avenues for further research. One question, explicitly asked as [8, Question 6.2], is whether CH together with the existence of an  $\omega_2$ -Kurepa tree implies the existence of an  $\aleph_2$ -Kurepa tree  $S \subseteq {}^{<\omega_2}\omega_2$  such that [S] is a continuous image of  ${}^{\omega_2}\omega_2$ . We provide a negative answer to this question (see Theorem B below).

We also undertake further explorations of topics connecting Kurepa trees, (variations on) the perfect set property, and continuous images of higher Baire spaces. Here we make use of a hierarchy of generalizations of perfectness introduced by Väänänen in [17] in terms of the existence of winning strategies for certain twoplayer games. In particular, given a regular cardinal  $\kappa$ , and an ordinal  $\delta$  with

<sup>&</sup>lt;sup>1</sup>See Section 2 for a precise definition.

 $\omega \leq \delta \leq \kappa$ , Väänänen defines a notion of  $\delta$ -perfectness for subsets of  $\kappa \kappa$ . These notions strengthen as  $\delta$  increases, with  $\omega$ -perfectness being classical perfectness (i.e., closed and having no isolated points), and  $\kappa$ -perfectness being as above.

Recall the classical Cantor-Bendixson theorem: If  $E \subseteq {}^{\omega}\omega$  is closed, then there is a countable set  $X \subseteq E$  such that  $E \setminus X$  is perfect. This generalizes straightforwardly to higher Baire spaces: if  $E \subseteq {}^{\kappa}\kappa$  is closed, then there is  $X \subseteq E$  such that  $|X| \leq {}^{\kappa}\kappa$ and  $E \setminus X$  is perfect. We prove in Section 3 that, if E is a continuous image of  ${}^{\kappa}\kappa$ , then we can obtain a stronger conclusion.<sup>2</sup>

In the other direction, recall that  $\kappa$ -Kurepa trees yield examples of closed subsets of  ${}^{\kappa}\kappa$  of cardinality greater than  $\kappa$  that do not contain  $\kappa$ -perfect subsets; in fact, if  $\kappa = \mu^+$  and  $T \subseteq {}^{<\kappa}\kappa$  is a  $\kappa$ -Kurepa tree, then [T] does not contain a  $(\mu+1)$ -perfect subset. In Section 4, we prove results showing that Kurepa trees are not necessary for the existence of such sets by producing models in which GCH holds, the Kurepa Hypothesis fails (i.e., there are no  $\omega_1$ -Kurepa trees), but there is a closed subset of  ${}^{\omega_1}\omega_1$  of cardinality  $\omega_2$  with no  $(\omega+1)$ -perfect subset. In fact, we prove the following stronger theorem:<sup>3</sup>

**Theorem A.** Suppose that there is an inaccessible cardinal. Then there is a forcing extension in which

- (1) GCH holds;
- (2) the Kurepa Hypothesis fails;
- (3) there is a weak Kurepa tree that does not contain a copy of  ${}^{<\omega+1}2$ .

We actually produce two models witnessing the conclusion of Theorem A. In the first, every weak Kurepa tree contains an Aronszajn (or even Suslin) subtree. In the second, there is a weak Kurepa tree that contains neither a copy of  $^{<\omega+1}2$  nor an Aronszajn subtree.

In Section 5, we provide a negative answer to the aforementioned question of Lücke and Schlicht. In addition, we show that, in the model we construct,  $^{\omega_2}\omega_2$  satisfies the strongest possible analogue of the Cantor-Bendixson theorem compatible with the existence of an  $\omega_2$ -Kurepa tree (recall that, if  $T \subseteq {}^{<\omega_2}\omega_2$  is an  $\omega_2$ -Kurepa tree), then [T] cannot contain an  $(\omega_1 + 1)$ -perfect subset. In particular, we prove the following theorem.

**Theorem B.** If there exists an inaccessible cardinal, then there is a forcing extension in which GCH holds, there exists and  $\omega_2$ -Kurepa tree, and, for every  $\omega_2$ -Kurepa tree  $S \subseteq {}^{<\omega_2}\omega_2$ , [S] is not a continuous image of  ${}^{\omega_2}\omega_2$ . In addition, in the forcing extension, for every closed subset  $E \subseteq {}^{\omega_2}\omega_2$ , there is  $X \subseteq E$  with  $|X| \leq \omega_2$  such that  $E \setminus X$  is  $\omega_1$ -perfect.

In Section 6, we prove some new results about full trees. A tree T is *full* if, for every limit ordinal  $\beta$  below the height of T, there is at most one branch through  $T \upharpoonright \beta$  that is not continued at level  $\beta$  (see Section 6 for a precise definition). Kunen asked whether there could consistently exist a full  $\kappa$ -Suslin tree for some regular uncountable cardinal  $\kappa$  (cf. [10]). This was answered positively by Shelah in [15] for inaccessible  $\kappa$  and recently by Rinot, Yadai, and You in [12] for successors

 $<sup>^{2}</sup>$ For more on stronger versions of the Cantor-Bendixson theorem for higher Baire spaces, see [17].

<sup>[17]. &</sup>lt;sup>3</sup>Recall that a *weak Kurepa tree* is a tree of height and size  $\omega_1$  with at least  $\omega_2$ -many cofinal branches.

of regular uncountable cardinals. Here, we are interested in full, splitting trees that may contain some cofinal branches. For example, in Section 6, we prove the following theorem about full trees of height  $\omega_1$ :

- **Theorem C.** (1) If  $\diamondsuit$  holds, then, for every cardinal  $\nu \in \omega \cup \{\omega, \omega_1, 2^{\omega_1}\}$ , there is a normal, full, splitting tree  $T \subseteq {}^{<\omega_1}\omega_1$  with exactly  $\nu$ -many cofinal branches.
  - (2) CH does not suffice for the conclusion of clause (1). In particular, CH is compatible with the assertion that every full, splitting tree of height  $\omega_1$  contains a copy of  ${}^{<\omega_1}2$ , and hence has  $2^{\omega_1}$ -many cofinal branches.

We then move up a cardinal to prove the consistency of the existence of a normal, splitting, full, superthin  $\omega_2$ -Kurepa tree. We refer the reader to Section 6 for the definition of *superthin*; we only note that superthin Kurepa trees play a central role in the investigations of Lücke and Schlicht in [8]. In particular, they prove there that if there exists a superthin  $\kappa$ -Kurepa tree, then there is a superthin  $\kappa$ -Kurepa tree  $T \subseteq {}^{<\kappa}\kappa$  such that [T] is a retract of  ${}^{\kappa}\kappa$ . In Section 7, we prove the following result, producing a model having  $\omega_2$ -Kurepa trees with very different behavior from that produced in Theorem B:

**Theorem D.** Suppose that GCH holds and  $\kappa$  is the successor of a regular uncountable cardinal. Then there is a cofinality-preserving forcing extension in which

- (1) GCH holds;
- (2) there exists a  $\kappa$ -Kurepa tree;
- (3) every  $\kappa$ -Kurepa tree contains a normal superthin  $\kappa$ -Kurepa subtree.

Finally, in Section 8, we record some closing remarks and a few questions that remain open.

1.1. Notation and conventions. Our notation is for the most part standard. We refer the reader to [4] for undefined notions and notations in set theory. We let Card denote the class of all cardinals. Given a well-ordered set X, we denote its order type by otp(X). If X and Y are sets,  $f : X \to Y$ , and  $A \subseteq X$ , then both f[A] and f "A denote the pointwise image of A, i.e.,  $\{f(a) \mid a \in A\}$ .

Given ordinals  $\alpha$  and  $\beta$ , we let  ${}^{\alpha}\beta$  denote the set of all functions  $f : \alpha \to \beta$ , and we let  ${}^{<\alpha}\beta$  denote  $\bigcup_{\eta < \alpha} {}^{\eta}\beta$ . We use  ${}^{\leq\alpha}\beta$  and  ${}^{<(\alpha+1)}\beta$  interchangeably. If  $\sigma, \tau \in {}^{<\alpha}\beta$ , then we let  $\sigma \sqsubseteq \tau$  denote the assertion that  $\sigma$  is an *initial segment* of  $\tau$ , i.e., dom $(\sigma) \le \text{dom}(\tau)$  and  $\sigma = \tau \upharpoonright \text{dom}(\sigma)$ . Given a nonempty set  $A \subseteq {}^{\leq\alpha}\beta$ , we let  $\bigwedge A \in {}^{\leq\alpha}\beta$  denote the (unique)  $\sqsubseteq$ -maximal element of  ${}^{\leq\alpha}\beta$  that is an initial segment of every element of A, i.e.,

$$\bigwedge A = \bigcup \{ \sigma \in {}^{\leq \alpha}\beta \mid \forall \tau \in A \ (\sigma \sqsubseteq \tau) \}.$$

If  $\sigma$  and  $\tau$  are two functions whose domains are ordinals  $\alpha$  and  $\beta$ , respectively, then  $\sigma \gamma \tau$  denotes the concatenation of  $\sigma$  and  $\tau$ , i.e., the function  $\rho$  with domain  $\alpha + \beta$  such that  $\rho(\eta) = \sigma(\eta)$  for all  $\eta < \alpha$  and  $\rho(\alpha + \xi) = \tau(\xi)$  for all  $\xi < \beta$ . We will sometimes think of functions with ordinal domains as sequences, e.g., a sequence of the form  $\langle \gamma \rangle$  will be thought of as a function  $\sigma$  with domain 1 and  $\sigma(0) = \gamma$ . We will sometimes write, e.g.,  $\sigma \gamma \gamma$  instead of  $\sigma \gamma \gamma$  when there is no risk of confusion.

We make use of various standard forcing notions throughout the paper. In particular, if  $\kappa$  is a regular infinite cardinal, then  $Add(\kappa, 1)$  is the forcing to add a single Cohen subset of  $\kappa$ . If, in addition,  $\mu > \kappa$  is a cardinal, then  $Coll(\kappa, <\mu)$  is the

Lévy collapse that collapses every cardinal in the interval  $(\kappa, \mu)$  to have cardinality  $\kappa$ . We assume that every forcing notion  $\mathbb{P}$  has a maximum element, which we denote  $1_{\mathbb{P}}$ . If we define a forcing notion  $\mathbb{P}$  that does not have a maximum element, then we implicitly add  $\emptyset$  as  $1_{\mathbb{P}}$ .

#### 2. Combinatorial and topological preliminaries

In this section, we introduce some of the basic notions forming the subject matter of this paper, particularly regarding trees, higher Baire spaces, and forcing.

**Definition 2.1.** A tree is a partial order  $(T, \leq_T)$  such that, for every  $t \in T$ , the set  $\operatorname{pred}_T(t) := \{s \in T \mid s <_T t\}$  is well-ordered by  $\leq_T$ . We will often abuse notation and simply refer to the tree by T, without explicitly mentioning the tree order  $\leq_T$ . Given  $t \in T$ , the *height* of t in T is  $\operatorname{ht}_T(t) := \operatorname{otp}(\operatorname{pred}_T(t))$ . Given an ordinal  $\alpha$ , the  $\alpha^{\operatorname{th}}$  level of T, denoted  $T_{\alpha}$ , is  $\{t \in T \mid \operatorname{ht}_T(t) = \alpha\}$ . The *height* of T, denoted  $\operatorname{ht}(T)$ , is the least ordinal  $\beta$  such that  $T_{\beta} = \emptyset$ . A subtree of T is a  $\leq_T$ -downward closed subset of T, with the tree order inherited from  $\leq_T$ .

- We say that a tree T is *normal* if it satisfies the following two conditions:
  - for all  $\alpha < \beta < \operatorname{ht}(T)$  and all  $s \in T_{\alpha}$ , there is  $t \in T_{\beta}$  such that  $s <_T t$ ; and
  - for all limit ordinals  $\alpha < \operatorname{ht}(T)$  and all  $s, t \in T_{\alpha}$ , if  $\operatorname{pred}_{T}(s) = \operatorname{pred}_{T}(t)$ , then s = t.

A branch through T is a maximal linearly ordered subset of T. We say that a branch b through T is a cofinal branch through T if  $b \cap T_{\alpha} \neq \emptyset$  for all  $\alpha < \operatorname{ht}(T)$ . The set of all cofinal branches through T is denoted by [T].

Given two nodes  $s, t \in T$ , we write  $s \perp t$  to denote the assertion that s and t are  $\leq_T$ -incomparable. An *antichain* of T is a set  $A \subseteq T$  such that  $s \perp t$  for all distinct  $s, t \in A$ .

**Definition 2.2.** Suppose that  $\kappa$  is an infinite cardinal. A tree *T* is a  $\kappa$ -tree if  $ht(T) = \kappa$  and  $|T_{\alpha}| < \kappa$  for every  $\alpha < \kappa$ . A  $\kappa$ -Aronszajn tree is a  $\kappa$ -tree with no cofinal branches, and a  $\kappa$ -Suslin tree is a  $\kappa$ -tree with no cofinal branches and no antichains of cardinality  $\kappa$ .

A  $\kappa$ -Kurepa tree is a  $\kappa$ -tree T with at least  $\kappa^+$ -many cofinal branches. A weak  $\kappa$ -Kurepa tree is a tree T of height and size  $\kappa$  with at least  $\kappa^+$ -many cofinal branches.

The  $\kappa$ -Kurepa Hypothesis (KH<sub> $\kappa$ </sub>) is the assertion that there exists a  $\kappa$ -Kurepa tree. The weak  $\kappa$ -Kurepa Hypothesis (wKH<sub> $\kappa$ </sub>) is the assertion that there exists a weak Kurepa tree. If the parameter  $\kappa$  is omitted in any of the above, then it should be understood that  $\kappa = \omega_1$ , e.g., a Suslin tree is an  $\omega_1$ -Suslin tree, a Kurepa tree is an  $\omega_1$ -Kurepa tree, and KH is KH<sub> $\omega_1$ </sub>.

**Definition 2.3.** Let T be a normal tree. For  $t, s \in T$ , the meet  $s \wedge t$  is the unique node  $r \leq s, t$  such that there is no r' > r with  $r' \leq s, t$ .

We will naturally interpret  ${}^{<\alpha}\beta$  as a tree by setting, for all  $\sigma, \tau \in {}^{<\alpha}\beta, \sigma \leq \tau$  if and only if  $\sigma$  is an initial segment of  $\tau$  (denoted  $\sigma \sqsubseteq \tau$ ). When we say that a tree T is a subtree of  ${}^{<\alpha}\beta$ , we will implicitly assume that T has height  $\alpha$ . In particular, in this context, the set [T] of cofinal branches through T can and will be identified with the set

$$\{b \in {}^{\alpha}\beta \mid \forall \eta < \alpha \ (b \upharpoonright \eta \in T)\}.$$

If  $\kappa$  and  $\lambda$  are cardinals, with  $\kappa$  regular, then we implicitly interpret  $^{\kappa}\lambda$  as a topological space, where each copy of  $\lambda$  is given the discrete topology and the

product is given the  $(\langle \kappa \rangle)$ -box topology. In other words, the topology on  ${}^{\kappa}\lambda$  is generated by all basic open sets of the form

$$N_{\sigma} = \{ x \in {}^{\kappa} \lambda \mid \sigma \sqsubseteq x \},\$$

where  $\sigma \in {}^{<\kappa}\lambda$ . We will be most interested in the space  ${}^{\kappa}\kappa$  for a regular uncountable cardinal  $\kappa$ ; this space is typically called the *higher Baire space at*  $\kappa$ .

Fix for now cardinals  $\kappa$  and  $\lambda$ , with  $\kappa$  regular. Given a nonempty set  $X \subseteq {}^{\kappa}\lambda$ , let

$$T(X) := \{ x \upharpoonright \alpha \mid x \in X \text{ and } \alpha < \kappa \}.$$

Note that T(X) is a subtree of  $\langle \kappa \lambda$ .

In the other direction, given a tree  $T \subseteq {}^{<\kappa}\lambda$ , it is a well-known and easily verified fact that [T] is a closed subset of  ${}^{\kappa}\lambda$  and, moreover, all closed subsets of  ${}^{\kappa}\lambda$  are of this form:

**Fact 2.4.** Suppose that  $\kappa$  and  $\lambda$  are cardinals, with  $\kappa$  regular. Then a subset  $X \subseteq {}^{\kappa}\lambda$  is closed if and only if it is of the form [T] for some tree  $T \subseteq {}^{<\kappa}\lambda$ .

We now recall a number of relevant definitions from [13].

**Definition 2.5.** Suppose that T is a subtree of  $\langle \kappa \rangle$ .

- (1) T is normal if for all  $\sigma \in T$  and all  $\alpha < \kappa$ , there is  $\tau \in T$  such that  $\sigma \sqsubseteq \tau$  and dom $(\tau) \ge \alpha$ .
- (2) Given  $\sigma, \tau \in T$ , we let  $\sigma \perp \tau$  denote the assertion that  $\sigma$  and  $\tau$  are not comparable in T.
- (3) T is splitting if, for all  $\sigma \in T$ , if  $|\sigma| + 1 < \operatorname{ht}(T)$ , then there are distinct  $i, j < \lambda$  such that  $\sigma^{\frown} i, \sigma^{\frown} j \in T$ . It is infinitely splitting if, for all such  $\sigma \in T$ , there are infinitely many  $i < \lambda$  such that  $\sigma^{\frown} i \in T$ .
- (4) T is cofinally splitting if, for every  $\sigma \in T$ , there are  $\tau_0, \tau_1 \in T$  such that  $\sigma \sqsubseteq \tau_0, \sigma \sqsubseteq \tau_1$ , and  $\tau_0 \perp \tau_1$ .
- (5) Given a cardinal  $\mu \leq \kappa$ , T is  $< \mu$ -closed if, for every  $\eta < \mu$  and every increasing sequence  $\langle \sigma_{\xi} | \xi < \eta \rangle$  from T, we have  $\bigcup \{ \sigma_{\xi} | \xi < \eta \} \in T$ .
- (6) T is  $\kappa$ -perfect if it is normal, cofinally splitting, and  $<\kappa$ -closed.
- (7) A nonempty closed subset  $X \subseteq {}^{\kappa}\lambda$  is said to be  $\kappa$ -perfect if T(X) is  $\kappa$ -perfect.

**Definition 2.6.** Suppose that S and T are trees and  $\iota: S \to T$ .

- (1)  $\iota$  is strict order preserving if, for all  $\sigma, \tau \in S$ , if  $\sigma < \tau$ , then  $\iota(\sigma) < \iota(\tau)$ .
- (2)  $\iota$  is  $\perp$ -preserving if, for all  $\sigma, \tau \in S$ , if  $\sigma \perp \tau$ , then  $\iota(\sigma) \perp \iota(\tau)$ .
- (3) If  $\iota$  is both strict order preserving and  $\perp$ -preserving, then we call  $\iota$  an *isomorphic embedding* of S into T. We say that T contains a copy of S if there is an isomorphic embedding  $\iota : S \to T$ ; in this situation, the image  $\iota[S]$  will be referred to as a copy of S in T.

**Remark 2.7.** We note that, in the above definition, the maps  $\iota: S \to T$  need not preserve levels, i.e., there may be  $\sigma \in S$  such that  $\operatorname{ht}_S(\sigma) \neq \operatorname{ht}_T(\iota(\sigma))$ . In particular, a copy of S in T need not be a subtree of T, as it need not be downward closed. Also note that S and T need not have the same height, e.g., it could be the case that S is a subtree of  ${}^{<\alpha}\beta$  and T is a subtree of  ${}^{<\gamma}\delta$  for cardinals  $\alpha \neq \gamma$  and  $\beta \neq \delta$ . Note, however, that if  $\iota: S \to T$  is strict order preserving, then we will have  $\operatorname{ht}_S(s) \leq \operatorname{ht}_T(\iota(s))$  for all  $s \in S$ . **Definition 2.8.** Let  $\kappa$  and  $\lambda$  be cardinals, with  $\kappa$  regular, and suppose that  $X \subseteq {}^{\kappa}\lambda$ . We say that X has the  $\kappa$ -perfect set property, denoted  $\mathsf{PSP}_{\kappa}(X)$ , if either  $|X| \leq \kappa$  or X contains a  $\kappa$ -perfect subset.

Note that, in the classical Baire space  ${}^{\omega}\omega$ , a nonempty closed subset of  ${}^{\omega}\omega$  is perfect in the classical sense (i.e., is closed with no isolated points) if and only if it is  $\omega$ -perfect in the sense of Definition 2.5(7), and a set  $X \subseteq {}^{\omega}\omega$  has the classical perfect set property if and only if it satisfies  $\mathsf{PSP}_{\omega}(X)$  as in Definition 2.8.

The following well-known proposition is easily proven.

**Proposition 2.9.** Let  $\kappa$  and  $\lambda$  be cardinals, with  $\kappa$  regular, and suppose that  $X \subseteq \kappa \lambda$  is closed. Then  $\mathsf{PSP}_{\kappa}(X)$  holds if and only if  $|X| \leq \kappa$  or T(X) contains a copy of  ${}^{<\kappa}2$ .

We end this section with some preliminaries on forcing. We first recall the definition of *strategic closure*.

**Definition 2.10.** Let  $\mathbb{P}$  be a partial order and let  $\beta$  be an ordinal.

- (1)  $\partial_{\beta}(\mathbb{P})$  is the two-player game in which Players I and II alternate playing conditions from  $\mathbb{P}$  to attempt to construct a  $\leq_{\mathbb{P}}$ -decreasing sequence  $\langle p_{\alpha} | \alpha < \beta \rangle$ . Player I plays at odd stages, and Player II plays at even stages (including limit stages). Player II is required to play  $p_0 = 1_{\mathbb{P}}$ . If, during the course of play, a limit ordinal  $\alpha < \beta$  is reached such that  $\langle p_{\eta} | \eta < \alpha \rangle$  has no lower bound in  $\mathbb{P}$ , then Player I wins. Otherwise, Player II wins.
- (2)  $\mathbb{P}$  is said to be  $\beta$ -strategically closed if Player II has a winning strategy in  $\Im_{\beta}(\mathbb{P})$ .

We now recall the following lemma, due to Silver. The lemma is usually stated with the hypothesis that  $\mathbb{R}$  is  $\tau^+$ -closed (cf. [16, Lemma 4]), but the standard proof is easily seen to work under the weaker assumption that  $\mathbb{R}$  is  $(\tau + 1)$ -strategically closed, so we leave it to the reader.

**Lemma 2.11.** Suppose that  $\tau < \nu$  are infinite regular cardinals, with  $2^{\tau} \geq \nu$ . Suppose that T is a  $\nu$ -tree and  $\mathbb{R}$  is a forcing poset that is  $(\tau + 1)$ -strategically closed. Then forcing with  $\mathbb{R}$  cannot add a branch of length  $\nu$  through T, i.e., every cofinal branch through T in  $V^{\mathbb{R}}$  is in V.

# 3. VÄÄNÄNEN'S GAME

The following game was introduced by Väänänen in [17] in order to generalize the notion of perfectness.

**Definition 3.1.** Suppose that  $\kappa$  is a regular cardinal, E is a subset of  ${}^{\kappa}\kappa, x_0 \in {}^{\kappa}\kappa$ , and  $\delta \leq \kappa$  is an infinite ordinal. Then the two-player game  $G_{\kappa}(E, x_0, \delta)$  is defined as follows. The game consists of rounds indexed by ordinals  $\xi$  with  $1 \leq \xi < \delta$ . In round  $\xi$ , Player I first plays an ordinal  $\alpha_{\xi} < \kappa$ , and then Player II plays an element  $x_{\xi} \in E$ . The plays must satisfy the following requirements:

- $\langle \alpha_{\xi} \mid 1 \leq \xi < \delta \rangle$  is an increasing, continuous sequence of ordinals;
- for all  $0 \le \eta < \xi < \delta$ , we have -  $x_{\eta} \ne x_{\xi}$ ;
  - $x_{\eta} \upharpoonright \alpha_{\eta+1} = x_{\xi} \upharpoonright \alpha_{\eta+1}.$

Player II wins if they can successfully play  $x_{\xi}$  for all  $1 \leq \xi < \delta$ ; otherwise, Player I wins.

**Definition 3.2.** Suppose that  $\kappa$  is a regular cardinal, E is a subset of  ${}^{\kappa}\kappa$ , and  $\delta \leq \kappa$  is an infinite ordinal. Then we say that E is  $\delta$ -perfect if it is closed and, for every  $x_0 \in E$ , Player II has a winning strategy in  $G_{\kappa}(E, x_0, \delta)$ .

Note that  $G_{\kappa}(E, x_0, \delta)$  becomes harder for Player II to win as  $\delta$  increases; thus, if  $\omega \leq \delta_0 \leq \delta_1 \leq \kappa$  and  $E \subseteq {}^{\kappa}\kappa$  is  $\delta_1$ -perfect, then it is also  $\delta_0$ -perfect.

**Remark 3.3.** Given a regular uncountable cardinal  $\kappa$ , there is a slight discrepancy between the notion of  $\kappa$ -perfect isolated in Definition 3.2 notion of  $\kappa$ -perfect isolated in Definition 2.5(7), which we will call *strongly*  $\kappa$ -*perfect* when we need to distinguish it from the notion in Definition 3.2. For example, as noted in [17], the set of all  $x \in {}^{\kappa}\kappa$  such that  $x(\alpha) = 0$  for only finitely many  $\alpha$  is  $\kappa$ -perfect but not strongly  $\kappa$ -perfect. However, it is readily verified that every strongly  $\kappa$ -perfect subset of  ${}^{\kappa}\kappa$  is  $\kappa$ -perfect and, conversely, every  $\kappa$ -perfect subset of  ${}^{\kappa}\kappa$  contains a strongly  $\kappa$ -perfect subset. In particular, the  $\kappa$ -perfect set property is equivalent when defined with either notion. Hence, for the purposes of this paper, it will not be necessarily to distinguish between the two.

In light of the above remark, the following definition generalizes Definition 2.8 in the setting of  $\kappa \kappa$ .

**Definition 3.4.** Suppose that  $\kappa$  is a regular infinite cardinal,  $\delta \leq \kappa$  is an infinite ordinal, and  $X \subseteq {}^{\kappa}\kappa$ . We say that X has the  $\delta$ -perfect set property, denoted  $\mathsf{PSP}_{\delta}(X)$ , if either  $|X| \leq \kappa$  or X contains a  $\delta$ -perfect subset.

The following proposition is readily verified; we leave the proof to the reader.

**Proposition 3.5.** Suppose that  $\kappa$  is a regular cardinal,  $\delta \leq \kappa$  is an infinite ordinal, and  $T \subseteq {}^{<\kappa}\kappa$  is a tree. If [T] is  $\delta$ -perfect, then T contains a copy of  ${}^{<\delta}2$ .

Note that a set  $E \subseteq {}^{\kappa}\kappa$  is  $\omega$ -perfect if and only if it is perfect in the classical sense. Unlike the case with  ${}^{\omega}\omega$ , it is not necessarily the case that perfect sets have full cardinality. For example, if  $\kappa$  is regular and uncountable and

$$E = \{ x \in {}^{\kappa}\kappa \mid |\{\alpha < \kappa \mid x(\alpha) \neq 0\}| < \kappa \},\$$

then E is readily seen to be a perfect subset of  $\kappa \kappa$ . However,  $|E| = \kappa^{<\kappa}$ , so if  $\kappa^{<\kappa} < 2^{\kappa}$ , then  $|E| < 2^{\kappa}$ . Nonetheless, we do recover a version of the Cantor-Bendixson theorem at higher  $\kappa$ :

**Proposition 3.6.** Suppose that  $E \subseteq {}^{\kappa}\kappa$  is closed. Then there is  $X \subseteq E$  such that  $|X| \leq \kappa^{<\kappa}$  and  $E \setminus X$  is perfect.

*Proof.* Let  $\Sigma = \{ \sigma \in {}^{<\kappa}\kappa \mid |E \cap N_{\sigma}| \le \kappa^{<\kappa} \}$ , and let  $X = \bigcup \{E \cap N_{\sigma} \mid \sigma \in \Sigma\}$ . Then X is as desired.

**Corollary 3.7.** If  $\kappa$  is an infinite regular cardinal such that  $\kappa^{<\kappa} = \kappa$ , then every closed set  $E \subseteq {}^{\kappa}\kappa$  satisfies the  $\omega$ -perfect set property.

As noted in the introduction, in  ${}^{\omega}\omega$ , every nonempty closed set is a continuous image of the entire space. For uncountable  $\kappa$ , this is no longer the case. However, if we know that a closed set E is a continuous image of  ${}^{\kappa}\kappa$ , then we can slightly improve upon the conclusion of the preceding proposition. The proof of the following theorem is a variation on that of [8, Theorem 1.1]

**Theorem 3.8.** Suppose that  $\kappa$  is an uncountable regular cardinal and  $E \subseteq {}^{\kappa}\kappa$  is a closed set that is a continuous image of  $\kappa \kappa$ . Then there is  $X \subseteq E$  with  $|X| \leq \kappa^{<\kappa}$ such that  $E \setminus X$  is closed and, for every  $x_0 \in E \setminus X$ , Player II has a winning strategy in  $G_{\kappa}(E, x_0, \omega + 1)$ .

*Proof.* Let  $f: {}^{\kappa}\kappa \to E$  be a continuous surjection. Let X be as in the proof of Proposition 3.6. Let  $X_0$  be the set of  $x \in E$  such that, for every  $y \in f^{-1}\{y\}$ , there is  $\alpha < \kappa$  such that  $|f[N_{y \upharpoonright \alpha}]| \leq \kappa^{<\kappa}$ .

Claim 3.9.  $|X_0| \leq \kappa^{<\kappa}$ .

Proof. Let

$$\Sigma = \{ \sigma \in {}^{<\kappa} \kappa \mid |f[N_{\sigma}]| \le \kappa^{<\kappa} \},$$
  
and let  $Y = \bigcup \{ f[N_{\sigma}] \mid \sigma \in \Sigma \}$ . Then  $|Y| \le \kappa^{<\kappa}$  and  $X_0 \subseteq Y$ .

**Claim 3.10.**  $X \subseteq X_0$ , and  $E \setminus X$  is the closure of  $E \setminus X_0$  in  ${}^{\kappa}\kappa$ .

*Proof.* The facts that  $X \subseteq X_0$  and  $E \setminus X$  is closed follow immediately from the definitions. To show that  $E \setminus X$  is the closure of  $E \setminus X_0$ , it suffices to show that, for every  $x \in E \setminus X$  and every open set U with  $x \in U$ , there is  $y \in U \cap (E \setminus X_0)$ . Thus, fix such x and U. Without loss of generality, we may assume that  $U = N_{x \uparrow \alpha}$ for some  $\alpha < \kappa$ . Since  $x \notin X$ , we know that  $|U \cap E| > \kappa^{<\kappa}$ . Thus, by Claim 3.9, we know that  $U \cap (E \setminus X_0) \neq \emptyset$ . 

It remains to show that Player II has a winning strategy in  $G_{\kappa}(E, x_0, \omega + 1)$  for all  $x_0 \in E \setminus X$ . We first establish the following claim.

**Claim 3.11.** Suppose that  $U \subseteq {}^{\kappa}\kappa$  is an open set such that  $|f[U]| > \kappa^{<\kappa}$ . Then there is  $y \in U$  such that  $|f[N_{y \restriction \alpha}]| > \kappa^{<\kappa}$  for all  $\alpha < \kappa$ .

*Proof.* If not, then, for every  $y \in U$ , there is  $\alpha_y < \kappa$  such that  $|f[N_y|_{\alpha_y}]| \le \kappa^{<\kappa}$ . Then we have  $|f[U]| \leq |\bigcup \{f[N_y|_{\alpha_y}] \mid y \in U\}| \leq \kappa^{<\kappa}$ , which is a contradiction.  $\Box$ 

**Claim 3.12.** Suppose that  $x \in E \setminus X$  and  $\alpha < \kappa$ . Then there is  $x' \in E \setminus X_0$  such that  $x \neq x'$  and  $x' \upharpoonright \alpha = x \upharpoonright \alpha$ .

*Proof.* If  $x \in X_0$  then we can choose any x' in the set  $(E \setminus X_0) \cap N_{x \upharpoonright \alpha}$ , which is nonempty by Claim 3.10. Thus, we can assume that  $x \notin X_0$ . Choose  $y \in {}^{\kappa}\kappa$  such that f(y) = x and  $|f[N_{y \upharpoonright \beta}]| > \kappa^{<\kappa}$  for all  $\beta < \kappa$ . By the continuity of f, we can find  $\beta < \kappa$  such that  $f[N_{y \restriction \beta}] \subseteq N_{x \restriction \alpha}$ . Let  $U = N_{y \restriction \beta} \setminus f^{-1}\{x\}$ . Then U is an open set such that  $|f[U]| > \kappa^{<\kappa}$ . Therefore, by Claim 3.11 we can find  $y' \in U$  such that  $|f[N_{y' \upharpoonright \gamma}]| > \kappa^{<\kappa}$  for all  $\gamma < \kappa$ . Then x' = f(y') is as desired. 

Fix  $x_0 \in E \setminus X$ . We will describe a winning strategy for Player II in  $G_{\kappa}(E \setminus E)$  $X, x_0, \omega + 1$ ). In the course of the game, as the players play the sequences  $\langle \alpha_n \rangle$  $1 \leq n \leq \omega$  and  $\langle x_n \mid 1 \leq n \leq \omega \rangle$ , Player II will also construct a sequence  $\langle y_n \mid 1 \leq n < \omega \rangle$  of elements of  ${}^{\kappa}\kappa$  and increasing sequences  $\langle \beta_n \mid 1 \leq n < \omega \rangle$  and  $\langle \gamma_n \mid 1 \leq n < \omega \rangle$  of ordinals below  $\kappa$  such that, for all  $1 \leq n < \omega$ , we have

- $\begin{array}{l} \bullet \ f(y_n) = x_n \in E \setminus X_0; \\ \bullet \ |f[N_{y_n \upharpoonright \gamma}]| > \kappa^{<\kappa} \ \text{for all} \ \gamma < \kappa; \end{array}$
- $f[N_{y_n \upharpoonright \beta_n}] \subseteq N_{x_n \upharpoonright \beta_n};$
- $\alpha_n \leq \beta_n;$
- $x_n \upharpoonright \beta_n \neq x_{n-1} \upharpoonright \beta_n;$
- $\max\{\alpha_{n+1}, \beta_n\} \le \gamma_n \le \beta_{n+1};$

- $f[N_{y_n \upharpoonright \gamma_n}] \subseteq N_{x_n \upharpoonright \alpha_{n+1}};$
- $y_{n+1} \upharpoonright \gamma_n = y_n \upharpoonright \gamma_n.$

We begin by describing Player II's first play. After Player I plays an ordinal  $\alpha_1$ , apply Claim 3.12 to find  $x_1 \in E \setminus X_0$  such that  $x_1 \neq x_0$  and  $x_1 \upharpoonright \alpha_1 = x_0 \upharpoonright \alpha_0$ . Then fix  $y_1 \in {}^{\kappa}\kappa$  such that  $f(y_1) = x_1$  and  $|f[N_{y_1 \upharpoonright \gamma}]| > \kappa^{<\kappa}$  for all  $\gamma < \kappa$ . Finally, using the continuity of f and the fact that  $\kappa$  has uncountable cofinality fix  $\beta_1$  with  $\alpha_1 \leq \beta_1 < \kappa$  such that  $f[N_{y_1 \upharpoonright \beta_1}] \subseteq N_{x_1 \upharpoonright \beta_1}$  and  $x_1 \upharpoonright \beta_1 \neq x_0 \upharpoonright \beta_0$ .

Now suppose that  $1 \leq n < \omega$  and  $\langle x_0, \alpha_1, x_1, \ldots, \alpha_{n+1} \rangle$  is a partial play of the game, with Player II playing so far according to the strategy that we are about to describe, and that Player II has also specified  $\langle (y_i, \beta_i) | 1 \leq i \leq n \rangle$ . We specify how to choose  $x_{n+1}$ , as well as  $\gamma_n$ ,  $\beta_{n+1}$ , and  $y_{n+1}$ . First, choose  $\gamma_n \geq \max\{\alpha_{n+1}, \beta_n\}$  such that  $f[N_{y_n \upharpoonright \gamma_n}] \subseteq N_{x_n \upharpoonright \alpha_{n+1}}$ , and let

$$U = N_{y_n \upharpoonright \gamma_n} \setminus \bigcup \{ f^{-1}\{x_i\} \mid i \le n \}.$$

Then U is an open set such that  $|f[U]| > \kappa^{<\kappa}$ , so, by Claim 3.11, we can choose  $y_{n+1} \in U$  such that  $|f[N_{y_{n+1} \upharpoonright \gamma}]| > \kappa^{<\kappa}$  for all  $\gamma < \kappa$ . Let  $x_{n+1} = f(y_{n+1})$ , and choose  $\beta_{n+1} > \gamma_n$  such that

- $f[N_{y_{n+1} \upharpoonright \beta_{n+1}}] \subseteq N_{x_{n+1} \upharpoonright \beta_{n+1}}$ ; and
- $x_{n+1} \upharpoonright \beta_{n+1} \neq x_n \upharpoonright \beta_{n+1}$ .

It is readily verified that this satisfies all of the requirements of the construction, and we can move on to the next round of the game.

This completely describes Player II's strategy at rounds indexed by natural numbers. It remains to show that, if they play according to this strategy, then they guarantee that they will be able to play in round  $\omega$ . To this end, suppose that  $\langle (\alpha_n, x_n) | 1 \leq n < \omega \rangle$  is an initial segment of the game of length  $\omega$ , with Player II playing according to the described strategy. Suppose that  $\langle (\beta_n, y_n) | 1 \leq n < \omega \rangle$ are the auxiliary objects specified by this strategy.

Now let  $\beta_{\omega} = \sup\{\beta_n \mid n < \omega\} = \sup\{\gamma_n \mid n < \omega\} \ge \alpha_{\omega} = \sup\{\alpha_n \mid n < \omega\}$ . We know that, for all  $1 \le n < \omega$ , we have  $y_{n+1} \upharpoonright \gamma_n = y_n \upharpoonright \gamma_n$ . We can therefore find a  $y_{\omega} \in {}^{\kappa}\kappa$  such that, for all  $1 \le n < \omega$ , we have  $y_{\omega} \upharpoonright \gamma_n = y_n \upharpoonright \gamma_n$ . Let  $x_{\omega} = f(y_{\omega})$ . Then  $x_{\omega} \in E$  and, moreover, for all  $1 \le n < \omega$ , we have

$$f[N_{y_{\omega} \upharpoonright \gamma_n}] = f[N_{y_n \upharpoonright \gamma_n}] \subseteq N_{x_n \upharpoonright \alpha_{n+1}}.$$

Therefore, for all  $n < \omega$  we have  $x_{\omega} \upharpoonright \alpha_{n+1} = x_n \upharpoonright \alpha_{n+1}$ . By a similar argument, using the fact that  $y_{\omega} \upharpoonright \beta_n = y_n \upharpoonright \beta_n$  for all  $1 \le n < \omega$ , we know that  $x_{\omega} \upharpoonright \beta_n = x_n \upharpoonright \beta_n \neq x_{n-1} \upharpoonright \beta_n$ , and hence  $x_{\omega} \notin \{x_n \mid n < \omega\}$ . Thus,  $x_{\omega}$  is a valid play for Player II in round  $\omega$ , completing our description of a winning strategy for Player II in  $G(E, x_0, \omega + 1)$ .

For some observations about this theorem and some remaining open questions, see Section 8 below.

#### 4. Kurepa trees and the perfect set property

As mentioned in the introduction,  $\kappa$ -Kurepa trees often provide natural examples of closed subsets of  $\kappa \kappa$  that fail to have various perfect set properties. Indeed, suppose that  $\kappa$  is a regular uncountable cardinal that is not strongly inaccessible and  $T \subseteq \kappa \kappa$  is a  $\kappa$ -Kurepa tree. Let  $\lambda < \kappa$  be the least cardinal such that  $2^{\lambda} \geq \kappa$ . Then we claim that [T] cannot contain a  $(\lambda + 1)$ -perfect subset. If it did, then,

by Proposition 3.5, we could find an isomorphic embedding  $\iota : {}^{<\lambda+1}2 \to T$ . Since  $2^{<\lambda} < \kappa$ , we can find  $\alpha < \kappa$  such that  $\iota[{}^{<\lambda}2] \subseteq T_{<\alpha}$ . Then  $\{\iota(x) \upharpoonright \alpha \mid x \in {}^{\lambda}2\}$  is a subset of  $T_{\alpha}$  of cardinality  $2^{\lambda} \ge \kappa$ , contradicting the fact that T is a  $\kappa$ -tree.

In this section, we show that  $\kappa$ -Kurepa trees are not needed to produce closed subsets of  $\kappa \kappa$  failing various perfect set properties. For concreteness and readability, we will focus on the important special case  $\kappa = \omega_1$ , but the techniques can readily be generalized to produce similar results at other cardinals.

Beginning in a model with an inaccessible cardinal, we will produce two models in which GCH holds, the Kurepa Hypothesis fails, and there is a closed subset of  ${}^{\omega_1}\omega_1$  that fails to have the  $(\omega + 1)$ -perfect set property.<sup>4</sup> In both models, the closed subset of  ${}^{\omega_1}\omega_1$  failing to have the  $(\omega_1)$ -perfect set property will be of the form [T]for some weak Kurepa tree  $T \subseteq {}^{<\omega_1}\omega_1$ . The fact that  $\mathsf{PSP}_{\omega+1}([T])$  fails will be implied by the fact that, in both case, T will fail to contain a copy of  ${}^{<\omega+1}2$ . In the first model, constructed in Theorem 4.4, every weak Kurepa tree will contain an Aronszajn subtree. In the second, constructed in Theorem 4.15, the weak Kurepa tree T we satisfying  $\neg \mathsf{PSP}_{\omega+1}([T])$  will not contain any Aronszajn subtree. Note that either theorem individually will establish Theorem A from the introduction.

We first need the following preliminary result.

**Theorem 4.1.** Assume that G is  $\mathbb{Q} = \text{Add}(\omega, 1)$ -generic over V. Suppose T is an arbitrary normal tree in V. Then T does not contain a copy of  ${}^{<\omega+1}2^{V[G]}$  in V[G].

*Proof.* We will proceed by contradiction and will assume that, in V[G], there is an isomorphic embedding from  $\langle \omega^{+1}2^{V[G]} \rangle$  into T. We first observe that it is suffices to consider embeddings which preserve meets and are continuous at limits:

**Claim 4.2.** Let T be a normal tree, and let f be an isomorphic embedding from  ${}^{<\omega+1}2$  to T.<sup>5</sup> Then there is an isomorphic embedding g which moreover preserves meets and is continuous at limits, i.e.:

(i) (preserves meets)<sup>6</sup> for all  $a, b \in {}^{<\omega+1}2$ ,  $g(a \wedge b) = g(a) \wedge g(b)$ ;

(ii) (continuous at limits) for all  $x \in {}^{\omega}2$ ,  $g(x) = \sup\{g(x \upharpoonright n) \mid n < \omega\}$ .

*Proof.* We first define g which preserves meets on  ${}^{<\omega}2$ , and then argue that we can extend g to the whole tree  ${}^{<\omega+1}2$ . For  $a \in {}^{<\omega}2$ , define

$$g(a) = f(a^{0}) \wedge f(a^{1}).$$

First note that  $f(a) \leq g(a)$  for all  $a \in {}^{<\omega}2$ ; from this it is easy to see that g is  $\bot$ -preserving because f is  $\bot$ -preserving. Moreover for all  $a \subset b \in {}^{<\omega}2$  we have g(a) < f(b) since  $g(a) < f(b \upharpoonright (\operatorname{dom}(a) + 1)) \leq f(b)$ . Therefore for  $a \subseteq b \in {}^{<\omega}2$  we have  $g(a) \leq f(b) \leq g(b)$ , so g preserves  $\leq$ . To see that g preserves meets, it is enough to verify that g preserves meets for incomparable nodes in  ${}^{<\omega}2$ , since we already verified that g is an isomorphic embedding from  ${}^{<\omega}2$  to T. Assume that  $a, b \in {}^{<\omega}2$  are incomparable. Then  $g(a) \wedge g(b) = f(a) \wedge f(b) = g(a \wedge b)$ : the first equality holds since  $f(a) \leq g(a)$  and  $f(b) \leq g(b)$  and the second holds by the definition of g (observe that for all  $a, a', b, b' \in T$ , with a, b incomparable,  $a \leq a', b \leq b'$  implies  $a \wedge b = a' \wedge b'$ ).

 $<sup>^{4}</sup>$ Note that this is sharp, by Corollary 3.7

<sup>&</sup>lt;sup>5</sup>For the purposes of the proof of Theorem 4.1, think of this Claim as being applied in V[G].

<sup>&</sup>lt;sup>6</sup>Note that if g is an isomorphic embedding then g preserves meets for all comparable nodes in  $^{<\omega+1}2$ .

We extend the definition of g to  $x \in {}^{\omega}2$  by letting  $g(x) = \sup\{g(x \upharpoonright n) \mid n < \omega\}$ . This makes g continuous at limits by definition provided we argue that the suprema exist: This is true because  $g(x \upharpoonright n) \leq f(x)$  for all  $n < \omega$ , and by normality of T, there is a unique node in T on the supremum of the levels of  $g(x \upharpoonright n)$ , and this node is  $\leq f(x)$ . Finally notice that if g is an isomorphic embedding from  ${}^{<\omega+1}2$  into T which preserves meets on  ${}^{<\omega}2$ , then it also preserves meets for all  $x, y \in {}^{\omega}2$ , so g is as required.

**Remark 4.3.** Note that, if T is a normal tree and f is an isomorphic embedding from  ${}^{<\omega+1}2$  to T which is continuous at limits, then f is determined by its restriction to  ${}^{<\omega}2$ .

Let us work in V[G]. Let T be a normal tree such that  $T \in V$ . Assume for a contradiction that  $f: ({}^{<\omega+1}2)^{V[G]} \to T$  is an isomorphic embedding; by Claim 4.2, we can assume that f preserve meets and is continuous at limits.

Let  $p \in G$  force this property about some name f for f. Let us work in V now. For each  $x \in ({}^{\omega}2)^V$ , there is a condition  $q_x \leq p$  in  $\mathbb{Q}$  which decides the value of  $\dot{f}(x)$ . Since the forcing is countable, the Baire category theorem implies that there is some  $q \in \mathbb{Q}$  such that

$$Y = \{ x \in (^{\omega}2)^V \mid q_x = q \}$$

is not nowhere dense (we say it is somewhere dense), i.e.

$$\exists a^* \in {}^{<\omega}2 \; \forall a \in {}^{<\omega}2 \; (a^* \subseteq a \to \exists x \in Y \; x \upharpoonright |a| = a),$$

equivalently

 $(1) \quad \exists a^* \in {}^{<\omega}2 \; \forall a \in {}^{<\omega}2 \; (a^* \subseteq a \to \exists x \in ({}^{\omega}2)^V \; x \upharpoonright |a| = a \text{ and } q \text{ decides } \dot{f}(x)).$ 

We will now show that the fact that  $\dot{f}$  is forced to be continuous allows us to prove that

$$q$$
 decides  $f[N_{a^*}^v]$ ,

where  $N_{a^*}^v$  denotes the set of all  $x \in ({}^{\omega}2)^V$  which extend  $a^*$ .

Consider the subset  $C = \{x \land y \mid x, y \in Y, x \neq y\}$  of  $\langle \omega 2$ . This set is in V, since Y is in V. More importantly, q decides the value of  $\dot{f}(a)$  for all  $a \in C$  since q decides  $\dot{f}(x)$  for all  $x \in Y$  and  $\dot{f}$  is forced to preserve meets. Since Y satisfies (1), it holds that

$$a^* \uparrow = \{ a \in {}^{<\omega}2 \, | \, a^* \subseteq a \}$$

is a subset of C. Since q decides  $\dot{f}(a)$  for all  $a \in C$ , q decides the values of  $\dot{f}(a)$  for all  $a \supseteq a^*$ . Since  $\dot{f}$  is forced to be continuous at limits, q also decides the values of  $\dot{f}(x)$  for all  $x \in ({}^{\omega}2)^V$  such that  $a^* \subseteq x$ .

Now we finish the proof by arguing that we can read off in V the Cohen subset  $c = \bigcup G$  added by  $\mathbb{Q}$ , which gives the desired contradiction. In V[G], c can be used to define a cofinal branch c' through  $a^* \uparrow$  as follows: Identify c with a function from  $\omega$  to 2 and define  $c' = (a^*)^{\frown}c$ ; i.e. letting  $n = \operatorname{dom}(a^*)$ ,  $c'(k) = a^*(k)$  for k < n and c'(k) = c(k-n) for  $k \ge n$ . Recall that f is an isomorphic embedding from  $(^{<\omega+1}2)^{V[G]}$  into T; therefore there is  $t^* \in T$  such that  $f(c') = t^*$ . Since  $t^*$  is in T,  $t^*$  is in V. Define  $d: \omega \to 2$  in V inductively as follows:

- (a) d(0) = 0, provided  $q \Vdash \dot{f}((a^*) \land (n, 0)) \le t^*$ ; otherwise let d(0) = 1.
- (b) If  $d \upharpoonright k$  is defined, set d(k) = 0, provided  $q \Vdash \dot{f}((a^*)^{\frown} d \upharpoonright k^{\frown} \langle k, 0 \rangle) \leq t^*$ ; otherwise let d(k) = 1.

Now  $q \Vdash \dot{f}((a^*) \cap d) \leq t^*$  since  $\dot{f}$  is forced to be an isomorphic embedding. We claim that  $q \Vdash d = \dot{c}$ : Since  $q \Vdash \dot{f}((a^*) \cap d) \leq t^*$  and  $q \Vdash \dot{f}(\dot{c}') = t^*$  and  $\dot{f}$  is forced to be an isomorphic embedding,  $q \Vdash (a^*) \cap d = \dot{c}'$  and hence by the definition of c',  $q \Vdash d = \dot{c}$ . This yields the desired contradiction.

We now prove the first of the two main theorems of this section.

**Theorem 4.4.** Suppose that there is an inaccessible cardinal  $\kappa$ . Then there is a forcing extension in which

- (1)  $\kappa = \omega_2;$
- (2) GCH;
- $(3) \neg \mathsf{KH};$
- (4) every weak Kurepa tree contains an Aronszajn subtree, moreover if we assume that ◊ holds in V, every weak Kurepa tree contains a Suslin subtree.
- (5) there is a weak Kurepa tree that does not contain a copy of  ${}^{<\omega+1}2$ .

*Proof.* Assume that GCH holds. Let  $\mathbb{P} = \operatorname{Coll}(\omega_1, <\kappa)$  and let  $\mathbb{Q} = \operatorname{Add}(\omega, 1)$ . We claim that the generic extension by  $\mathbb{P} \times \mathbb{Q}$  is the desired forcing extension in which (1)–(5) hold. Item (1) and item (2) are clear, and item (3) follows by standard arguments for the tree property as in [16] or [3].

The proof of item (4). To prove item (4), it suffices to show that every weak Kurepa tree in a generic extension by  $\mathbb{P} \times \mathbb{Q}$  contains a copy of  $({}^{<\omega_1}2)^V$ . To see this, let  $G \times H$  be  $\mathbb{P} \times \mathbb{Q}$ -generic over V. In V, we can construct a special  $\omega_1$ -Aronszajn tree as a subtree of  $({}^{<\omega_1}2)^V$  and this tree is preserved in all forcing extensions which preserve  $\omega_1$ . In particular, it is still a special  $\omega_1$ -Aronszajn tree in V[G][H]. Moreover, note that  $\mathbb{P} \times \mathbb{Q}$  does not add cofinal branches to any  $\omega_1$ -tree in V since  $\mathbb{P}$ is  $\omega_1$ -closed in V and  $\mathbb{Q}$  is  $\omega_1$ -Knaster in V[G]. It follows that all (not only special)  $\omega_1$ -Aronszajn subtrees of  $({}^{<\omega_1}2)^V$  remain Aronszajn in V[G][H]. In particular, if S is an  $\omega_1$ -Suslin subtree of  $({}^{<\omega_1}2)^V$  in V, it is still Aronszajn in V[G][H]. In fact it is still a Suslin tree in V[G][H]: by Easton's Lemma it is Suslin in V[G] since  $\mathbb{P}$ is  $\omega_1$ -closed and it is still Suslin in V[G][H] since  $\mathbb{Q}$  is  $\omega_1$ -Knaster and S is ccc in V[G].

Let us now proceed to show that every weak Kurepa tree in a generic extension by  $\mathbb{P} \times \mathbb{Q}$  contains a copy of  $({}^{<\omega_1}2)^V$ . If  $\dot{T}$  is a  $\mathbb{P} \times \mathbb{Q}$ -name for a weak Kurepa tree then, since  $\mathbb{P} \times \mathbb{Q}$  is  $\kappa$ -cc,  $\dot{T}$  is a  $\mathbb{P}_{\theta} \times \mathbb{Q}$ -name for some regular cardinal  $\theta < \kappa$ , where we denote  $\mathbb{P}_{\theta} = \operatorname{Coll}(\omega_1, <\theta)$ . Let  $G_{\theta}$  be  $\mathbb{P}_{\theta}$ -generic over V and H be  $\mathbb{Q}$ -generic over  $V[G_{\theta}]$ . We will work in  $V[G_{\theta}]$  and we will show that T contains a copy of  $({}^{<\omega_1}2)^{V[G_{\theta}]}$  in  $V[G_{\theta}][H]$ . Note that  $({}^{<\omega_1}2)^V = ({}^{<\omega_1}2)^{V[G_{\theta}]}$  since  $\mathbb{P}_{\theta}$  is  $\omega_1$ -closed.

Note that, in  $V[G_{\theta}][H]$ , we have  $2^{\omega_1} < \kappa$ , and hence T has fewer than  $\kappa$ -many cofinal branches in  $V[G_{\theta}][H]$ . Since  $\dot{T}$  is a  $\mathbb{P} \times \mathbb{Q}$ -name for a weak Kurepa tree, it is forced to have  $\kappa$ -many branches in the extension by  $\mathbb{P} \times \mathbb{Q} \cong \mathbb{P}_{\theta} * (\mathbb{Q} \times \mathbb{P}^{\theta})$ , where  $\mathbb{P}^{\theta} = \operatorname{Coll}(\omega_1, [\theta, <\kappa))^{V[G_{\theta}]}$ . Therefore, we can fix in  $V[G_{\theta}]$  conditions  $p^* \in \mathbb{P}^{\theta}$  and  $q^* \in H$  and a  $\mathbb{Q} \times \mathbb{P}^{\theta}$ -name  $\dot{b}$  for a cofinal branch through  $\dot{T}$  such that

(2) 
$$(q^*, p^*) \Vdash b \notin V[G_\theta][H]$$

Without loss of generality, we can assume that the underlying set of T is forced to be a subset of  $\omega_1 \times \omega_1$  and that if  $(\alpha, \gamma) \in \dot{T}$ , then  $(\alpha, \gamma) \in \dot{T}_{\gamma}$ . In  $V[G_{\theta}]$ , we will build by induction on  $\omega_1$  the following objects:

- a labeled tree  $\mathcal{T} = \{p_s \mid s \in {}^{<\omega_1}2\}$  of conditions in  $\mathbb{P}^{\theta}$ , all of them extending  $p^*$  from (2);
- a labeled tree  $\{\gamma_s \mid s \in {}^{<\omega_1}2\}$  of ordinals below  $\omega_1$ ;
- a maximal antichain  $A_s$  of conditions in  $\mathbb{Q}$  below  $q^*$  from (2) for each  $s \in {}^{<\omega_1}2;$

such that the following hold for each  $s \in {}^{<\omega_1}2$ :

- (a)  $p_t \leq p_s$  for each  $s \subseteq t$  in  ${}^{<\omega_1}2$ ;
- (b)  $\gamma_s < \gamma_t$  for each  $s \subset t$  in  ${}^{<\omega_1}2$ ;
- (c) for each  $q \in A_s$ , the conditions  $(q, p_{s^{\frown}0})$  and  $(q, p_{s^{\frown}1})$  decide  $\dot{b}$  up to  $\gamma_s$  differently; i.e., there are  $\gamma \leq \gamma_s$  and  $\tau_{q,p_{s^{\frown}0}} \neq \tau_{q,p_{s^{\frown}1}}$  both forced by q to be in  $\dot{T}_{\gamma}$  such that  $(q, p_{s^{\frown}0}) \Vdash \dot{b}(\gamma) = \tau_{q,p_{s^{\frown}0}}$  and  $(q, p_{s^{\frown}1}) \Vdash \dot{b}(\gamma) = \tau_{q,p_{s^{\frown}1}}$ .

The construction of  $\mathcal{T}$  uses the standard method of diagonalizing over antichains in  $\mathbb{Q}$  while taking lower bounds in  $\mathbb{P}^{\theta}$ , using the  $\omega_1$ -closure of  $\mathbb{P}^{\theta}$ , but we will give details to make the argument self-contained.

Set  $p_{\emptyset} = p^*$ . First assume that  $\alpha$  is a limit ordinal and for every  $\beta < \alpha$  and every  $s \in {}^{\beta}2$  the conditions  $p_s$  have been constructed. For  $s \in {}^{\alpha}2$  let  $p_s$  be a lower bound of  $\langle p_{s \restriction \beta} | \beta < \alpha \rangle$ . Note that  $A_s$  and  $\gamma_s$  for  $s \in {}^{\alpha}2$  will be constructed in the successor stage.

Now, assume that  $\alpha$  is a successor ordinal  $\alpha = \beta + 1$  and for every  $s \in {}^{\beta}2$ ,  $p_s$  has been constructed, and for every  $s \in {}^{<\beta}2$ ,  $A_s$  and  $\gamma_s$  have been constructed. Given  $s \in {}^{\beta}2$ , we describe the construction of  $p_{s \sim 0}$ ,  $p_{s \sim 1}$ ,  $A_s$  and  $\gamma_s$ .

**Claim 4.5.** For every  $q \leq q^*$  in  $\mathbb{Q}$ , all  $r^0, r^1 \in \mathbb{P}^\theta$  with  $r^0, r^1 \leq p^*$ , and all  $\gamma' < \omega_1$ , there are  $\gamma' < \gamma < \omega_1$ ,  $(q', p^0) \leq (q, r^0)$  and  $(q', p^1) \leq (q, r^1)$  such that  $(q', p^0)$  and  $(q', p^1)$  decide  $\dot{b}(\gamma)$  differently.

Proof. Let  $q \in \mathbb{Q}$ ,  $r^0, r^1 \in \mathbb{P}^{\theta}$  and  $\gamma' < \omega_1$  be given. Since  $\dot{b}$  is forced by  $(p^*, q^*)$  to be a cofinal branch through  $\dot{T}$  that is not in  $V[G_{\theta}][\dot{H}]$ , there are  $(\bar{q}, \bar{p}^0) \leq (q, r^0)$ ,  $(\bar{q}, \bar{p}^1) \leq (q, r^0)$  and  $\gamma > \gamma'$  such that  $(\bar{q}, \bar{p}^0)$  and  $(\bar{q}, \bar{p}^1)$  decide  $\dot{b}(\gamma)$  differently; i.e. there are  $\tau^0 \neq \tau^1$ , both forced by  $\bar{q}$  to be in  $\dot{T}_{\gamma}$ , such that  $(\bar{q}, \bar{p}^0) \Vdash \dot{b}(\gamma) = \tau^0$  and  $(\bar{q}, \bar{p}^1) \Vdash \dot{b}(\gamma) = \tau^1$ . Now, consider the condition  $(\bar{q}, r^1)$ : since  $\dot{b}$  is a  $\mathbb{P}^{\theta} \times \mathbb{Q}$  name for a cofinal branch through  $\dot{T}$ , there is an extension  $(q', p^1) \leq (\bar{q}, r^1)$  which decides  $\dot{b}(\gamma)$ . Since  $\tau^0 \neq \tau^1$ ,  $(q', p^1)$  cannot decide  $\dot{b}(\gamma)$  as being equal to both of them. Let  $p^0$  be  $\bar{p}^i$ , for some i < 2, such that  $(q', p^1)$  and  $(q', \bar{p}^i)$  disagree on  $\dot{b}(\gamma)$ . Then  $q', p^0, p^1$  and  $\gamma'$  are as required.

We use the previous claim to inductively construct in  $\delta$ -many stages for some  $\delta < \omega_1$  a maximal antichain  $A_s = \{q_i \in \mathbb{Q} \mid i < \delta\}$  below  $q^*$ , an increasing sequence of ordinals  $\langle \gamma_i < \omega_1 \mid i < \delta \rangle$  whose supremum will be  $\gamma_s$ , and decreasing sequences  $\langle p_i^0 \in \mathbb{P}^{\theta} \mid i < \delta \rangle$  and  $\langle p_i^1 \in \mathbb{P}^{\theta} \mid i < \delta \rangle$  with lower bounds  $p_{s^{-0}}$  and  $p_{s^{-1}}$ , respectively. Let us initialize the construction and define the required objects for i = 0. First set  $\gamma'_s = \sup\{\gamma_{s \restriction \beta'} \mid \beta' < \beta\}$  (in case  $\beta = 0$ , take  $\gamma'_s = 0$ ). By Claim 4.5 there are  $\gamma'_s < \gamma_0 < \omega_1$  and  $(q_0, p_0^0), (q_0, p_0^1) \leq (q^*, p_s)$  such that  $(q_0, p_0^0)$  and  $(q_0, p_0^1)$  decide  $\dot{b}(\gamma_0)$  differently. The condition  $q_0, p_0^0, p_0^1$  and the ordinal  $\gamma_0$  are as required.

Now assume that  $0 < i < \omega_1$  and for all j < i we already have  $q_j$ ,  $p_j^0$ ,  $p_j^1$  and  $\gamma_j$ .

If there is  $q \in \mathbb{Q}$  below  $q^*$  such that q is incompatible with all  $q_j$  for j < i, let us fix such q. If i is a limit ordinal, fix some lower bounds  $r^0$  and  $r^1$  of  $\langle p_j^0 \in \mathbb{P}^{\theta} | j < i \rangle$ and  $\langle p_j^1 \in \mathbb{P}^{\theta} | j < i \rangle$ , respectively, and a supremum  $\gamma'$  of  $\langle \gamma_j < \omega_1 | j < i \rangle$ . If i

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is a successor of j, set  $r^0 = p_j^0$ ,  $r^1 = p_j^1$  and  $\gamma' = \gamma_j$ . By Claim 4.5, there are  $\gamma' < \gamma_i < \omega_1$  and  $(q_i, p_i^0) \le (q, r^0), (q_i, p_i^1) \le (q, r^1)$  such that  $(q_i, p_i^0)$  and  $(q_i, p_i^1)$ decide  $\dot{b}(\gamma_i)$  differently. The conditions  $q_i, p_i^0, p_i^1$  and the ordinal  $\gamma_i$  are as required.

If there is no  $q \in \mathbb{Q}$  below  $q^*$  such that q is incompatible with  $q_j$  for all j < i, we stop the construction and set  $\delta = i$  and  $A_s = \{q_j \in \mathbb{Q} \mid j < \delta\}$ . If *i* is a limit ordinal, we set  $p_{s \cap 0}$  and  $p_{s \cap 1}$  to be lower bounds of  $\langle p_j^0 \in \mathbb{P}^{\theta} | j < \delta \rangle$  and  $\langle p_j^1 \in \mathbb{P}^{\theta} | j < \delta \rangle$ , respectively, and  $\gamma_s$  to be a supremum of  $\langle \gamma_j < \omega_1 | j < \delta \rangle$ . If *i* is a successor of *j*, then we set  $p_{s \frown 0} = p_j^0$ ,  $p_{s \frown 1} = p_j^1$  and  $\gamma_s = \gamma_j$ . Note that the construction will end after countably many steps since  $\mathbb{Q}$  is ccc, and hence  $\delta < \omega_1$ .

This ends the construction of the labeled tree  ${\mathcal T}$  and the related objects.

In  $V[G_{\theta}][H]$ , we define a copy of  $({}^{<\omega_1}2)^{V[G_{\theta}]}$  in  $T = \dot{T}^{V[G_{\theta}][H]}$  using the tree  $\mathcal{T}$ . This copy is given by an embedding  $h: ({}^{<\omega_1}2)^{V[G_{\theta}]} \to T$  which maps sequences  $s \cap 0$ and  $s^{1}$  to the nodes  $\tau_{q,p_{s^{1}}}$  and  $\tau_{q,p_{s^{1}}}$ , respectively where q is the unique element of  $H \cap A_s$  (see item (c) in the properties of  $\mathcal{T}$  for definitions). The definition of h extends continuously the limit levels: if  $s \in (\gamma 2)^{V[G_{\theta}]}$  for a limit ordinal  $\gamma < \omega_1$ , then let h(s) be the supremum of  $\{h(s \upharpoonright \alpha) \mid \alpha < \gamma\}$ . This supremum exists in T by normality and the fact that, for instance,  $h(s \cap 0)$  is above  $h(s \cap \alpha)$  for all  $\alpha < \delta$ . Since the q's are chosen from H, the forcing statements from item (c)

$$(q, p_{s^\frown 0}) \Vdash b(\gamma) = \tau_{q, p_{s^\frown 0}} \text{ and } (q, p_{s^\frown 1}) \Vdash b(\gamma) = \tau_{q, p_{s^\frown 1}}$$

respect the tree T, i.e. the embedding h preserves the strict ordering and the incompatibility of nodes between the trees  $(\langle \omega_1 2, \subseteq \rangle^{V[G_{\theta}]})$  and  $(T, \langle T)$ . It follows that T contains a copy of  $({}^{<\omega_1}2)^{V[G_{\theta}]} = ({}^{<\omega_1}2)^V$  as required.

The proof of item (5). Item (5) is a consequence of Theorem 4.1: In V[G] there are normal weak Kurepa trees, e.g.,  $({}^{<\omega_1}2)^V$ . Such a tree remains a weak Kurepa tree in V[G][H] and, by Theorem 4.1, cannot contain a copy of  $(\langle \omega+12 \rangle^{V[G][H]})$ .

We now turn to proving the second main theorem of this section, which will produce a model similar to that of Theorem 4.4, except we will obtain a weak Kurepa tree which does not contain a copy of  ${}^{<\omega+1}2$  and does not contain an Aronszajn subtree. First we define a forcing which adds a weak Kurepa tree with these properties and establish some basic facts about the forcing.

**Definition 4.6.** Let  $\lambda$  be an uncountable cardinal. We define a poset  $\mathbb{K}_{\lambda}$  which adds a tree with size and height  $\omega_1$  with  $\lambda$ -many cofinal branches. Conditions  $q \in \mathbb{K}_{\lambda}$  are pairs  $(T_q, f_q)$  such that

- there is  $\eta_q < \omega_1$  such that  $T_q$  is a normal, infinitely splitting subtree of  $<\eta_q+1\omega_1$  that does not contain a copy of  $<\omega+12$ ;
- $f_q$  is a countable partial function from  $\lambda$  to  $T_q \cap {}^{\eta_q}\omega_1$ .

If  $q_0, q_1 \in \mathbb{Q}$ , then  $q_1 \leq q_0$  if and only if

- $\eta_{q_1} \ge \eta_{q_0};$   $T_{q_1} \cap {}^{<\eta_{q_0}+1}\omega_1 = T_{q_0};$
- $\operatorname{dom}(f_{q_1}) \supseteq \operatorname{dom}(f_{q_0});$
- for all  $\alpha \in \text{dom}(f_{q_0}), f_{q_1}(\alpha) \supseteq f_{q_0}(\alpha)$ .

We also include the pair  $(\emptyset, \emptyset)$  in  $\mathbb{K}_{\lambda}$  as  $1_{\mathbb{K}_{\lambda}}$ .

Note that  $\mathbb{K}_{\lambda}$  is not  $\omega_2$ -cc: in fact, it collapses  $2^{\omega_1}$  to  $\omega_1$  since we can code subsets of  $\omega_1$  in the ground model into the levels of the generic tree added by  $\mathbb{K}_{\lambda}$ . Therefore if  $\lambda \leq 2^{\omega_1}$ , then the generic tree added by  $\mathbb{K}_{\lambda}$  is not a weak Kurepa tree.

**Lemma 4.7.** Let  $\lambda$  and  $\mu$  be uncountable cardinals such that  $\mu > 2^{\omega_1}$  is regular and  $\mu > \gamma^{\omega}$  for all  $\gamma < \mu$ . Then  $\mathbb{K}_{\lambda}$  is  $\mu$ -Knaster. In particular  $\mathbb{K}_{\lambda}$  is  $(2^{\omega_1})^+$ -Knaster.

*Proof.* Let a set of conditions  $\{q_{\alpha} = (T_{\alpha}, f_{\alpha}) \in \mathbb{K}_{\lambda} | \alpha < \mu\}$  be given. Since  $\mu > 2^{\omega_1}$  is regular and there are only  $2^{\omega_1}$ -many possibilities for  $T_{\alpha}$ 's, there is a tree  $T \subseteq {}^{<\eta+1}\omega_1$  of countable height and  $I \subseteq \mu$  of size  $\mu$  such that  $T_\alpha = T$  for all  $\alpha \in I$ .

Since  $\gamma^{\omega} < \mu$  for all  $\gamma < \mu$ , there is  $I' \subseteq I$  of size  $\mu$  such that the set  $\{\operatorname{dom}(f_{\alpha}) \mid \alpha \in I'\}$  forms a  $\Delta$ -system with root  $a \subseteq \lambda$ . Since a is at most countable, there are at most  $2^{\omega} < \mu$  many functions from a to  $T_{\eta}$  and therefore there is a countable f from a to  $T_{\eta}$  and  $J \subseteq I$  of size  $\mu$  such  $f = f_{\alpha} \cap f_{\beta}$  for all  $\alpha \neq \beta \in J$ . Then all conditions in  $\{q_{\alpha} \mid \alpha \in J\}$  are compatible. 

**Lemma 4.8.** Let  $\lambda$  be an uncountable cardinal. If CH holds, then  $\mathbb{K}_{\lambda}$  is  $\omega_1$ -closed.

*Proof.* Let  $\langle q_n \mid n < \omega \rangle$  be a decreasing sequence from  $\mathbb{K}_{\lambda}$ . To avoid trivialities, assume that the sequence  $\langle \eta_{q_n} \mid n < \omega \rangle$  is strictly increasing. Let  $\eta := \sup\{\eta_{q_n} \mid n < \omega \}$  $n < \omega$ . We will construct a lower bound q for  $\langle q_n \mid n < \omega \rangle$  such that  $\eta_q = \eta$ . Let  $T = \bigcup_{n < \omega} T_{q_n}$ . To define  $T_q$ , we simply need to decide which cofinal branches through T should continue.

We first note that there may be countably many branches that we are obliged to extend because of the functions  $\{f_{q_n} \mid n < \omega\}$ . Namely, let  $a = \bigcup_{n < \omega} \operatorname{dom}(f_{q_n})$ and, for each  $\alpha \in a$ , let

$$b_{\alpha} := \bigcup \{ f_{q_n}(\alpha) \mid n < \omega \land \alpha \in \operatorname{dom}(f_{q_n}) \}.$$

Then each  $b_{\alpha}$  is a cofinal branch through T, and we are obliged to put  $b_{\alpha}$  in  $T_q$ .

We may need to extend additional branches through T in order to ensure that  $T_q$  is normal; i.e., for each  $\sigma \in T$ , we need to ensure that there is  $\tau \in T_q \cap {}^{\eta}\omega_1$  with  $\sigma \subseteq \tau$ . However, when doing so, we must be careful not to add a copy of  ${}^{<\omega+1}2$  to  $T_q$ . We will do so through the use of the following bookkeeping device.

Let  $\langle \iota_{\xi} | \xi < \omega_1 \rangle$  enumerate all isomorphic embeddings  $\iota$  from  $\langle \omega_2 \rangle$  to T such that, for every  $x \in {}^{\omega}2$ , the union  $\bigcup \{ \iota(x \upharpoonright n) \mid n < \omega \}$  is a cofinal branch through T. Note that this is possible, due to the fact that CH holds and  $|T| \leq \omega_1$ . For each  $\xi < \omega_1$ , let

$$[\iota_{\xi}] := \left\{ \bigcup \{ \iota_{\xi}(x \upharpoonright n) \mid n < \omega \} \mid x \in {}^{\omega}2 \right\}.$$

Note that  $[\iota_{\xi}] \subseteq [T]$ , and  $|[\iota_{\xi}]| = 2^{\omega} = \omega_1$ . Let us enumerate T as  $\langle \sigma_{\xi} | \xi < \omega_1 \rangle$ , with repetitions if T is countable. We now recursively construct disjoint subsets  $\{c_{\xi} \mid \xi < \omega_1\}$  and  $\{d_{\xi} \mid \xi < \omega_1\}$  of [T]. Suppose that  $\xi < \omega_1$  and we have constructed  $c_{\zeta}$  and  $d_{\zeta}$  for all  $\zeta < \xi$ . First, choose  $c_{\xi} \in [T]$  such that

- σ<sub>ξ</sub> ⊆ c<sub>ξ</sub>;
  c<sub>ξ</sub> ∉ {d<sub>ζ</sub> | ζ < ξ}.</li>

Note that this is possible to do: T is normal and splitting, and  $cf(\gamma) = \omega$ , so there are  $2^{\omega}$  many elements of [T] extending  $\sigma_{\xi}$ . Next, choose  $d_{\xi} \in [T]$  such that

- $d_{\xi} \in [\iota_{\xi}];$
- $d_{\xi} \notin \{b_{\alpha} \mid \alpha \in a\} \cup \{c_{\zeta} \mid \zeta \leq \xi\}.$

At the end of the construction, set  $T_q := T \cup \{b_\alpha \mid \alpha \in a\} \cup \{c_\xi \mid \xi < \omega_1\}$ . Let  $f_q$ be such that dom $(f_q) = a$  and, for all  $\alpha \in a$ ,  $f_q(a) = b_{\alpha}$ .

We claim that q is a condition in  $\mathbb{K}_{\lambda}$  and it is a lower bound of  $\langle q_n \mid n < \omega \rangle$ . The only nontrivial thing to verify is the fact that  $T_q$  does not contain a copy of  $^{<\omega+1}2$ . Assume for a contradiction that  $T_q$  contains a copy of  $^{<\omega+1}2$ . Since, for every  $n < \omega$ , we know that  $T_{q_n}$  does not contain a copy of  ${}^{<\omega+1}2$ , there must be a  $\xi < \omega_1$  such that  $[\iota_{\xi}] \subseteq (T_q)_{\eta} = \{b_{\alpha} \mid \alpha \in a\} \cup \{c_{\zeta} \mid \zeta < \omega_1\}$ . However, this means that  $d_{\xi} \in \{b_{\alpha} \mid \alpha \in a\} \cup \{c_{\zeta} \mid \zeta < \omega_1\}$  which contradicts our choice of  $d_{\xi}$ .  $\Box$ 

**Lemma 4.9.** Let  $\lambda < \kappa$  be uncountable cardinals. Then there is a projection from  $\mathbb{K}_{\kappa}$  to  $\mathbb{K}_{\lambda}$ .

*Proof.* We define  $\pi$  from  $\mathbb{K}_{\kappa}$  to  $\mathbb{K}_{\lambda}$  by letting  $\pi(T_q, f_q) = (T_q, f_q \upharpoonright \lambda)$  for all  $q \in \mathbb{K}_{\kappa}$ . It is routine to verify that  $\pi$  is order-preserving and that  $\pi(\mathbb{1}_{\mathbb{K}_{\kappa}}) = \mathbb{1}_{\mathbb{K}_{\lambda}}$ .

Let  $q \in \mathbb{K}_{\kappa}$  and  $r \in \mathbb{K}_{\lambda}$  be such that  $r \leq \pi(q) = (T_q, f_q \upharpoonright \lambda)$ . Define  $r' \leq q$ by first letting  $T_{r'} = T_r$  (note that  $T_r$  is an end-extension of  $T_q$ ). Let  $\operatorname{dom}(f_{r'}) = \operatorname{dom}(f_q) \cup \operatorname{dom}(f_r)$  and define

 $-f_{r'}(\alpha) = f_r(\alpha) \text{ for every } \alpha \in \operatorname{dom}(f_r) \text{ and} \\ -f_{r'}(\alpha) = \tau, \text{ where } \tau \in (T_{r'})_{\eta_{r'}} \text{ with } \tau \supseteq f_q(\alpha), \text{ for every } \alpha \in \operatorname{dom}(f_q) \setminus \operatorname{dom}(f_r).$ It is easy to check that  $\pi(r') = r$ . Therefore  $\pi$  is a projection.  $\Box$ 

Let H be a  $\mathbb{K}_{\lambda}$ -generic filter,  $T = \bigcup \{T_q \mid q \in H\}$ , and let  $\mathbb{K}_{\kappa}/H = \{r \in \mathbb{K}_{\kappa} \mid \pi(r) \in H\}$  be the quotient given by H and the projection  $\pi$ . Then  $\mathbb{K}_{\kappa}$  is forcing equivalent to a two step iteration  $\mathbb{K}_{\lambda} * \mathbb{K}_{\kappa}/H$ . It is easy to see that in V[H],  $\mathbb{K}_{\kappa}/H$  is forcing equivalent to the forcing notion  $\mathbb{K}_{\lambda,\kappa}$ , where conditions in  $\mathbb{K}_{\lambda,\kappa}$  are pairs  $r = (\eta_r, f_r)$  such that  $f_r$  is a countable partial function from  $\kappa \setminus \lambda$  to  $T_{\eta_r}$  and  $r \leq q$  if and only if  $\eta_r \geq \eta_q$ ,  $\operatorname{dom}(f_q) \subseteq \operatorname{dom}(f_r)$ , and  $f_q(\alpha) \subseteq f_r(\alpha)$  for all  $\alpha \in \operatorname{dom}(f_q)$ .

**Lemma 4.10.** Let  $\lambda < \kappa$  be uncountable cardinals and H be a  $\mathbb{K}_{\lambda}$ -generic filter. Then  $\mathbb{K}_{\kappa}/H$  is  $\omega_1$ -distributive in V[H].

*Proof.* The forcing  $\mathbb{K}_{\kappa}$  is forcing equivalent to a two step iteration  $\mathbb{K}_{\lambda} * \mathbb{K}_{\kappa}/\dot{H}$ . Since  $\mathbb{K}_{\kappa}$  is  $\omega_1$ -closed by Lemma 4.8,  $\mathbb{K}_{\kappa}/H$  cannot add new countable sequences of ordinals over V[H], hence it is  $\omega_1$ -distributive in V[H].

We fix the following notation: for  $r \in \mathbb{K}_{\kappa}$  and  $\lambda < \kappa$ , let  $r \upharpoonright \lambda$  denote  $(T_r, f_r \upharpoonright \lambda)$ , i.e.  $r \upharpoonright \lambda = \pi(r)$  for the projection  $\pi$  defined in Lemma 4.9.

**Lemma 4.11.** Let  $\lambda < \kappa$  be uncountable cardinals and H be a canonical  $\mathbb{K}_{\lambda}$ -name for the  $\mathbb{K}_{\lambda}$ -generic filter. Let  $q \in \mathbb{K}_{\lambda}$  and  $r \in \mathbb{K}_{\kappa}$ . Then the following are equivalent.

- (i)  $q \Vdash r \in \mathbb{K}_{\kappa}/\dot{H};$
- (*ii*)  $q \leq r \upharpoonright \lambda$ .

*Proof.* To see that (i) implies (ii), note that if  $q \leq r \upharpoonright \lambda$ , then by the separativity of  $\mathbb{K}_{\lambda}$  there is  $q' \leq q$  which is incompatible with  $r \upharpoonright \lambda$ , and hence q does not force r into  $\mathbb{K}_{\kappa}/\dot{H}$ . The other direction is clear, since  $q \leq r \upharpoonright \lambda$  means that  $q \leq \pi(r)$ .  $\Box$ 

**Lemma 4.12.** Assume GCH and let  $\lambda > \omega_2$  be a cardinal. Let H be a  $\mathbb{K}_{\lambda}$ -generic filter over V. Then the generic tree  $T = \bigcup \{T_q \mid q \in H\}$  is a weak Kurepa tree with  $\lambda$ -many cofinal branches which does not contain a copy of  ${}^{<\omega+1}2$ .

Proof. Since  $\mathbb{K}_{\lambda}$  is  $\omega_1$ -closed,  $\omega_1$  is preserved by  $\mathbb{K}_{\lambda}$  and the generic tree  $T = \bigcup\{T_q \mid q \in H\}$  is thus a tree with height and size  $\omega_1$ . By a standard density argument T has  $\lambda$ -many cofinal branches in V[H]. Since GCH holds in the ground model,  $\mathbb{K}_{\lambda}$  is  $\omega_3$ -Knaster by Lemma 4.7, hence all cardinals greater than  $\omega_2$  are preserved (recall that  $2^{\omega_1}$  is always collapsed); in particular  $\lambda$  is preserved. Since

 $\lambda > \omega_2$  in the ground model,  $\lambda > \omega_1$  in V[H]; therefore the generic tree T is a weak Kurepa tree in V[H].

By Claim 4.2, if there is a copy of  ${}^{<\omega+1}2$  in T, there is one which is bounded in the height of T and therefore there is a condition  $q \in H$  such that  $T_q$  contains a copy of  ${}^{<\omega+1}2$ ; this would contradict the definition of the forcing  $\mathbb{K}_{\lambda}$ .

We show now that the generic tree added by  $\mathbb{K}_{\lambda}$  does not contain an Aronszajn subtree and that this property is preserved by all  $\omega_1$ -closed forcings.

**Lemma 4.13.** Assume CH. Let  $\lambda > \omega_1$  be a cardinal and let H be a  $\mathbb{K}_{\lambda}$ -generic filter over V. Assume that  $\mathbb{P}$  is an  $\omega_1$ -closed forcing in V[H]. Then the generic tree  $T = \bigcup \{T_q \mid q \in H\}$  does not contain an Aronszajn subtree in the generic extension of V[H] by  $\mathbb{P}$ . In particular T does not contain an Aronszajn subtree in V[H].

*Proof.* We work in V. Let T denote a canonical  $\mathbb{K}_{\lambda}$ -name for the generic tree added by  $\mathbb{K}_{\lambda}$ . Assume for a contradiction that  $\dot{S}$  is a  $\mathbb{K}_{\lambda} * \dot{\mathbb{P}}$ -name for an Aronszajn subtree of  $\dot{T}$  and  $(q, \dot{p}) \in \mathbb{K}_{\lambda} * \dot{\mathbb{P}}$  is a condition which forces this.

We begin with some easy observations. For every  $r \in \mathbb{K}_{\lambda}$  and all  $\sigma, \tau \in {}^{<\omega_1}\omega_1$ , it holds that  $r \Vdash \sigma \leq_{\dot{T}} \tau$  if and only if  $\sigma \subseteq \tau \in T_r$ . Moreover, if  $(r, \dot{p}') \leq (q, \dot{p})$ decides that level  $\dot{S}_{\gamma} = x$  for some  $\gamma < \omega_1$ , then  $\operatorname{ht}(T_r) \geq \gamma + 1$  and  $x \subseteq (T_r)_{\gamma}$ .

**Claim 4.14.** Let  $(r, \dot{p}') \leq (q, \dot{p})$  decide that level  $\dot{S}_{\gamma} = x$  for some  $\gamma < \omega_1$ . Then there is a condition  $(r^*, \dot{p}^*) = ((T_{r^*}, f_{r^*}), \dot{p}^*)$  stronger than  $(r, \dot{p}')$  such that the following holds:

(i) for each  $\sigma \in x$  there is  $\alpha \in \text{dom}(f_{r^*})$  such that  $f_{r^*}(\alpha) \supseteq \sigma$ ; (ii) for each  $\alpha \in \text{dom}(f_{r^*})$ ,  $(r^*, \dot{p}^*) \Vdash f_{r^*}(\alpha) \notin \dot{S}$ .

*Proof.* Note that we can assume that  $(r, \dot{p}')$  is such that for each  $\sigma \in x$  there is  $\alpha \in \text{dom}(f_r)$  with  $f_r(\alpha) \supseteq \sigma$ ; in particular  $(r, \dot{p}')$  forces that  $\sigma$  is in some cofinal branch of T. If this is not the case, then we can extend  $f_r$  appropriately using that  $T_r$  is normal. We now build by induction on  $\omega$  a decreasing sequence of conditions  $\langle (r_n, \dot{p}_n) \in \mathbb{K}_{\lambda} * \dot{\mathbb{P}} | n < \omega \rangle$  such that the desired  $(r^*, \dot{p}^*)$  will be a lower bound of this sequence.)

Begin by letting  $(r_0, \dot{p}_0) = (r, \dot{p}')$ . Now fix  $n < \omega$  and suppose that we have constructed  $(r_n, \dot{p}_n)$ . Note that, for each  $\alpha \in \operatorname{dom}(f_{r_n})$ ,  $\bigcup \{f_s \mid s \in \dot{H}\}$  is forced to be a cofinal branch through  $\dot{T}$ , where  $\dot{H}$  is a name for the  $\mathbb{K}_{\lambda}$ -generic filter. Therefore, since  $\dot{S}$  is forced by  $(r_n, \dot{p}_n)$  to be an Aronszajn subtree of  $\dot{T}$ , and since  $\operatorname{dom}(f_{r_n})$  is countable, there is  $(r_{n+1}, \dot{p}_{n+1}) \leq (r_n, \dot{p}_n)$  such that, for all  $\alpha \in \operatorname{dom}(f_{r_n})$ , we have  $(r_{n+1}, \dot{p}_{n+1}) \Vdash f_{r_{n+1}}(\alpha) \notin \dot{S}$ . Let  $(q^*, \dot{p}^*)$  be a lower bound of  $\langle (r_n * \dot{p}_n) \in \mathbb{K}_{\lambda} * \dot{\mathbb{P}} | n < \omega \rangle$  such that  $\operatorname{dom}(f_{q^*}) = \bigcup \{\operatorname{dom}(f_{r_n}) | n < \omega\}$ . Then it is easy to see that  $(q^*, \dot{p}^*)$  satisfy condition (i) and (ii) above.  $\Box$ 

Now, we build a decreasing sequence of conditions  $\langle (r_n^*, \dot{p}_n^*) \in \mathbb{K}_{\lambda} * \dot{\mathbb{P}} | n < \omega \rangle$ such that each condition will satisfy an instance of Claim 4.14. Let  $(r_0, \dot{p}_0) \leq (q, \dot{p})$ decide that level  $\dot{S}_{\gamma_0} = x_{\gamma_0}$  for some  $\gamma_0 < \omega_1$ . Then take  $(r_0^*, \dot{p}_0^*) \leq (r_0, \dot{p}_0)$  to be a condition which satisfies Claim 4.14 for  $\gamma_0$ . Note that  $\operatorname{ht}(T_{r_0^*}) \geq \gamma_0$ . Let  $(r_n^*, \dot{p}_n^*)$ be constructed for some  $n < \omega$ . Let  $(r_{n+1}, \dot{p}_{n+1})$  be a condition which decides  $\dot{S}_{\gamma_{n+1}} = x_{\gamma_{n+1}}$  for some  $\gamma_{n+1} > \operatorname{ht}(T_{r_n^*})$ , and then take  $(r_{n+1}^*, \dot{p}_{n+1}^*) \leq (r_{n+1}, \dot{p}_{n+1})$ to be a condition which satisfies Claim 4.14 for  $\gamma_{n+1}$ . Note that  $\operatorname{ht}(T_{r_{n+1}^*}) \geq \gamma_{n+1}$ . It now follows from the construction that  $T^{\gamma} = \bigcup_{n < \omega} T_{r_n^*}$  is a countable subtree of  $\langle \gamma \omega_1 \rangle$  of limit height  $\gamma$ , where  $\gamma = \sup\{\gamma_n \mid n < \omega\}$ . Let

$$S^{\gamma} = \bigcup \{ \sigma \in {}^{<\gamma} \omega_1 \, | \, \exists n < \omega \ (r_n^*, \dot{p}_n^*) \Vdash \sigma \in \dot{S} \}.$$

Note that  $S^{\gamma}$  is a subtree of  $T^{\gamma}$  with countable levels and height  $\gamma$  in V, and that any lower bound for  $\langle (r_n^*, \dot{p}_n^*) \mid n < \omega \rangle$  will force that  $\dot{S} \cap {}^{<\gamma}\omega_1 = S^{\gamma}$ .

Let  $a = \bigcup_{n < \omega} \operatorname{dom}(f_{r_n^*})$ , then for each  $\alpha \in a$ 

$$b_{\alpha} = \bigcup \{ f_{r_n^*}(\alpha) \, | \, n < \omega \land \alpha \in \operatorname{dom}(f_{r_n^*}) \}.$$

is a cofinal branch through  $T^{\gamma}$ . However,  $b_{\alpha}$  is not a cofinal branch through  $S^{\gamma}$  since there is  $n < \omega$  such that  $(r_n^*, \dot{p}_n^*) \Vdash f_{r_n^*}(\alpha) \notin \dot{S}$ . Hence we are not obligated to put any cofinal branch through  $S^{\gamma}$  into lower bound of  $\langle r_n^* \in \mathbb{K}_{\lambda} | n < \omega \rangle$ .

We claim that we can find a lower bound  $r^*$  of  $\langle r_n^* | n < \omega \rangle$  such that  $T_{r^*}$  has height  $\gamma + 1$  and for each  $\alpha \in \operatorname{dom}(f_{r^*})$ ,  $f_{r^*}(\alpha)$  is not a cofinal branch through  $S^{\gamma}$ . Consider  $r = (T_r, f_r)$ , where  $T_r = T^{\gamma} \cup \{b_{\alpha} | \alpha \in a\}$ ,  $\operatorname{dom}(f_r) = a$  and for each  $\alpha \in a$ ,  $f_r(\alpha) = b_{\alpha}$ . Each lower bound of  $\langle r_n^* \in \mathbb{K}_{\lambda} | n < \omega \rangle$  has to extend r, but ritself does not have to be a condition in  $\mathbb{K}_{\lambda}$ , since  $T_r$  does not have to be normal. However for each  $\sigma \in S^{\gamma}$  there is  $\alpha \in a$  such that  $\sigma \subseteq b_{\alpha}$ , hence to extend  $T_r$  to a normal tree we only need to extend nodes in  $T^{\gamma} \setminus S^{\gamma}$ . As in the proof that  $\mathbb{K}_{\lambda}$ is  $\omega_1$ -closed, extend  $T_r$  to a normal tree  $T^* = T \cup \{c_{\tau} | \tau \in T^{\gamma} \setminus S^{\gamma}\}$  by carefully picking cofinal branches  $c_{\tau}$  in  $T^{\gamma}$  for each  $\tau \in T^{\gamma} \setminus S^{\gamma}$  to ensure that  $T^*$  does not contain a copy of  ${}^{<\omega+1}2$ . Since we add only nodes to level  $\gamma$  above  $T^{\gamma} \setminus S^{\gamma}$ , we did not add any cofinal branch through  $S^{\gamma}$  into  $T^*$ . Therefore every cofinal branch bthrough  $S^{\gamma}$  is vanishing in  $T^*$ ; i.e. there is no node on level  $\gamma$  above b.

Let  $r^* = (T^*, f_r)$ . Then  $r^*$  is a condition in  $\mathbb{K}_{\lambda}$  which is a lower bound of  $\langle r_n^* \in \mathbb{K}_{\lambda} | n < \omega \rangle$ . Consider  $(r^*, \dot{p}^*)$ , where  $\dot{p}^*$  is chosen so that  $(r^*, \dot{p}^*)$  is a lower bound of  $\langle (r_n^*, \dot{p}_n^*) \in \mathbb{K}_{\lambda} * \dot{\mathbb{P}} | n < \omega \rangle$ . Then the condition  $(r^*, \dot{p}^*)$  forces that  $\dot{S} \cap \dot{T}_{\gamma} = \emptyset$ , which is a contradiction since  $(q, \dot{p})$  forces that  $\dot{S}$  is an Aronszajn subtree of  $\dot{T}$  and in particular unbounded in  $\dot{T}$ .

We are now ready to prove the following variation on Theorem 4.4.

**Theorem 4.15.** Suppose that there is an inaccessible cardinal  $\kappa$ . Then there is a forcing extension in which

- (1)  $\kappa = \omega_2;$
- (2) GCH;
- $(3) \neg \mathsf{KH};$
- (4) there is a weak Kurepa tree  $T \subseteq {}^{<\omega_1}\omega_1$  that does not contain a copy of  ${}^{<\omega+1}2$  and does not contain an Aronszajn subtree.

Proof. Let  $\kappa$  be an inaccessible cardinal, and assume that GCH holds. Let  $\mathbb{P} = \operatorname{Coll}(\omega_1, < \kappa)$  and let  $G \times H$  be  $\mathbb{P} \times \mathbb{K}_{\kappa}$ -generic over V.  $\mathbb{P} \times \mathbb{K}_{\kappa}$  is  $\omega_1$ -closed and  $\kappa$ -Knaster and hence preserves  $\omega_1$  and all cardinals greater or equal  $\kappa$ . Since  $\mathbb{P}$  collapses cardinals between  $\omega_1$  and  $\kappa$  to  $\omega_1$ ,  $\kappa$  is  $\omega_2$  in V[G][H]. Since  $\mathbb{P} \times \mathbb{K}_{\kappa}$  is  $\omega_1$ -closed and has cardinality  $\kappa$ , it follows that GCH holds in V[G][H]. By Lemma 4.12,  $T = \bigcup \{T_q \mid q \in H\}$  is a weak Kurepa tree with  $\kappa$ -many cofinal branches which does not contain a copy of  ${}^{<\omega+1}2$  in V[H]. Note that T is still a weak Kurepa tree in V[H][G] since  $\mathbb{P}$  preserves  $\omega_1$  and  $\kappa$  over V[H]. Moreover, since  $\mathbb{P}$  is  $\omega_1$ -closed in V[H], T does not contain any Aronszajn subtree in V[H][G] = V[G][H] by Lemma

4.13. To see that T does not contain a copy of  ${}^{<\omega+1}2$  in V[H][G], we will prove the stronger assertion that no  $\omega_1$ -distributive forcing can add a copy of  ${}^{<\omega+1}2$  to a tree of height  $\omega_1$  which does not contain such a copy in the ground model.

**Claim 4.16.** Let T be a normal tree of height  $\omega_1$  which does not contain a copy of  ${}^{<\omega+1}2$  and let P be an  $\omega_1$ -distributive forcing. Then P does not add a copy of  ${}^{<\omega+1}2$  to T.

Proof. This follows from Claim 4.2 which implies that we can assume that a copy of  ${}^{<\omega+1}2$  is determined by its restriction to  ${}^{<\omega}2$ . Let us give some details. Let F be P-generic over V and let f be an isomorphic embedding from  ${}^{<\omega+1}2$  to T in V[F]. Then since  ${}^{<\omega}2$  has size  $\omega$  and P is  $\omega_1$ -distributive,  $f \upharpoonright {}^{<\omega}2$  is in the ground model and we can extend it to an isomorphic embedding g from  ${}^{<\omega+1}2$  by defining  $g(x) = \sup\{x \upharpoonright n \mid n < \omega\}$  for  $x \in {}^{\omega}2$  and g(a) = f(a) for  $a \in {}^{<\omega}2$ . Note that g(x) is well defined for all  $x \in {}^{\omega}2$  since f(x) is above  $\{x \upharpoonright n \mid n < \omega\}$  in V[F] and T is normal. The mapping g determines a copy of  ${}^{<\omega+1}2$  in the ground model, which is a contradiction.

Since  $\mathbb{P}$  is  $\omega_1$ -closed in V[H], it follows by the previous claim that T does not contain a copy of  $\langle \omega+12 \rangle$  in V[H][G] because it has no such copy in V[H].

To finish the proof we need to show that there are no Kurepa trees in V[G][H]. Assume that S is an  $\omega_1$ -tree in V[G][H]. Since S has size  $\omega_1$  and  $\mathbb{P} \times \mathbb{K}_{\kappa}$  is  $\kappa$ -Knaster, there is a nice  $\mathbb{P} \times \mathbb{K}_{\kappa}$ -name  $\dot{S}$  of size less than  $\kappa$  for S. Since  $\dot{S}$  has size less than  $\kappa$ ,  $\dot{S}$  is a  $\mathbb{P}_{\theta} \times \mathbb{K}_{\theta}$ -name for some regular cardinal  $\theta < \kappa$ , where  $\mathbb{P}_{\theta} = \operatorname{Coll}(\omega_1, <\theta)$ . Let  $G_{\theta}$  denote the  $\mathbb{P}_{\theta}$ -generic over V determined by G; i.e.  $G_{\theta} = \{p \upharpoonright \theta \mid p \in G\}$ , and let  $H_{\theta}$  denote the  $\mathbb{K}_{\theta}$ -generic filter over  $V[G_{\theta}]$  determined by H and  $\pi$ , where  $\pi$  is the projection from  $\mathbb{K}_{\kappa}$  to  $\mathbb{K}_{\theta}$  from Lemma 4.9. The  $\omega_1$ -tree S is an element of  $V[G_{\theta}][H_{\theta}]$  and has at most  $(2^{\omega_1})^{V[G_{\theta}][H_{\theta}]}$ -many cofinal branches here, which is less than  $\kappa$ , since  $\kappa$  is still inaccessible in  $V[G_{\theta}][H_{\theta}]$ . We show that the quotient forcing  $\mathbb{P}^{\theta} \times \mathbb{K}_{\kappa}/H_{\theta}$ , where  $\mathbb{P}^{\theta}$  denotes  $\operatorname{Coll}(\omega_1, [\theta, <\kappa))$ , cannot add a cofinal branch to S over  $V[G_{\theta}][H_{\theta}]$ . It will follow that S has  $<\kappa = \omega_2$ -many cofinal branches in V[G][H], i.e., it is not a Kurepa tree. Since S was an arbitrary  $\omega_1$ -tree, this will conclude the proof of Theorem 4.15. The proof that there are no Kurepa trees in V[G][H] is relatively long; for easier reading, it is divided into several Claims (Claims 4.18 to 4.21).

First we observe that we can express the generic extension V[G][H] as a forcing extension given by  $(\mathbb{P}_{\theta} \times \mathbb{K}_{\theta}) * (\mathbb{K}_{\kappa}/\dot{H}_{\theta} \times \mathbb{P}^{\theta})$ : Since  $\mathbb{P}$  is forcing equivalent to  $\mathbb{P}_{\theta} \times \mathbb{P}^{\theta}$ , we can view G as a filter  $G_{\theta} \times g$  which is  $\mathbb{P}_{\theta} \times \mathbb{P}^{\theta}$ -generic over V. Similarly, since  $\mathbb{K}_{\kappa}$  is forcing equivalent to  $\mathbb{K}_{\theta} * \mathbb{K}_{\kappa}/\dot{H}_{\theta}$ , we can view H as a filter  $H_{\theta} * h$ which is generic over  $V[G_{\theta}][g]$ . Therefore V[G][H] is equal to the generic extension  $V[G_{\theta}][g][H_{\theta}][h]$ . Moreover, since  $\mathbb{P}^{\theta}$  and  $\mathbb{K}_{\theta}$  live in  $V[G_{\theta}]$  and  $H_{\theta}$  is generic over  $V[G_{\theta}][g], V[G_{\theta}][g][H_{\theta}][h] = V[G_{\theta}][H_{\theta}][g][h]$ . Similarly, we can exchange g and hsince both  $\mathbb{P}^{\theta}$  and  $\mathbb{K}_{\kappa}/H_{\theta}$  live in  $V[G_{\theta}][H_{\theta}]$  and h is generic over  $V[G_{\theta}][H_{\theta}][g]$ . It follows that  $V[G_{\theta}][H_{\theta}][g][h] = V[G_{\theta}][H_{\theta}][h][g]$ .

Since  $\mathbb{P}^{\theta}$  and  $\mathbb{P}_{\theta} \times \mathbb{K}_{\kappa}$  are both  $\omega_1$ -closed in V,  $\mathbb{P}^{\theta}$  is still  $\omega_1$ -closed in  $V[G_{\theta}][H_{\theta}][h]$ . By Lemma 2.11,  $\omega_1$ -closed forcing cannot add new cofinal branches to an  $\omega_1$ -tree and hence any cofinal branch through S is already in  $V[G_{\theta}][H_{\theta}][h]$ .

To show that  $\mathbb{K}_{\kappa}/H_{\theta}$  cannot add a cofinal branch to S over  $V[G_{\theta}][H_{\theta}]$ , we will work in  $V[G_{\theta}]$ . Note that  $\mathbb{K}_{\kappa}/H_{\theta}$  is only  $\omega_1$ -distributive in  $V[G_{\theta}][H_{\theta}]$  and therefore we cannot use Fact 2.11. We will show that if there is a  $\mathbb{K}_{\theta} * (\mathbb{K}_{\kappa}/\dot{H}_{\theta})$ -name  $\dot{b}$  and

a condition  $(q^*, r^*) \in \mathbb{K}_{\theta} * (\mathbb{K}_{\kappa}/\dot{H}_{\theta})$  which forces that  $\dot{b}$  is a cofinal branch through  $\dot{S}$  that is not in  $V[G_{\theta}][\dot{H}_{\theta}]$ , then there is a condition  $q \leq q^*$  which forces that  $\dot{S}$  has an uncountable level. This is a contradiction, since we can assume that  $(q^*, r^*)$  is in  $H_{\theta} * h$  and that  $q^*$  forces that  $\dot{S}$  is an  $\omega_1$ -tree.

The proof is similar to Silver's argument for  $\omega_1$ -closed forcings: we build a tree  $\mathcal{T}$  of conditions in  $\mathbb{K}_{\kappa}$  labeled by  ${}^{<\omega}2$ , but since we are working with the quotient  $\mathbb{K}_{\kappa}/\dot{H}_{\theta}$  and we work in  $V[G_{\theta}]$  and not in  $V[G_{\theta}][H_{\theta}]$ , we will guide this construction by a decreasing sequence of length  $\omega$  of conditions in  $\mathbb{K}_{\theta}$ , which will force conditions from  $\mathcal{T}$  into the quotient  $\mathbb{K}_{\kappa}/\dot{H}_{\theta}$ .

We make a natural identification and view  $(\dot{S}, <_{\dot{S}})$  as a name for a tree with underlying set  $\omega_1$ .

Work in  $V[G_{\theta}]$ . Assume that  $\dot{b}$  is a  $\mathbb{K}_{\theta} * (\mathbb{K}_{\kappa}/\dot{H}_{\theta})$ -name and  $(q^*, \dot{r}^*) \in \mathbb{K}_{\theta} * (\mathbb{K}_{\kappa}/\dot{H}_{\theta})$  forces that  $\dot{b}$  is a cofinal branch through  $\dot{S}$  that is not in  $V[G_{\theta}][\dot{H}_{\theta}]$ . We will build by induction on  $\omega$  the following objects:

- a decreasing sequence  $\langle q_n | n < \omega \rangle$  of conditions in  $\mathbb{K}_{\theta}$  with  $q_0 \leq q^*$ ;
- a labeled tree  $\mathcal{T} = \{r_s \mid s \in {}^{<\omega}2\}$  of conditions in  $\mathbb{K}_{\kappa}$ , with  $r_s \leq r^*$  for all s;
- a strictly increasing sequence  $\langle \gamma_n | n < \omega \rangle$  of ordinals below  $\omega_1$ ,
- such that the following hold for all n and all  $s \in {}^{<\omega}2$  of length n:
- (a)  $q_n = r_s \upharpoonright \theta$ ; in particular  $q_n \Vdash r_s \in \mathbb{K}_{\kappa}/\dot{H}_{\theta}$ ;
- (b) the conditions  $(q_{n+1}, r_{s^{\frown}0})$  and  $(q_{n+1}, r_{s^{\frown}1})$  decide  $\dot{b}$  up to  $\gamma_{n+1}$  differently; i.e. there are  $\delta \leq \gamma_{n+1}$  and  $\tau_{s^{\frown}0} \neq \tau_{s^{\frown}1}$  forced to be in  $\dot{S}_{\delta}$  such that  $(q_n, r_{s^{\frown}0}) \Vdash \dot{b}(\delta) = \tau_{s^{\frown}0}$  and  $(q_n, r_{s^{\frown}1}) \Vdash \dot{b}(\delta) = \tau_{s^{\frown}1}$ ;
- (c)  $(q_{|t|}, r_t) \leq (q_{|s|}, r_s)$  for all  $s \subseteq t$  in  ${}^{<\omega}2$ ;
- (d)  $\operatorname{rng}(f_{q_s} \upharpoonright [\theta, \kappa)) \cap \operatorname{rng}(f_{q_t} \upharpoonright [\theta, \kappa)) = \emptyset$  for distinct s, t in  ${}^{<\omega}2$  of the same length.

**Definition 4.17.** Let us call a system above, satisfying conditions (a)–(d), a *labeled* system for  $\dot{b}$ . We say just a labeled system if  $\dot{b}$  is clear from the context.

A labeled system will be constructed below, using Claims 4.18 and 4.19. First we prove an auxiliary claim which will be useful for the definition of the construction.

**Claim 4.18.** For every  $(q, r^0), (q, r^1) \leq (q^*, r^*) \in \mathbb{K}_{\theta} * (\mathbb{K}_{\kappa}/\dot{H}_{\theta})$  and  $\gamma' < \omega_1$  there are  $\gamma' < \gamma < \omega_1, (q', p^0) \leq (q, r^0)$  and  $(q', p^1) \leq (q, r^1)$  such that  $(q', p^0)$  and  $(q', p^1)$  decide  $\dot{b}(\gamma)$  differently and the condition q' extends both  $p^0 \upharpoonright \theta$  and  $p^1 \upharpoonright \theta$ .

Proof. Let  $(q, r^0), (q, r^1) \leq (q^*, r^*)$  and  $\gamma' < \omega_1$  be given. Fix for the moment a  $\mathbb{K}_{\theta}$ -generic F over  $V[G_{\theta}]$  such that  $q \in F$  and work in  $V[G_{\theta}][F]$ . Since  $\dot{b}$  is forced by  $(q^*, r^*)$  to be a new cofinal branch through  $\dot{S}$  and  $q^* \in F$ , there are  $\bar{p}^0 \leq r^0$ ,  $\bar{p}^1 \leq r^0$  and  $\gamma > \gamma'$  such that  $\bar{p}^0$  and  $\bar{p}^1$  decide  $\dot{b}(\gamma)$  differently; i.e. there are  $\tau^0 \neq \tau^1$  in  $\dot{S}_{\gamma}$  such that  $\bar{p}^0 \Vdash \dot{b}(\gamma) = \tau^0$  and  $\bar{p}^1 \Vdash \dot{b}(\gamma) = \tau^1$ .

Since  $\bar{p}^0$  and  $\bar{p}^1$  are in  $\mathbb{K}_{\kappa}/F$ ,  $\bar{p}^0 \upharpoonright \theta$  and  $\bar{p}^1 \upharpoonright \theta$  are in F and hence they are compatible. Let  $\bar{q}$  be a common extension of  $\bar{p}^0 \upharpoonright \theta$ ,  $\bar{p}^1 \upharpoonright \theta$  and q such that, for each i < 2, we have  $(\bar{q}, \bar{p}^i) \Vdash \dot{b}(\gamma) = \tau^i$ .

Now, we return back to working in  $V[G_{\theta}]$ . Since  $\bar{q}$  extends both  $\bar{p}^0 \upharpoonright \theta$  and  $\bar{p}^1 \upharpoonright \theta$ , it forces both of them into the quotient  $\mathbb{K}_{\kappa}/\dot{H}_{\theta}$ . It follows  $(\bar{q}, \bar{p}^0)$  and  $(\bar{q}, \bar{p}^1)$  are conditions in  $\mathbb{K}_{\theta} * (\mathbb{K}_{\kappa}/\dot{H}_{\theta})$  which extend  $(q, r^0)$  and decide  $\dot{b}(\gamma)$  differently.

Now, consider the condition  $(\bar{q}, r^1)$ . Since  $\dot{b}$  is a  $\mathbb{K}_{\theta} * (\mathbb{K}_{\kappa}/\dot{H}_{\theta})$  name for a cofinal branch through  $\dot{S}$ , there is an extension  $(q', p^1) \leq (\bar{q}, r^1)$  which decides  $\dot{b}(\gamma)$ . Since

 $\tau^0 \neq \tau^1$ ,  $(q', p^1)$  cannot decide  $\dot{b}(\gamma)$  as being equal to both of them. Let  $p^0$  be  $\bar{p}^i$ , for i < 2, such that  $(q', p^1)$  and  $(q', \bar{p}^i)$  disagree on  $\dot{b}(\gamma)$ . Then  $q', p^0, p^1$  and  $\gamma$  are as required. Note that  $q' \leq p^1 \upharpoonright \theta$  since q' forces  $p^1$  into  $\mathbb{K}_{\kappa}/\dot{H}_{\theta}$  and  $\mathbb{K}_{\theta}$  is separative.

Now, we are ready to construct our labeled system. The construction is by induction on  $\omega$ .

Let  $\gamma_0$  be an arbitrary ordinal below  $\omega_1$  and let  $(q_0, r_{\emptyset})$  be  $(q^*, r^*)$ . By Lemma 4.11,  $q^* \leq r^* \upharpoonright \theta$  since  $q^*$  forces  $r^*$  into  $\mathbb{K}_{\kappa}/\dot{H}_{\theta}$  and  $\mathbb{K}_{\theta}$  is separative. If  $q^* \neq r^* \upharpoonright \theta$ , we can extend  $r^*$  appropriately to ensure the condition (a); for more details, see Claim 4.19 below in the successor step of the construction.

Now fix  $n < \omega$  and assume that we have constructed  $\gamma_n$ ,  $q_n$  and  $r_s$  for all  $s \in {}^{n}2$ . Let  $\langle s_i | i < 2^n \rangle$  enumerate  ${}^{n}2$ . We describe how to construct  $\gamma_{n+1}$ ,  $q_{n+1}$ , and  $r_s$  for  $s \in {}^{n+1}2$ .

We proceed by induction on  $2^n = m$ . Let us start with  $s_0$ . By Claim 4.18 there are  $q^0 \leq q_n, r'_{s_0 \frown 0}, r'_{s_0 \frown 1} \leq r_{s_0}$  and  $\gamma^0 \geq \gamma_n$  such that  $(q^0, r'_{s_0 \frown 0}), (q^0, r'_{s_0 \frown 1})$  decide  $\dot{b}(\gamma^0)$  differently and  $q^0$  extends both  $r'_{s_0 \frown 0} \upharpoonright \theta$  and  $r'_{s_0 \frown 1} \upharpoonright \theta$ . Now fix  $1 \leq i < m$ , and suppose that  $\gamma^{i-1}$  and  $q^{i-1}$  have been constructed. By Claim 4.18 there are  $q^i \leq q^{i-1}, r'_{s_i \frown 0}, r'_{s_i \frown 1} \leq r'_{s_i}$  and  $\gamma^i \geq \gamma^{i-1}$  such that  $(q^i, r'_{s_i \frown 0}), (q^i, r'_{s_i \frown 1})$  decide  $\dot{b}(\gamma^i)$  differently and  $q^i$  extends both  $r'_{s_i \frown 0} \upharpoonright \theta$  and  $r'_{s_i \frown 1} \upharpoonright \theta$ . Let  $q'_{n+1}$  be  $q^{m-1}$  and  $\gamma_{n+1}$  be  $\gamma^{m-1}$ . It follows by the construction that the

Let  $q'_{n+1}$  be  $q^{m-1}$  and  $\gamma_{n+1}$  be  $\gamma^{m-1}$ . It follows by the construction that the objects  $q'_{n+1}$ ,  $\gamma_{n+1}$ ,  $r'_{s^{-j}}$ , for j < 2 and all  $s \in {}^{n}2$  satisfy the desired conditions (b) and (c).

However, we have only ensured that  $q'_{n+1}$  extends  $r'_s \upharpoonright \theta$  for  $s \in {}^{n+1}2$ , but not that they are equal as is required in condition (a), and we have not ensured condition (d) either. To ensure conditions (a) and (d), we define appropriate extensions of  $q'_{n+1}$  and  $r'_s$  for all  $s \in {}^{n+1}2$ .

**Claim 4.19.** The objects constructed above can be extended to satisfy conditions (a)-(d) of a labeled system in Definition 4.17.

Proof. Since  $q'_{n+1}$  extends  $r'_s \upharpoonright \theta$  for all  $s \in {}^{n+1}2$ ,  $T_{q'_{n+1}}$  is an end-extension of  $T_{r'_s}$  for all  $s \in {}^{n+1}2$ . Let  $\eta + 1$  be the height of  $T_{q'_{n+1}}$  and let T' be a one-level extension of  $T_{q'_{n+1}}$  such that for every node  $\tau$  in  $T_{q'_{n+1}}$  on level  $\eta$  we add at least countably many new nodes above  $\tau$  into T'. The height of T' is  $\eta + 2$ , and T' is normal and infinitely splitting since  $T_{q'_{n+1}}$  is normal and infinitely splitting. Let f' be a function such that dom $(f') = \text{dom}(f_{q'_{n+1}})$  and for each  $\alpha \in \text{dom}(f')$  let  $f'(\alpha) \supseteq f_{q'_{n+1}}(\alpha)$  be some node of T' on level  $\eta + 1$ . Then  $q_{n+1} = (T', f')$  is a condition in  $\mathbb{K}_{\theta}$  and it extends  $q'_{n+1}$ .

Now, we define extensions of  $r'_s$  for  $s \in {}^{n+1}2$ . For  $s \in {}^{n+1}2$ , let  $f_s$  be a function such that dom $(f_s) = \text{dom}(f_{r'_s}) \cap [\theta, \kappa)$  and for each  $\alpha \in \text{dom}(f_s)$ , let  $f_s(\alpha) \supseteq f_{r'_s}(\alpha)$ be some node of T' on level  $\eta + 1$ . Moreover, ensure that  $\text{rng}(f_s) \cap \text{rng}(f_t) = \emptyset$  for all  $s \neq t \in {}^{n+1}2$ . Note that we can find such functions since T' is infinitely splitting and the sets dom $(f_{r'_s})$  for  $s \in {}^{n+1}2$  are at most countable. Set  $r_s = (T', f' \cup f_s)$  for  $s \in {}^{n+1}2$ . Clearly,  $r_s$  are conditions in  $\mathbb{K}_{\kappa}$  for all  $s \in {}^{n+1}2$  such that  $q_{n+1}$  is equal to  $r_s \upharpoonright \theta$ , hence condition (a) holds. Condition (d) follows from the definition of  $f_s$ for  $s \in {}^{n+1}2$ .

Since  $q_{n+1}$  extends  $q'_{n+1}$  and  $r_s$  extends  $r'_s$  for all  $s \in {}^{n+1}2$ , conditions (b) and (c) are still satisfied for  $q_{n+1}$ ,  $\gamma_{n+1}$ ,  $r_{s^{-}j}$ , for j < 2 and all  $s \in {}^{n}2$ .

This completes our construction of a labeled system. Let  $\gamma$  be the supremum of  $\langle \gamma_n | n < \omega \rangle$ . To finish the proof of the whole Theorem 4.15, we would like to find a lower bound q of the sequence  $\langle q_n | n < \omega \rangle$  and a lower bound  $r_x$  of sequences  $\langle r_{x \uparrow n} | n < \omega \rangle$  for all  $x \in {}^{\omega}2$  such that q forces  $r_x$  into the quotient  $\mathbb{K}_{\kappa}/\dot{H}_{\theta}$  for every x, thus ensuring that every  $(q, r_x)$  is a condition in  $\mathbb{K}_{\theta} * (\mathbb{K}_{\kappa}/\dot{H}_{\theta})$ . If this is the case, then q forces that level  $\gamma$  of  $\dot{S}$  has size  $2^{\omega}$ , and hence q forces that  $\dot{S}$  is not an  $\omega_1$ -tree, finishing the proof.

In the rest of the proof we will construct such conditions. Before we start, note that we do not need to ensure that q forces  $r_x$  into quotient for all  $x \in {}^{\omega}2$ , it is enough to have this for uncountably many  $x \in {}^{\omega}2$ .

Let  $T^* = \bigcup_{n < \omega} T_{q_n}$  and  $a = \bigcup_{n < \omega} \operatorname{dom}(f_{q_n})$ . Then  $T^*$  is a normal tree with limit height which does not contain a copy of  ${}^{<\omega+1}2$ . Let  $\eta < \omega_1$  denote the height of  $T^*$ . By condition (a),  $T^* = \bigcup_{n < \omega} T_{r_{x \restriction n}}$  and  $a = \bigcup_{n < \omega} \operatorname{dom}(f_{r_{x \restriction n}}) \cap \theta$  for all  $x \in {}^{\omega}2$ .

Note that if we want to ensure that a lower bound of  $\langle q_n | n < \omega \rangle$  forces a lower bound of  $\langle r_{x \uparrow n} | n < \omega \rangle$ , for some  $x \in {}^{\omega}2$ , into the quotient, we are obliged not only to extend all cofinal branches through  $T^*$  which are given by functions  $\{f_{q_n} | n < \omega\}$ (recall the proof that  $\mathbb{K}_{\kappa}$  is  $\omega_1$ -closed), but also to extend all cofinal branches given by  $\{f_{r_{x \uparrow n}} | n < \omega\}$ , otherwise it can happen that the lower bounds will be incompatible. Therefore if we want to ensure that a lower bound of  $\langle q_n | n < \omega \rangle$ forces lower bounds of  $\langle r_{x \restriction n} | n < \omega \rangle$  for uncountably many  $x \in {}^{\omega}2$  into the quotient, we are obliged to extend uncountably many cofinal branches of  $T^*$ . This can be done because the trees in the conditions can be wide, but we need to make sure that while doing it, we do not add a copy of  ${}^{\langle \omega+1}2$ . To argue that  $T_q$  does not contain a copy of  ${}^{\langle \omega+1}2$ , we use following claim which says that if we add only countably many cofinal branches to a tree which does not contain a copy of  ${}^{\langle \omega+1}2$ , then we do not add a copy of  ${}^{\langle \omega+1}2$ .

**Claim 4.20.** Let T be a tree with countable height  $\eta + 1$ , where  $\eta$  is limit. Let D be a countable set of cofinal branches of  $T \upharpoonright \eta$ . If T does not contain a copy of  ${}^{<\omega+1}2$ , then T' also does not contain a copy of  ${}^{<\omega+1}2$ , where  $T' = T \cup \{d \mid d \in D\}$ .

*Proof.* This is a straightforward consequence of the following well-known fact: If P is a perfect subset of  ${}^{\omega}2$  and  $Q \subseteq P$  is countable, then  $P \setminus Q$  contains a perfect set.  $\Box$ 

To build suitable lower bounds, we will proceed similarly as in the proof of Lemma 4.8. First apply the construction in Lemma 4.8 to find q' which is a lower bound of  $\langle q_n | n < \omega \rangle$  such that  $T_{q'}$  has height  $\eta + 1$  and  $\operatorname{dom}(f_{q'}) = a$ . Note that each node  $\tau \in T_{q'}$  on level  $\eta$  determines a cofinal branch through  $T^*$  which we denote  $b_{\tau}$ . Note also that  $T_{q'} = T^* \cup \{b_{\tau} | \tau \in (T_{q'})_{\eta}\}$ .

For each  $x \in {}^{\omega}2$ , let  $a_x = \bigcup_{n < \omega} \operatorname{dom}(f_{r_x \restriction n}) \setminus \theta$ . In order to ensure that q is compatible with a lower bound of  $\langle r_{x \restriction n} | n < \omega \rangle$ , we are obliged to extend cofinal branches given by  $\{r_{x \restriction n} | n < \omega\}$ . For  $\alpha \in a_x$ , let

(3) 
$$d_{\alpha}^{x} := \bigcup \{ f_{r_{x \uparrow n}}(\alpha) \mid n < \omega \land \alpha \in \operatorname{dom}(f_{r_{x \restriction n}}) \}$$

and let  $D_x = \{d_\alpha^x \mid \alpha \in a_x\}$ . Note that  $D_x \cap D_y = \emptyset$  for all  $x \neq y \in {}^{\omega}2$  by condition (d). Moreover, let as define  $f_x$  to be a function with domain  $a_x$  such that for each  $\alpha \in a_x$ ,  $f_x(\alpha) = d_\alpha^x$ .

We will use the following bookkeeping device to extend cofinal branches in uncountably many  $D_x$  (but not in all  $D_x$  because this might add a copy of  ${}^{<\omega+1}2$ ).

Let  $\langle \iota_{\xi} | \xi < \omega_1 \rangle$  enumerate all isomorphic embeddings  $\iota$  from  ${}^{<\omega}2$  to  $T^*$  such that, for every  $x \in {}^{\omega}2$ , the function  $\bigcup \{\iota(x \upharpoonright n) | n < \omega\}$  is a cofinal branch through  $T^*$ . Note that this is possible, due to the fact that CH holds and  $|T^*| \leq \omega_1$ . For each  $\xi < \omega_1$ , let

(4) 
$$[\iota_{\xi}] := \left\{ \bigcup \{ \iota_{\xi}(x \upharpoonright n) \mid n < \omega \} \mid x \in {}^{\omega}2 \right\}.$$

Note that  $[\iota_{\xi}] \subseteq [T^*]$ , and  $|[\iota_{\xi}]| = 2^{\omega} = \omega_1$ .

We now recursively build two disjoint sequences  $\langle x_{\xi} | \xi < \omega_1 \rangle$  and  $\langle y_{\xi} | \xi < \omega_1 \rangle$  of functions in  ${}^{\omega}2$ . The first sequence picks functions x in  ${}^{\omega}2$  for which we will extend the corresponding cofinal branches  $D_x$ . The second sequence picks functions y in  ${}^{\omega}2$  for which we will forbid the extension of corresponding cofinal branches in  $D_y$ to ensure that  $T_q$  does not contain a copy of  ${}^{\langle \omega+1}2$ . Note that we need here that the  $D_x$ 's are pairwise disjoint.

Suppose that we have constructed  $x_{\zeta}$  and  $y_{\zeta}$  for all  $\zeta < \xi$ , where  $\xi < \omega_1$ . First, set  $x_{\xi}$  to be any function such that  $x_{\xi} \notin \{x_{\zeta} \mid \zeta < \xi\} \cup \{y_{\zeta} \mid \zeta < \xi\}$ . Next, if there is  $y \notin \{x_{\zeta} \mid \zeta \leq \xi\} \cup \{y_{\zeta} \mid \zeta < \xi\}$  such that

(5) 
$$([\iota_{\xi}] \setminus \{b_{\tau} \mid \tau \in (T_{q'})_{\eta}\}) \cap D_{y} \neq \emptyset,$$

then let  $y_{\xi}$  be any such y. If there is no such y, then let  $y_{\xi}$  be any y which is not in  $\{x_{\zeta} \mid \zeta \leq \xi\} \cup \{y_{\zeta} \mid \zeta < \xi\}$ .

At the end of the construction, set  $T_q = T_{q'} \cup \bigcup_{\xi < \omega_1} D_{x_{\xi}}$  and  $f_q = f'_q$ . Set  $T_{r_{x_{\xi}}} = T_q$  and  $f_{r_{x_{\xi}}} = f_{q'} \cup f_{x_{\xi}}$  for every  $\xi < \omega_1$ .

Claim 4.21. The following hold:

- (i) q is a condition in  $\mathbb{K}_{\theta}$ ;
- (ii)  $r_{x_{\xi}}$  is a condition in  $\mathbb{K}_{\kappa}$  for all  $\xi < \omega_1$ ;
- (iii) q forces  $r_{x_{\varepsilon}}$  into the quotient  $\mathbb{K}_{\kappa}/\dot{H}_{\theta}$ , for all  $\xi < \omega_1$ .

Proof. Note that  $T_q$  is a normal tree since  $T_{q'}$  is normal and  $T_q = T_{q'} \cup \bigcup_{\xi < \omega_1} D_{x_{\xi}}$ . The only nontrivial thing to check is that  $T_q$  does not contain a copy of  ${}^{<\omega+1}2$ . Assume for contradiction that  $T_q$  contains a copy of  ${}^{<\omega+1}2$ ; hence there is  $\xi < \omega_1$ such that  $[\iota_{\xi}] \subseteq (T_q)_{\eta}$ . Let I be the set of all  $\zeta < \omega_1$  such that  $(D_{x_{\zeta}} \setminus (T_{q'})_{\eta}) \cap [\iota_{\xi}] \neq \emptyset$ ; in particular,  $[\iota_{\xi}] \subseteq (T_{q'})_{\eta} \cup (\bigcup_{\zeta \in I} D_{x_{\zeta}})$ . Note that I cannot be countable by Claim 4.20, since  $T_{q'}$  does not contain copy of  ${}^{<\omega+1}2$ . Hence I is unbounded in  $\omega_1$  which means that there is  $\zeta > \xi$  such that  $(D_{x_{\zeta}} \setminus (T_{q'})_{\eta}) \cap [\iota_{\xi}] \neq \emptyset$ . Therefore in step  $\xi$  of the induction, condition (5) was satisfied and we chose  $y_{\xi}$  such that the intersection  $(D_{y_{\xi}} \setminus (T_{q'})_{\eta}) \cap [\iota_{\xi}]$  is nonempty. This is a contradiction since  $D_{y_{\xi}}$  should be disjoint from  $D_{x_{\zeta}}$  for all  $\zeta \in I$ , and we did not add any cofinal branches from  $D_{y_{\xi}} \setminus (T_{q'})_{\eta}$ into  $T_q$ . This shows that q is a condition in  $\mathbb{K}_{\theta}$  and  $r_{x_{\xi}}$  are conditions in  $\mathbb{K}_{\kappa}$  for all  $\xi < \omega_1$ .

By the definition of  $r_{x_{\xi}}$ ,  $r_{x_{\xi}} \upharpoonright \theta = q$  and hence q forces  $r_{x_{\xi}}$  into the quotient  $\mathbb{K}_{\kappa}/\dot{H}_{\theta}$  for every  $\xi < \omega_1$ .

It is straightforward to check that q forces  $\dot{S}_{\gamma}$  to be uncountable: Let F be any  $\mathbb{K}_{\theta}$ -generic which contains q. Then  $r_{x_{\xi}}$  is a condition in the quotient  $\mathbb{K}_{\theta}/F$  for every  $\xi < \omega_1$ . Let  $r_{x_{\xi}}^*$  be an extension of  $r_{x_{\xi}}$  which decides  $\dot{b}(\gamma) = \tau_{\xi}$ . Then for

every  $\xi \neq \zeta < \omega_1, \tau_{\xi} \neq \tau_{\zeta}$  since  $(q, r_{x_{\xi}})$  and  $(q, r_{x_{\zeta}})$  decide b differently below  $\gamma$ . This concludes the proof of Theorem 4.15. 

### 5. Kurepa trees and continuous images

In this section, we prove Theorem B, answering a question of Lücke and Schlicht from [8]. In particular, we will construct a model of ZFC in which GCH holds, there exist  $\omega_2$ -Kurepa trees, and, for every  $\omega_2$ -Kurepa tree  $S \subseteq \langle \omega_2 \omega_2, [S]$  is not a continuous image of  $\omega_2 \omega_2$ .

We will also prove that, in the model that we construct, closed subsets of  $\omega_2 \omega_2$ satisfy the strongest possible perfect set property compatible with the existence of  $\omega_2$ -Kurepa trees. Note that, by the discussion at the beginning of Section 4, if  $S \subseteq {}^{<\omega_2}\omega_2$  is a Kurepa tree, then  $\mathsf{PSP}_{\omega_1+1}([S])$  necessarily fails. In the model we construct, this failure will be sharp: clause (5) in the statement of Theorem 5.1 below will imply that, for every closed  $E \subseteq {}^{\omega_2}\omega_2$ , we have  $\mathsf{PSP}_{\omega_1}(E)$ .

In what follows, when referring to trees, we will sometimes write *countably closed* in place of  $(\langle \omega_1 \rangle)$ -closed.

**Theorem 5.1.** Suppose that there is an inaccessible cardinal  $\kappa$ . Then there is a forcing extension in which

- (1)  $\kappa = \omega_3$ ;
- (2) GCH holds;
- (3) there is an  $\omega_2$ -Kurepa tree;
- (4) for every  $\omega_2$ -Kurepa tree  $S \subseteq {}^{<\omega_2}\omega_2$ , [S] is not a continuous image of  ${}^{\omega_2}\omega_2$ ;
- (5) for every closed subset  $E \subseteq \omega_2 \omega_2$ , there is  $X \subseteq E$  with  $|X| \leq \omega_2$  such that  $E \setminus X$  is  $\omega_1$ -perfect.

*Proof.* We can assume that GCH holds in the ground model. Let  $\mathbb{P} := \operatorname{Coll}(\omega_2, < \kappa)$ , and let  $\mathbb{Q}$  be the forcing to add a countably closed  $\omega_2$ -Kurepa tree with  $\kappa$ -many cofinal branches. More precisely,  $\mathbb{Q}$  consists of all pairs  $q = (T_q, f_q)$  such that

- there is an  $\eta_q < \omega_2$  such that  $T_q$  is a normal, splitting, countably closed subtree of  $<\eta_q+1\omega_2$ ;
- for all  $\xi \leq \eta_q$ ,  $|T_q \cap {}^{\xi}\omega_2| \leq \omega_1$ ;
- $f_q$  is a partial function of size  $\omega_1$  from  $\kappa$  to  $T_q \cap {}^{\eta_q}\omega_2$ .

If  $q_0, q_1 \in \mathbb{Q}$ , then  $q_1 \leq q_0$  if

- $\eta_{q_1} \ge \eta_{q_0};$   $T_{q_1} \cap {}^{<\eta_{q_0}+1}\omega_2 = T_{q_0};$
- $\operatorname{dom}(f_{q_1}) \supseteq \operatorname{dom}(f_{q_0});$
- for all  $\alpha \in \text{dom}(f_{q_0}), f_{q_1}(\alpha) \supseteq f_{q_0}(\alpha)$ .

We also include  $(\emptyset, \emptyset)$  in  $\mathbb{Q}$  as  $1_{\mathbb{Q}}$ . By standard arguments,  $\mathbb{P} \times \mathbb{Q}$  is  $\omega_2$ -closed and has the  $\kappa$ -cc.

For a cardinal  $\delta < \kappa$ , let  $\mathbb{P}_{\delta} := \operatorname{Coll}(\omega_2, < \delta)$ , and let  $\mathbb{Q}_{\delta}$  be the set of  $q \in \mathbb{Q}$  for which dom $(q) \subseteq \delta$ . It is routine to verify that, for all  $\delta < \kappa$ , the map  $\pi_{\delta} : \mathbb{P} \times \mathbb{Q} \to \mathbb{Q}$  $\mathbb{P}_{\delta} \times \mathbb{Q}_{\delta}$  defined by letting  $\pi_{\delta}(p,q) = (p \upharpoonright \delta, (T_q, f_q \upharpoonright \delta))$  is a projection.

**Claim 5.2.** Let  $\delta < \kappa$  be a cardinal. Then, in  $V^{\mathbb{P}_{\delta} \times \mathbb{Q}_{\delta}}$ , the quotient forcing  $\mathbb{R}_{\delta} :=$  $(\mathbb{P} \times \mathbb{Q})/(\mathbb{P}_{\delta} \times \mathbb{Q}_{\delta})$  is countably closed.

*Proof.* Let  $G_{\delta} \times H_{\delta}$  be  $\mathbb{P}_{\delta} \times \mathbb{Q}_{\delta}$ -generic over V, and move to  $V[G_{\delta} \times H_{\delta}]$ . Let  $\langle (p_n, q_n) \mid n < \omega \rangle$  be be a decreasing sequence from  $\mathbb{P} \times \mathbb{Q}$  such that, for all  $n < \omega$ ,

we have  $\pi_{\delta}(p_n, q_n) \in G_{\delta} \times H_{\delta}$ . Note that this sequence is in fact in V, by the distributivity of  $\mathbb{P} \times \mathbb{Q}$ .

Let  $p = \bigcup \{p_n \mid n < \omega\}$ . It is evident that  $p \in \mathbb{P}$  and  $p \upharpoonright \delta \in G_{\delta}$ . We next define a condition  $q \in \mathbb{Q}$ . If the sequence  $\langle T_{q_n} \mid n < \omega \rangle$  is eventually constant, then, by removing an initial segment of the sequence, we can assume it is constant. Let  $T_q$ be this constant value, and let  $f_q = \bigcup \{f_{q_n} \mid n < \omega\}$ . Then (p,q) is a lower bound for  $\langle (p_n, q_n) \mid n < \omega \rangle$  and  $\pi_{\delta}(p,q) \in G_{\delta} \times H_{\delta}$ .

So suppose that  $\langle T_{q_n} | n < \omega \rangle$  is not eventually constant. Let  $T^* = \bigcup \{T_{q_n} | n < \omega \}$ . Then  $T^*$  is a normal subtree of  ${}^{<\eta}\omega_2$  for some  $\eta < \omega_2$  of countable cofinality. Moreover, by CH, we have  $|[T^*]| \leq \omega_1$ . Let  $T_q := T^* \cup [T^*]$ . By construction and the fact that each  $T_{q_n}$  is countably closed, it follows that  $T_q$  is a normal countably closed subtree of  ${}^{<\eta+1}\omega_2$ . Since the trees in the first coordinate of conditions in  $\mathbb{Q}$  are required to be countably closed, we must have  $(T_q, \emptyset) \in H_{\delta}$ . Let  $D = \bigcup \{ \operatorname{dom}(f_{q_n}) | n < \omega \}$ . Define  $f_q : D \to B$  by letting

$$f_q(\alpha) = \bigcup \{ f_{q_n}(\alpha) \mid n < \omega \land \alpha \in \operatorname{dom}(f_{q_n}) \}$$

for all  $\alpha \in D$ . Then (p,q) is a lower bound for  $\langle (p_n,q_n) \mid n < \omega \rangle$  and  $\pi_{\delta}(p,q) \in G_{\delta} \times H_{\delta}$ .

Let  $G \times H$  be  $\mathbb{P} \times \mathbb{Q}$ -generic over V and, for all cardinals  $\delta < \kappa$ , let  $G_{\delta} \times H_{\delta}$  be the  $\mathbb{P}_{\delta} \times \mathbb{Q}_{\delta}$ -generic filter induced by  $G \times H$ . We claim that  $V[G \times H]$  is the desired forcing extension. We have  $(\omega_2)^V = (\omega_2)^{V[G \times H]}$ , and  $\kappa = (\omega_3)^{V[G \times H]}$ ; moreover, GCH holds in  $V[G \times H]$ . Let  $T = \bigcup \{T_q \mid q \in H\}$  and, for each  $\alpha < \kappa$ , let

$$b_{\alpha} = \bigcup \{ f_q(\alpha) \mid q \in H \land \alpha \in \operatorname{dom}(f_q) \}.$$

Then T is a normal, countably closed subtree of  ${}^{<\omega_2}\omega_2$ , all of whose levels have size at most  $\omega_1$ . Moreover, for all  $\alpha < \kappa$ ,  $b_{\alpha}$  is a cofinal branch through T and, by genericity, for all  $\alpha < \beta < \kappa$ , we have  $b_{\alpha} \neq b_{\beta}$ . Therefore, T is an  $\omega_2$ -Kurepa tree in  $V[G \times H]$ .

To verify clause (4) in the statement of the theorem, we must show that, in  $V[G \times H]$ , there is no  $\omega_2$ -Kurepa tree  $S \subseteq {}^{<\omega_2}\omega_2$  such that [S] is a continuous image of  ${}^{\omega_2}\omega_2$ . Suppose for the sake of contradiction that S is such a tree, and let  $g: {}^{\omega_2}\omega_2 \to [S]$  be a continuous surjection. Define a function  $h: {}^{<\omega_2}\omega_2 \to S \cup [S]$  by letting  $h(\sigma) = \bigwedge \{g(x) \mid x \in N_{\sigma}\}$  for all  $\sigma \in {}^{<\omega_2}\omega_2$ , and let

$$W = \{ \sigma \in {}^{<\omega_2}\omega_2 \mid \operatorname{dom}(h(\sigma)) \ge \operatorname{dom}(\sigma) \}.$$

Note that h is monotone, i.e., for all  $\sigma, \tau \in {}^{<\omega_2}\omega_2$ , if  $\sigma \sqsubseteq \tau$ , then  $h(\sigma) \sqsubseteq h(\tau)$ . It follows that W is  $(<\omega_2)$ -closed, i.e., if  $\eta < \omega_2$  and  $\langle \sigma_{\xi} | \xi < \eta \rangle$  is a  $\sqsubseteq$ -increasing sequence of elements of W, then  $\bigcup \{\sigma_{\xi} | \xi < \eta\} \in W$ .

# **Claim 5.3.** For all $x \in {}^{\omega_2}\omega_2$ , $C_x := \{\alpha < \omega_2 \mid x \upharpoonright \alpha \in W\}$ is a club in $\omega_2$ .

*Proof.* Fix  $x \in {}^{\omega_2}\omega_2$ . The fact that  $C_x$  is closed follows immediately from the fact that h is monotone. To see that it is unbounded, fix an  $\alpha_0 < \omega_2$ . We will find  $\alpha \in C_x \setminus \alpha_0$ . Starting with  $\alpha_0$ , recursively define an increasing sequence  $\langle \alpha_n \mid n < \omega \rangle$  of ordinals below  $\omega_2$  such that, for all  $n < \omega$  and all  $y \in N_{x \upharpoonright \alpha_{n+1}}$ , we have  $g(y) \upharpoonright \alpha_n = g(x) \upharpoonright \alpha_n$ . It is straightforward to build such a sequence due to the continuity of g.

Let  $\alpha = \sup\{\alpha_n \mid n < \omega\}$ . For all  $y \in N_{x \restriction \alpha}$  and all  $n < \omega$ , we have  $g(y) \restriction \alpha_n = g(x) \restriction \alpha_n$ ; therefore,  $g(y) \restriction \alpha = g(x) \restriction \alpha$ . In particular,  $h(x \restriction \alpha) \sqsupseteq g(x) \restriction \alpha$ , so  $\alpha \in C_x$ .

Note that S, h, and W are objects of size  $\omega_2$ , so, by the chain condition of  $\mathbb{P} \times \mathbb{Q}$ , we can fix a  $\delta < \kappa$  so that  $S, h, W \in V[G_{\delta} \times H_{\delta}]$  and, in that model, the quotient forcing  $\mathbb{R}_{\delta}$  forces them to have the relevant properties isolated above. In particular, letting  $\dot{g}$  be an  $\mathbb{R}_{\delta}$ -name for g, it is forced by every condition in  $\mathbb{R}_{\delta}$  that S is an  $\omega_2$ -Kurepa tree,  $\dot{g}$  is a continuous surjection from  $\omega_2 \omega_2$  to [S], and h and W are as defined above from S and  $\dot{g}$ .

Work now in  $V[G_{\delta} \times H_{\delta}]$ . In this model,  $2^{\omega_2} < \kappa$ ; therefore, forcing with  $\mathbb{R}_{\delta}$  must add new cofinal branches to S. Let  $\dot{b}$  be an  $\mathbb{R}_{\delta}$ -name for a cofinal branch through S such that  $\Vdash_{\mathbb{R}_{\delta}} \dot{b} \notin V[G_{\delta} \times H_{\delta}]$ . Let  $\dot{x}$  be an  $\mathbb{R}_{\delta}$ -name for an element of  $\omega_2 \omega_2$  such that  $\Vdash_{\mathbb{R}_{\delta}} \dot{g}(\dot{x}) = \dot{b}$ .

Now, by recursion on dom( $\sigma$ ), we construct a labeled tree  $\langle (r_{\sigma}, b_{\sigma}, x_{\sigma}) | \sigma \in \langle \omega_1 2 \rangle$  of elements of  $\mathbb{R}_{\delta} \times S \times \langle \omega_2 \omega_2 \rangle$ . We will arrange so that the following requirements are satisfied.

- (1) For all  $\sigma \in {}^{<\omega_1}2$ , we have
  - (a)  $r_{\sigma} \Vdash_{\mathbb{R}_{\delta}} ``\dot{b} \sqsupseteq b_{\sigma} \text{ and } \dot{x} \sqsupseteq x_{\sigma}";$
  - (b)  $x_{\sigma} \in W$ ;
  - (c)  $h(x_{\sigma}) \supseteq b_{\sigma}$ .
- (2) For all  $\sigma \sqsubseteq \tau \in {}^{<\omega_1}2$ , we have  $r_{\tau} \leq r_{\sigma}$ ,  $b_{\tau} \sqsupseteq b_{\sigma}$ , and  $x_{\tau} \sqsupseteq x_{\sigma}$ .
- (3) For all  $\sigma \in {}^{<\omega_1}2, b_{\sigma \frown \langle 0 \rangle}$  and  $b_{\sigma \frown \langle 1 \rangle}$  are incomparable in S.

Begin by letting  $r_{\emptyset} = (\emptyset, \emptyset)$  and  $b_{\emptyset} = x_{\emptyset} = \emptyset$ . Next, suppose that  $\sigma \in {}^{<\omega_1}2$  is of limit length, and suppose that  $(r_{\sigma \restriction \eta}, b_{\sigma \restriction \eta}, x_{\sigma \restriction \eta})$  has been defined for all  $\eta < \operatorname{dom}(\sigma)$ . By Lemma 5.2,  $\mathbb{R}_{\delta}$  is countably closed, so we can let  $r_{\sigma}$  be a lower bound of  $\langle r_{\sigma \restriction \eta} \mid$  $\eta < \operatorname{dom}(\sigma) \rangle$ . Let  $b_{\sigma} = \bigcup \{ b_{\sigma \restriction \eta} \mid \eta < \operatorname{dom}(\sigma) \}$  and  $x_{\sigma} = \bigcup \{ x_{\sigma \restriction \eta} \mid \eta < \operatorname{dom}(\sigma) \}$ . For all  $\eta < \operatorname{dom}(\sigma)$ , we have  $r_{\sigma} \Vdash_{\mathbb{R}_{\delta}} "\dot{b} \sqsupseteq b_{\sigma \restriction \eta}$  and  $\dot{x} \sqsupseteq x_{\sigma \restriction \eta}$ ". As a result, we get  $r_{\sigma} \Vdash_{\mathbb{R}_{\delta}} "\dot{b} \sqsupseteq b_{\sigma}$  and  $\dot{x} \sqsupseteq x_{\sigma}$ ". In particular, it follows that  $b_{\sigma}$  is in fact in S. The fact that  $x_{\sigma} \in W$  and  $h(x_{\sigma}) \sqsupseteq b_{\sigma}$  follows from the closure of W and the monotonicity of h. We have therefore satisfied all of the requirements of the construction.

Finally, suppose that  $\sigma \in {}^{<\omega_1}2$  and we have defined  $(r_{\sigma}, b_{\sigma}, x_{\sigma})$ . We describe how to define  $(r_{\sigma} \langle i \rangle, b_{\sigma} \langle i \rangle, x_{\sigma} \langle i \rangle)$  for i < 2. First, since  $\Vdash_{\mathbb{R}_{\delta}} \dot{b} \notin V[G_{\delta} \times H_{\delta}]$ , we can find  $r_i^* \leq r_{\sigma}$  and  $b_i^* \supseteq b_{\sigma}$  for i < 2 such that  $b_0^*$  and  $b_1^*$  are incomparable in Sand, for i < 2, we have  $r_i^* \Vdash_{\mathbb{R}_{\delta}} \dot{b} \supseteq b_i^*$ .

By Claim 5.3, there are forced to be unboundedly many  $\xi < \omega_2$  such that  $\dot{x} \upharpoonright \xi \in W$ . Therefore, for each i < 2, we can find  $r_{\sigma \frown \langle i \rangle} \leq r_i^*$  and  $x_{\sigma \frown \langle i \rangle} \in W$  such that

- $x_{\sigma \frown \langle i \rangle} \supseteq x_{\sigma};$
- dom $(x_{\sigma \frown \langle i \rangle}) \ge \text{dom}(b_i^*);$
- $r_{\sigma \frown \langle i \rangle} \Vdash_{\mathbb{R}_{\delta}} \dot{x} \sqsupseteq x_{\sigma \frown \langle i \rangle}.$

Finally, for i < 2, let  $b_{\sigma \frown \langle i \rangle} = h(x_{\sigma \frown \langle i \rangle})$ . Note first that, since

$$r_{\sigma^{\frown}\langle i\rangle} \Vdash \dot{g}(\dot{x}) = b \land \dot{x} \sqsupseteq x_{\sigma^{\frown}\langle i\rangle},$$

it follows that  $r_{\sigma \frown \langle i \rangle} \Vdash b \sqsupseteq b_{\sigma \frown \langle i \rangle}$ . Next observe that  $b_{\sigma \frown \langle i \rangle}$  must be in S, since if it were in [S], then we would have  $r_{\sigma \frown \langle i \rangle} \Vdash \dot{b} = b_{\sigma \frown \langle i \rangle}$ , contradicting the fact that  $\dot{b}$  is forced not to be in  $V[G_{\delta} \times H_{\delta}]$ . We have therefore satisfied requirements (1)–(3) above, and we can continue with the construction.

At the end of the construction, since CH holds,  $\{b_{\sigma} \mid \sigma \in {}^{<\omega_1}2\}$  is a subset of S of size  $\omega_1$ . Therefore, we can fix a  $\xi < \omega_2$  such that  $b_{\sigma} \in {}^{<\xi}\omega_2$  for all  $\sigma \in {}^{<\omega_1}2$ . For each  $\nu \in {}^{\omega_1}2$ , let  $b_{\nu} = \bigcup \{b_{\nu \upharpoonright \eta} \mid \eta < \omega_1\}$ .

Claim 5.4. For all  $\nu \in {}^{\omega_1}2, b_{\nu} \in S$ .

*Proof.* Fix  $\nu \in {}^{\omega_1}2$ . Let  $x_{\nu} = \bigcup \{x_{\nu \upharpoonright \eta} \mid \eta < \omega_1\}$ . By monotonicity of h, we have  $h(x_{\nu}) \supseteq h(x_{\nu \upharpoonright \eta}) \supseteq b_{\nu \upharpoonright \eta}$  for all  $\eta < \omega_1$ . Therefore, by definition of  $b_{\nu}$ , it follows that  $h(x_{\nu}) \supseteq b_{\nu}$ . Since h maps into  $S \cup [S]$ , we get  $b_{\nu} \in S$ , as desired.  $\Box$ 

**Claim 5.5.** For all distinct  $\nu_0, \nu_1 \in {}^{\omega_1}2$ , we have  $b_{\nu_0} \neq b_{\nu_1}$ .

*Proof.* Fix distinct  $\nu_0, \nu_1 \in {}^{\omega_1}2$ . Let  $\eta < \omega_1$  be least such that  $\nu_0(\eta) \neq \nu_1(\eta)$ . Without loss of generality, assume that  $\nu_i(\eta) = i$  for i < 2. Let  $\sigma = \nu_0 \upharpoonright \eta$ . Then, for i < 2, we have  $b_{\nu_i} \sqsupseteq b_{\sigma^\frown \langle i \rangle}$ . Since  $b_{\sigma^\frown \langle 0 \rangle}$  and  $b_{\sigma^\frown \langle 1 \rangle}$  are incomparable in  $S, b_{\nu_0}$  and  $b_{\nu_1}$  are incomparable, as well; in particular, they are not equal.

It follows that  $\{b_{\nu} \mid \nu \in {}^{\omega_1}2\}$  is a collection of  $\omega_2$ -many elements of  $S_{\leq \xi}$ . This contradicts the fact that S is an  $\omega_2$ -tree, thus completing the verification of clause (4).

We finally verify clause (5) in the statement of the theorem. To this end, fix a closed set  $E \subseteq {}^{\omega_2}\omega_2$  in  $V[G \times H]$ . Let  $\Sigma = \{\sigma \in {}^{<\omega_2}\omega_2 \mid |E \cap N_{\sigma}| \leq \omega_2\}$ . Equivalently, by the chain condition of  $\mathbb{P} \times \mathbb{Q}$ ,  $\Sigma$  is the set of all  $\sigma \in {}^{<\omega_2}\omega_2$  for which there exists  $\delta < \kappa$  such that  $E \cap N_{\sigma} \subseteq V[G_{\delta} \times H_{\delta}]$ . Let  $X = \bigcup \{E \cap N_{\sigma} \mid \sigma \in \Sigma\}$ . Then  $|X| \leq \omega_2$  and, since E is closed and  $\bigcup \{N_{\sigma} \mid \sigma \in \Sigma\}$  is open, it follows that  $E \setminus X$  is closed. It remains to show that, for all  $x_0 \in E \setminus X$ , Player II has a winning strategy in  $G_{\omega_2}(E \setminus X, x_0, \omega_1)$ . To this end, fix an arbitrary  $x_0 \in E \setminus X$ .

Recall that  $T(E) = \{x \upharpoonright \eta \mid x \in E \text{ and } \eta < \omega_2\}$  is a subtree of  ${}^{<\omega_2}\omega_2$ ; since E is closed, it follows that E = [T(E)]. Moreover,  $T(E) \setminus \Sigma$  is a subtree of  ${}^{<\omega_2}\omega_2$ , and  $E \setminus X = [T(E) \setminus \Sigma]$ . In V, let  $\dot{E}$  be a  $\mathbb{P} \times \mathbb{Q}$ -name for E and  $\dot{X}$  be a  $\mathbb{P} \times \mathbb{Q}$ -name for X. By the chain condition of  $\mathbb{P} \times \mathbb{Q}$ , we can find a nonzero cardinal  $\gamma < \kappa$  such that

- $x_0, T(E), \Sigma, X \in V[G_{\gamma} \times H_{\gamma}];$
- interpreting  $\dot{E}$  in  $V[G_{\gamma} \times H_{\gamma}]$  as an  $\mathbb{R}_{\gamma}$ -name, the empty condition in  $\mathbb{R}_{\gamma}$  forces all of the following statements:
  - $\dot{E} = [T(E)];$
  - $\dot{E} \setminus X = [T(E) \setminus \Sigma];$  $- \forall \sigma \in T(E) \setminus \Sigma (|\dot{E} \cap N_{\sigma}| = \kappa).$

To describe a winning strategy for Player II in  $G_{\omega_2}(E \setminus X, x_0, \omega_1)$ , we look more carefully at the quotient forcing  $\mathbb{R}_{\gamma}$  in  $V[G_{\gamma} \times H_{\gamma}]$ . Note that T, the subtree of  $\langle \omega_2 \omega_2 \rangle$  added by H, is in  $V[G_{\gamma} \times H_{\gamma}]$ . It is not hard to show that  $\mathbb{R}_{\gamma}$  is forcing equivalent to  $\mathbb{S} \times \mathbb{T}$ , where  $\mathbb{S} = \operatorname{Coll}(\omega_2, [\gamma, \kappa))$  and  $\mathbb{T}$  is the set of all pairs  $(\eta, f)$ such that  $\eta < \omega_2$  and f is a partial function of size  $\omega_1$  from  $\kappa \setminus \gamma$  to  $T \cap {}^{\eta}\omega_2$ . Given two elements  $(\eta, f), (\xi, g) \in \mathbb{T}$ , we have  $(\xi, g) \leq_{\mathbb{T}} (\eta, f)$  if and only if  $\xi \geq \eta$ ,  $\operatorname{dom}(g) \supseteq \operatorname{dom}(f)$ , and, for all  $\alpha \in \operatorname{dom}(f)$ , we have  $g(\alpha) \supseteq f(\alpha)$ .

Recall that elements of  $\operatorname{Coll}(\omega_2, [\gamma, \kappa))$  are all functions s such that dom(s) is a subset of  $\operatorname{Card} \cap [\gamma, \kappa)$  of size  $\leq \omega_1$  and, for each  $\nu \in \operatorname{dom}(s)$ ,  $s(\nu)$  is a partial function from  $\omega_2$  to  $\nu$  of size  $\leq \omega_1$ . Given  $I \subseteq \kappa \setminus \gamma$ , let  $\mathbb{S}_I$  be the set of all  $s \in \mathbb{S}$ such that dom $(s) \subseteq I$ , and let  $\mathbb{T}_I$  be the set of all  $(\eta, f) \in \mathbb{T}$  such that dom $(f) \subseteq I$ . Note that, if  $I_0, I_1 \subseteq \kappa \setminus \gamma$  and  $\pi : I_0 \to I_1$  is a bijection, then  $\mathbb{T}_{I_0} \cong \mathbb{T}_{I_1}$  via the map  $(\eta, f) \mapsto (\eta, \hat{\pi}(f))$ , where dom $(\hat{\pi}(f)) = \pi[\text{dom}(f)]$  and, for all  $\alpha \in \text{dom}(f)$ , we let  $\hat{\pi}(f)(\alpha) = f(\alpha)$ . If, moreover,  $\pi$  is a bijection between  $\text{Card} \cap I_0$  and  $\text{Card} \cap I_1$  such that  $\pi(\nu) \leq \nu$  for all  $\nu \in \text{Card} \cap I_0$ , then  $\pi$  introduces a projection  $\tilde{\pi} : \mathbb{S}_{I_0} \to \mathbb{S}_{I_1}$  defined as follows: for each  $s \in \mathbb{S}_{I_0}$ , let  $\tilde{\pi}(s) \in \mathbb{S}_{I_1}$  be such that

- dom $(\tilde{\pi}(s)) = \pi[\text{dom}(s)];$
- for all  $\nu \in \operatorname{dom}(s)$ , we have  $\operatorname{dom}(\tilde{\pi}(s)(\nu)) = \operatorname{dom}(s(\nu))$ ;
- for all  $\nu \in \operatorname{dom}(s)$  and all  $\eta \in \operatorname{dom}(s(\nu))$ ,

$$\tilde{\pi}(s)(\pi(\nu))(\eta) = \begin{cases} s(\nu)(\eta) & \text{if } s(\nu)(\eta) < \pi(\nu) \\ 0 & \text{otherwise.} \end{cases}$$

Suppose in particular that  $\delta \in (\gamma, \kappa)$  is a cardinal, say  $\delta = \gamma^{+\varepsilon}$  for some  $\varepsilon < \kappa$ . Then we can partition the interval  $[\gamma, \kappa)$  into intervals  $\{I_{\zeta} \mid \zeta < \kappa\}$ , where  $I_{\zeta} = [\gamma^{+\varepsilon \cdot \zeta}, \gamma^{+\varepsilon \cdot (\zeta+1)})$  for all  $\zeta < \kappa$ . The preceding discussion then implies that

- $\mathbb{R}_{\gamma}$  is forcing equivalent to the  $\langle \omega_2$ -support product of  $\langle \mathbb{S}_{I_{\zeta}} \times \mathbb{T}_{I_{\zeta}} | \zeta < \kappa \rangle$ ;
- for all  $\zeta < \kappa$ , there is a projection from  $\mathbb{S}_{I_{\zeta}} \times \mathbb{T}_{I_{\zeta}}$  to  $\mathbb{S}_{I_0} \times \mathbb{T}_{I_0}$ .

In particular, forcing with  $\mathbb{R}_{\gamma}$  adds  $\kappa$ -many pairwise mutually ( $\mathbb{S}_{I_0} \times \mathbb{T}_{I_0}$ )-generic filters.

We are now ready to describe Player II's winning strategy in  $G_{\omega_2}(E \setminus X, x_0, \omega_1)$ . Some aspects of the strategy can only be precisely specified after Player I makes their first move, so suppose that, in round 1 of the game, Player I plays the ordinal  $\alpha_1 < \omega_2$ . Let  $\sigma_1 = x_0 \upharpoonright \alpha_1$ . Since  $\sigma_1 \in T(E) \setminus \Sigma$ , moving to  $V[G_{\gamma} \times H_{\gamma}]$ , we can find a nice  $\mathbb{R}_{\gamma}$ -name  $\dot{y}$  that is forced to be an element of  $\dot{E} \cap N_{\sigma_1} \setminus V[G_{\gamma} \times H_{\gamma}]$ . By the chain condition of  $\mathbb{R}_{\gamma}$ , we can find a limit ordinal  $\varepsilon$  such that, letting  $I_0 = [\gamma, \gamma^{+\varepsilon})$ ,  $\dot{y}$  is an  $\mathbb{S}_{I_0} \times \mathbb{T}_{I_0}$ -name. Let  $\mathbb{U}$  denote the  $<\omega_2$ -support product of  $\kappa$ -many copies of  $\mathbb{S}_{I_0} \times \mathbb{T}_{I_0}$ ; for  $\zeta < \kappa$ , let  $\mathbb{U}(\zeta)$  denote its  $\zeta^{\text{th}}$  factor. By the preceding discussion, there is a projection from  $\mathbb{R}_{\gamma}$  to  $\mathbb{U}$ , so we can view  $\mathbb{R}_{\gamma}$  as a two-step iteration of the form  $\mathbb{U} * \dot{\mathbb{W}}$ . For each  $\zeta < \kappa$ , let  $\dot{y}_{\zeta}$  denote the name for the interpretation of  $\dot{y}$  with respect to the generic filter for  $\mathbb{U}(\zeta)$ . By the product lemma, for all  $\zeta_0 < \zeta_1 < \kappa$ ,  $\dot{y}_{\zeta_0}$  and  $\dot{y}_{\zeta_1}$  are forced to be distinct elements of  $N_{\sigma_1} \cap (\dot{E} \setminus X)$ .

Let K \* L be  $\mathbb{U} * \mathbb{W}$ -generic over  $V[G_{\gamma} \times H_{\gamma}]$  such that  $V[G \times H] = V[G_{\gamma} \times H_{\gamma}][K * L]$ . For  $\zeta < \kappa$ , let  $K_{\zeta}$  denote the  $\mathbb{S}_{I_0} \times \mathbb{T}_{I_0}$ -generic filter induced by the  $\zeta^{\text{th}}$  factor of  $\mathbb{U}$ . In the course of the run of  $G_{\omega_2}(E \setminus X, x_0, \omega_1)$  in which Player II plays according to the strategy we will specify here, producing the play,  $\langle (\alpha_i, x_i) | 1 \leq i < \omega_1 \rangle$ , Player II will construct a strictly increasing sequence  $\langle \zeta_i | 1 \leq i < \omega_1 \rangle$  of ordinals below  $\kappa$  and a decreasing sequence  $\langle p_i | 1 \leq i < \omega_1 \rangle$  of conditions in  $\mathbb{S}_{I_0} \times \mathbb{T}_{I_0}$  satisfying the following requirements for all  $1 \leq i < \omega_1$ :

- (1)  $x_i$  is the interpretation of  $\dot{y}_{\zeta_i}$  in  $V[G_{\gamma} \times H_{\gamma}][K * L];$
- (2)  $p_{i+1} \in K_{\zeta_i};$
- (3)  $p_i$  decides the value of  $\dot{y} \upharpoonright \alpha_i$ .

We first describe Player II's first move. Let  $\zeta_1 = 0$ , let their play  $x_1$  be the interpretation of  $\dot{y}_0$  in  $V[G_{\gamma} \times H_{\gamma}][K * L]$ , and, let  $p_1 = 1_{\mathbb{S}_{I_0} \times \mathbb{T}_{I_0}}$ .

Now suppose that  $1 \leq j < \omega_1$  and, in our run of  $G_{\omega_2}(E \setminus X, x_0, \omega_1)$ , the players have played  $\langle \alpha_i | i \leq j \rangle$  and  $\langle x_i | i < j \rangle$ , with Player II playing according to their winning strategy and also specifying  $\langle (p_i, \zeta_i) | i < j \rangle$ .

Suppose first that j is a successor ordinal, say  $j = j_0 + 1$ . First, choose  $p_j \leq p_{j_0}$  such that  $p_j \in K_{\zeta_{j_0}}$  and  $p_j$  decides the value of  $\dot{y} \upharpoonright \alpha_j$ . Note that, in  $V[G_{\gamma} \times H_{\gamma}]$ ,

the set

# $\{f \in \mathbb{U} \mid \exists \zeta \in (\zeta_{j_0}, \kappa) \ (f(\zeta) \le p_j)\}$

is a dense open subset of U. Therefore, by genericity, Player II can choose  $\zeta_j \in (\zeta_{j_0}, \kappa)$  such that  $p_j \in K_{\zeta_j}$ . Then Player II plays the interpretation of  $\dot{y}_{\zeta_j}$  as  $x_j$ .

Next, suppose that j is a limit ordinal. In this case, we have  $\alpha_j = \sup\{\alpha_i \mid i < j\}$ . Since  $\mathbb{S}_{I_0} \times \mathbb{T}_{I_0}$  is countably closed, we can fix a lower bound  $p_j$  for  $\langle p_i \mid i < j \rangle$ . As in the previous paragraph, by genericity we can choose a  $\zeta_j < \kappa$  such that  $p_j \in K_{\zeta_j}$ and  $\zeta_j > \zeta_i$  for all i < j. Then Player II plays the interpretation of  $\dot{y}_{\zeta_i}$  as  $x_j$ .

It is readily verified that this describes a winning strategy for Player II in  $G_{\omega_2}(E \setminus X, x_0, \omega_1)$ , thus completing the proof that  $E \setminus X$  is  $\omega_1$ -perfect in  $V[G \times H]$ .

### 6. Full trees

Recall that, given a tree T and an ordinal  $\beta < \operatorname{ht}(T)$ , we let  $[T_{<\beta}]$  denote the set of cofinal branches through  $T_{<\beta}$ , i.e., the set of all elements of  $b \in \prod_{\alpha < \beta} T_{\alpha}$  such that the range of b is linearly ordered by  $<_T$ . Note that we can identify each element of  $T_{\beta}$  with an element of  $[T_{<\beta}]$ , namely the branch given by its predecessors; if T is normal, then this identification is injective. With a slight abuse of notation, then, we let  $[T_{<\beta}] \setminus T_{\beta}$  denote the set of *vanishing* branches through T of length  $\beta$ , i.e., the set of  $b \in [T_{<\beta}]$  such that the range of b does not have an upper bound in  $T_{\beta}$ .

**Definition 6.1.** A tree T is *full* if, for every limit ordinal  $\beta < ht(T)$ , there is at most one vanishing branch through T of length  $\beta$ .

There has been some research in recent years into the existence of full  $\kappa$ -Suslin trees. For example, in [15], Shelah establishes the consistency of the existence of full  $\kappa$ -Suslin trees for a Mahlo cardinal  $\kappa$ , answering a question of Jech (cf. [10]). In [12], Rinot, Yadai, and You prove the consistency of the existence of full  $\kappa$ -Suslin trees at accessible cardinals  $\kappa$ ; for example, they can consistently exist at all successors of regular uncountable cardinals.

Here, we are interested in full, splitting trees that may contain some cofinal branches. We begin by investigating full trees of height and size  $\omega_1$ . We first show that, under  $\diamond$ , we have considerable control over the number of cofinal branches through such trees, establishing Theorem C(1).

**Theorem 6.2.** Suppose that  $\diamond$  holds. Then, for every cardinal  $\nu \in \omega \cup \{\omega, \omega_1, 2^{\omega_1}\}$ , there is a normal, full, splitting tree  $T \subseteq {}^{<\omega_1}\omega_1$  such that  $|[T]| = \nu$ .

*Proof.* If  $\nu = 2^{\omega_1}$ , we can simply let  $T = {}^{<\omega_1}\omega_1$ . For concreteness, we will prove the theorem in case  $\nu = \omega_1$ ; the proof for smaller values of  $\nu$  is similar but easier.

We will construct a tree T by recursively specifying its  $\alpha^{\text{th}}$  level  $T_{\alpha} = T \cap^{\alpha} \omega_1$  for  $\alpha < \omega_1$ . When specifying  $T_{\alpha}$ , we will also specify an injective function  $f_{\alpha} : \alpha \to T_{\alpha}$  with the requirement that, for all  $\eta < \alpha < \beta < \omega_1$ , we have  $f_{\alpha}(\eta) \subseteq f_{\beta}(\eta)$ . The idea is that, at the end of the construction, for each  $\eta < \omega_1$ ,  $b_{\eta} := \bigcup \{f_{\alpha}(\eta) \mid \eta < \alpha < \omega_1\}$  will be a cofinal branch through T, and we will arrange so that every cofinal branch through T is equal to  $b_{\eta}$  for some  $\eta < \omega_1$ .

Since  $\diamondsuit$  holds, we can fix a sequence  $\langle a_{\alpha} \mid \alpha < \omega_1 \rangle$  such that

• for all  $\alpha < \omega_1, a_\alpha : \alpha \to \alpha;$ 

• for all  $b: \omega_1 \to \omega_1$ , there are stationarily many  $\alpha < \omega_1$  for which  $b \upharpoonright \alpha = a_\alpha$ . We now describe the construction of T. We must set  $T_0 = f_0 = \emptyset$ . Given  $T_\alpha$  and  $f_\alpha$ , first form  $T_{\alpha+1}$  by splitting maximally, i.e.,  $T_{\alpha+1} = \{\sigma^{-1} \mid \sigma \in T_\alpha, i < \omega_1\}$ .

For each  $\eta < \alpha$ , let  $f_{\alpha+1}(\eta) = f_{\alpha}(\eta) \cap 0$ , and let  $f_{\alpha+1}(\alpha)$  be any element of  $T_{\alpha+1}$  not equal to  $f_{\alpha+1}(\eta)$  for some  $\eta < \alpha$ .

Now suppose that  $\beta < \omega_1$  is a limit ordinal and  $T_{<\beta}$ , together with  $\langle f_\alpha \mid \alpha < \beta \rangle$ , has been specified. For each  $\eta < \beta$ , let  $b_{\eta}^{\beta} := \bigcup \{f_{\alpha}(\eta) \mid \eta < \alpha < \beta\}$ . By construction, each  $b_{\eta}^{\beta}$  is in  $[T_{<\beta}]$ . Now consider the function  $a_{\beta}$  given by our  $\diamond$ sequence. There are two cases to consider.

**Case 1:**  $a_{\beta}$  is a branch through  $T_{<\beta}$  and, for all  $\eta < \beta$ , we have  $a_{\beta} \neq b_{\eta}^{\beta}$ . In this case, let  $T_{\beta} = [T_{<\beta}] \setminus \{a_{\beta}\}$ . For all  $\eta < \beta$ , let  $f_{\beta}(\eta) = b_{\eta}^{\beta}$ .

**Case 2:** otherwise. In this case, let  $T_{\beta} = [T_{<\beta}]$  and, again, let  $f_{\beta}(\eta) = b_{\eta}^{\beta}$  for all  $\eta < \beta$ .

This completes the construction of T and  $\langle f_{\alpha} \mid \alpha < \omega_1 \rangle$ . It is easily verified that T is a normal, full, splitting tree and  $\langle b_{\eta} \mid \eta < \omega_1 \rangle$  is an injective sequence of cofinal branches through T. It remains to show that every cofinal branch through T is equal to  $b_{\eta}$  for some  $\eta < \omega_1$ .

To this end, fix  $b \in {}^{\omega_1}\omega_1$  such that, for all  $\eta < \omega_1$ , we have  $b \neq b_{\eta}$ . We will find  $\beta < \omega_1$  such that  $b \upharpoonright \beta \notin T$ . First, let C be the set of limit ordinals  $\beta < \omega_1$  such that

• 
$$b[\beta] \subseteq \beta;$$

• for all  $\eta < \beta$ , we have  $b_{\eta} \upharpoonright \beta \neq b \upharpoonright \beta$ .

Then C is a club in  $\omega_1$ , so we can fix  $\beta \in C$  such that  $b \upharpoonright \beta = a_\beta$ . Now consider stage  $\beta$  of the construction of T. If  $a_\beta \notin [T_{<\beta}]$ , then we are done, since this immediately implies that  $b \upharpoonright \beta \notin T$ . If  $a_\beta$  is a branch through  $T_{<\beta}$ , then, for all  $\eta < \beta$ , we have  $a_\beta \neq b_\eta \upharpoonright \beta = b_\eta^\beta$ . We are therefore in Case 1 of the construction, and hence we have  $b \upharpoonright \beta = a_\beta \notin T$ , as desired.

CH is a necessary condition for the existence of full, splitting trees of height  $\omega_1$  with few cofinal branches as it is easily seen that, if CH fails, then every full, splitting tree of height  $\omega_1$  has at least  $2^{\aleph_0}$ -many cofinal branches. However, we now show that CH is not a sufficient condition for this, i.e., the hypothesis of  $\diamond$  in Theorem 6.2 cannot be weakened to CH. Recall that a forcing notion  $\mathbb{P}$  is *totally proper* if it is proper and adds no reals. We will need an iterable strengthening of total properness known as *complete properness*, introduced in [11] (cf. also [2]). Let us recall the relevant definitions, beginning with  $\alpha$ -properness.

**Definition 6.3.** Suppose that  $\mathbb{Q}$  is a forcing notion and  $\theta$  is a sufficiently large regular cardinal. A countable elementary submodel M of  $H(\theta)$  is said to be *suitable* for  $\mathbb{Q}$  if  $\mathbb{Q}, \mathscr{P}(\mathbb{Q}) \in M$ .

If M is suitable for  $\mathbb{Q}$  and  $q \in \mathbb{Q}$ , then we say that q is  $(M, \mathbb{Q})$ -generic if, for every dense open subset D of  $\mathbb{Q}$  with  $D \in M$  and every  $r \leq_{\mathbb{Q}} q$ , r is compatible with an element of  $D \cap M$ . If  $G \subseteq \mathbb{Q} \cap M$  is a filter, then we say that G is  $(M, \mathbb{Q})$ -generic if  $G \cap D \neq \emptyset$  for every dense open subset D of  $\mathbb{Q}$  with  $D \in M$ . A condition  $q \in \mathbb{Q}$ is totally  $(M, \mathbb{Q})$ -generic if the set  $\{p \in M \cap \mathbb{Q} \mid q \leq p\}$  is an  $(M, \mathbb{Q})$ -generic filter.

**Definition 6.4.** Suppose that  $\mathbb{Q}$  is a forcing notion and  $\alpha < \omega_1$ . A suitable  $\alpha$ -tower for  $\mathbb{Q}$  is a continuous,  $\subseteq$ -increasing sequence  $\langle M_\eta \mid \eta < \alpha \rangle$  of countable elementary submodels of  $H(\theta)$  for some sufficiently large regular cardinal  $\theta$  such that

- $M_0$  is suitable for  $\mathbb{Q}$ ;
- for all  $\xi$  with  $\xi + 1 < \alpha$ , we have  $\langle M_{\eta} | \eta \leq \xi \rangle \in M_{\xi+1}$ .

We say that  $\mathbb{Q}$  is  $\alpha$ -proper if, for every suitable  $\alpha$ -tower  $\langle M_{\eta} \mid \eta < \alpha \rangle$  for  $\mathbb{Q}$  and every  $q_0 \in M_0 \cap \mathbb{Q}$ , there is  $q \leq_{\mathbb{Q}} q_0$  such that q is  $(M_{\eta}, \mathbb{Q})$ -generic for all  $\eta < \alpha$ . We say that  $\mathbb{Q}$  is *totally*  $\alpha$ -proper if we can additionally require that q is totally  $(M_{\eta}, \mathbb{Q})$ -generic for all  $\eta < \alpha$ .

We now turn to Moore's notion of *complete properness*.

**Definition 6.5.** If M and N are sets, then the notation  $M \to N$  denotes the existence of an elementary embedding  $\varepsilon : (M, \in) \to (N, \in)$  such that  $\varepsilon \in N$  and  $N \models "M$  is countable", i.e., N contains an injection of M into  $\omega$ . If  $X \in M$  and  $M \to N$ , as witnessed by  $\varepsilon$ , then we will let  $X^N$  denote  $\varepsilon(X)$ . If  $X \subseteq M$  is not an element of M, then  $X^N$  denotes the pointwise image  $\varepsilon[X]$ .

**Definition 6.6.** Suppose that  $\mathbb{Q}$  is a forcing notion, M is suitable for  $\mathbb{Q}$ , and  $M \to N$ . Then a filter  $G \subseteq \mathbb{Q} \cap M$  is  $\overrightarrow{MN}$ -prebounded if whenever  $N \to P$  and  $G \in P$ , then  $p \models "G^P$  has a lower bound in  $\mathbb{Q}^P$ , where  $G^P$  is defined via the composition  $M \to N \to P$ .

**Definition 6.7.** Suppose that  $\mathbb{Q}$  is a forcing notion. We say that  $\mathbb{Q}$  is *completely* proper if whenever M is suitable for  $\mathbb{Q}$ ,  $q \in \mathbb{Q} \cap M$ , and  $M \to N_i$  for i < 2, there is an  $(M, \mathbb{Q})$ -generic filter  $G \subseteq \mathbb{Q} \cap M$  that is  $\overrightarrow{MN_i}$ -prebounded for all i < 2 with  $q \in G$ .

Moore proved in [11, Lemma 4.11] that completely proper forcings are 2-complete with respect to some completeness system  $\mathbb{D}$  (cf. [14, §V.5]). Combined with Shelah's iteration theorem concerning such forcings ([14, Theorem VIII.4.5]), this yields the following theorem (cf. [2, Main Theorem]).

**Theorem 6.8.** Suppose that  $\langle \mathbb{P}_{\eta}, \hat{\mathbb{Q}}_{\xi} | \eta \leq \delta, \xi < \delta \rangle$  is a countable support iteration of totally proper forcing notions such that, for all  $\eta < \delta$ , we have

 $\Vdash_{\mathbb{P}_n} ``\dot{\mathbb{Q}}_\eta is completely proper and totally <math>\alpha$ -proper for all  $\alpha < \omega_1$ ".

Then  $\mathbb{P}_{\delta}$  is totally proper.

We are now ready to prove that the assumption of  $\diamond$  cannot be weakened to CH in Theorem 6.2. In fact, we will prove that it is consistent with CH that every full, splitting tree of height  $\omega_1$  contains a copy of  ${}^{<\omega_1}2$ , thus establishing Theorem C(2).

**Theorem 6.9.** It is consistent that CH holds and every full, splitting tree of height  $\omega_1$  contains a copy of  ${}^{<\omega_1}2$ .

*Proof.* We first note that it is enough to consider trees of height and size  $\omega_1$ . To see this, suppose that CH holds and T is a tree of height  $\omega_1$ . Let  $\theta$  be a sufficiently large regular cardinal and let  $M \prec H(\theta)$  be such that

- $|M| = \omega_1;$
- ${}^{\omega}M \subseteq M;$
- $T \in M$ .

Then  $T \cap M$  is a full, splitting subtree of T of size  $\omega_1$ ; if  $T \cap M$  contains a copy of  ${}^{<\omega_1}2$  then, a fortiori, so does T.

Suppose that GCH holds. Fix for now a full, splitting tree T of height and size  $\omega_1$ . Without loss of generality, we can assume that T is a subtree of  ${}^{<\omega_1}\omega_1$ . We will describe a totally proper forcing  $\mathbb{P}(T)$  of size  $\omega_1$  that adds a copy of  ${}^{<\omega_1}2$  to T and then show that this forcing can be iterated without adding reals.

Given a subtree  $S \subseteq {}^{<\omega_1}\omega_1$ , let  $\partial S$  be the set of  $\sigma \in {}^{<\omega_1}\omega_1$  such that

- $dom(\sigma)$  is a limit ordinal;
- $\sigma \notin S;$

• for all  $\eta < \operatorname{dom}(\sigma), \sigma \upharpoonright \eta \in S$ .

Conditions of  $\mathbb{P}(T)$  are all pairs of the form  $p = (S_p, f_p)$  such that

- (1)  $S_p$  is a countable subtree of  ${}^{<\omega_1}2$ ;
- (2)  $f_p: S_p \to T$  is an isomorphic embedding;
- (3) for all  $\sigma \in \partial S_p$ , we have  $\bigcup \{ f_p(\sigma \upharpoonright \eta) \mid \eta < \operatorname{dom}(\sigma) \} \in T$ .

If  $p, q \in \mathbb{P}(T)$ , then  $q \leq p$  if and only if  $S_q \supseteq S_p$  and  $f_q \upharpoonright S_p = f_p$ .

Since CH holds,  $\mathbb{P}(T)$  is of size  $\omega_1$ . Given  $p = (S_p, f_p) \in \mathbb{P}(T)$ , let  $T_p$  denote the  $\leq_T$ -downward closure of  $f_p$  " $S_p$ . Condition (3) above can then be expressed as the assertion that  $\partial T_p \subseteq T$ .

Let  $\mathcal{L}$  be the set of  $\beta \in \lim(\omega_1)$  such that  $[T_{\leq\beta}] \setminus T_{\beta} \neq \emptyset$  and, for each  $\beta \in \mathcal{L}$ , let  $b_{\beta}$  be the unique element of  $[T_{\leq\beta}] \setminus T_{\beta}$ . When constructing elements of  $\mathbb{P}(T)$ , we need to be careful to avoid the branches  $b_{\beta}$  for  $\beta \in \mathcal{L}$ . The following lemmas show that this can be done.

**Lemma 6.10.** Suppose that  $\beta \in \lim(\omega_1)$  and  $b \in {}^{\beta}\omega_1$ . Suppose moreover that

- $p = (S_p, f_p) \in \mathbb{P}(T);$
- $\sigma \in {}^{<\omega_1}2$  is such that  $\sigma, \sigma \cap 0, \sigma \cap 1 \in S_p$ ;
- $f_p(\sigma) \sqsubseteq b$  but, for all i < 2, we have  $f_p(\sigma^{\frown}i) \perp b$ .

Let  $\xi < \beta$  be such that  $f_p(\sigma \cap i) \perp (b \upharpoonright \xi)$  for all i < 2. Then, for every  $q \leq p$  and every  $\tau \in S_q$ , it is not the case that  $f_q(\tau) \supseteq b \upharpoonright \xi$ .

*Proof.* Note that  $f_p(\sigma) = b \upharpoonright \eta$  for some  $\eta < \xi$ . Fix  $q \leq p$  and  $\tau \in S_q$ . There are three cases to consider.

First, if  $\tau \sqsubseteq \sigma$ , then  $f_q(\tau) = f_p(\tau) \sqsubseteq b \upharpoonright \eta$ , so  $f_q(\tau) \not\supseteq b \upharpoonright \xi$ .

Second, if  $\sigma \sqsubseteq \tau$ , then there is i < 2 such that  $\sigma^{\frown}\langle i \rangle \sqsubseteq \tau$ , and hence  $f_p(\sigma^{\frown}\langle i \rangle) \sqsubseteq f_q(\tau)$ . Since  $f_p(\sigma^{\frown}\langle i \rangle) \perp b \upharpoonright \xi$ , it follows that  $f_q(\tau) \perp b \upharpoonright \xi$ .

The remaining case is that in  $\tau \perp \sigma$ . In this case,  $f_q(\tau) \perp f_p(\sigma) = b \upharpoonright \eta$ .

**Lemma 6.11.** Suppose that M is suitable for  $\mathbb{P}(T)$ ,  $p \in M \cap \mathbb{P}(T)$ ,  $\beta = M \cap \omega_1$ , and  $\beta \in \mathcal{L}$ . Let

$$\xi := \sup\{\eta < \beta \mid \exists \sigma \in S_p \ [f_p(\sigma) = b_\beta \upharpoonright \eta]\},\$$

and assume that  $b_{\beta} \upharpoonright \xi \in M$ . Then there is  $q \leq p$  with  $q \in M$  and  $\sigma \in S_q$  such that

- $f_q(\sigma) \sqsubseteq b_\beta;$
- for all i < 2,  $\sigma^{\frown} i \in S_q$  and  $f_q(\sigma^{\frown} i) \perp b_{\beta}$ .

*Proof.* We may assume that p itself does not satisfy the conclusion of the theorem. Since  $p \in \mathbb{P}(T)$  and  $b_{\beta} \notin T$ , we must have  $\xi < \beta$ . Assume first that there is  $\sigma \in S_p$  such that  $f_p(\sigma) = b_{\beta} \upharpoonright \xi$ . By the definition of  $\xi$ , if i < 2 and  $\sigma^{\frown} i \in S_p$ , then  $f_p(\sigma^{\frown} i) \perp b_{\beta}$ . We will define a condition  $q \leq p$  in M such that  $S_q = S_p \cup \{\sigma^{\frown} 0, \sigma^{\frown} 1\}$ . It suffices to define  $f_q(\sigma^{\frown} i)$  for i < 2. Using the fact that T is splitting, we know that there are two distinct nodes  $t_0, t_1 \in T_{\xi+2} \cap M$  such that, for all i < 2, we have  $b_{\beta} \upharpoonright \xi \sqsubseteq t_i$  but  $b_{\beta} \upharpoonright (\xi+2) \perp t_i$ . For i < 2, if  $\sigma^{\frown} i \notin S_p$ , set  $f_q(\sigma^{\frown} i) = t_i$ . It is readily verified that  $q \leq p$  is as desired.

Now suppose that there is no  $\sigma \in S_p$  such that  $f_p(\sigma) = b_\beta \upharpoonright \xi$ . By our assumptions about p, there must be an increasing sequence of ordinals  $\langle \eta_n \mid n < \omega \rangle$  and a  $\sqsubseteq$ -increasing sequence  $\langle \sigma_n \mid n < \omega \rangle$  from  $S_p$  such that

•  $\sup\{\eta_n \mid n < \omega\} = \xi;$ 

- $\bigcup \{ \sigma_n \mid n < \omega \} \notin S_p; \text{ and }$
- for all  $n < \omega$ ,  $f_p(\sigma_n) = b_\beta \upharpoonright \eta_n$ .

Let  $\sigma := \bigcup \{ \sigma_n \mid n < \omega \}$ . Then  $\sigma \in M$ , since it is definable in M as the unique element  $\tau$  of  $\partial S_p$  such that  $\bigcup \{ f_p(\tau \upharpoonright \eta) \mid \eta < \operatorname{dom}(\tau) \} = b_\beta \upharpoonright \xi$ . We will define a condition  $q \leq p$  in M such that  $S_q = S_p \cup \{\sigma, \sigma^{\frown} 0, \sigma^{\frown} 1\}$ . Work entirely in M. First, set  $f_q(\sigma) = b_\beta \upharpoonright \xi$ . Then define  $f_q(\sigma^{-1})$  for i < 2 exactly as in the previous case. It is again readily verified that  $q \leq p$  is as desired. 

**Lemma 6.12.**  $\mathbb{P}(T)$  is totally  $\alpha$ -proper for all  $\alpha < \omega_1$ .

*Proof.* The proof is by induction on  $\alpha$ . We will in fact prove the following stronger statement, which will be used in the inductive step:

For every suitable  $(\alpha + 1)$ -tower  $\langle M_{\eta} \mid \eta \leq \alpha \rangle$  and every  $p_0 \in$  $M_0 \cap \mathbb{P}(T)$ , there is  $p \leq p_0$  such that

- p is totally  $(M_{\eta}, \mathbb{P}(T))$ -generic for all  $\eta \leq \alpha$ ;
- $S_p, f_p \subseteq M_\alpha$ .

Fix  $\beta < \omega_1$  and suppose that we have established the induction hypothesis for all  $\alpha < \beta$ . Let  $\langle M_n \mid \eta \leq \beta \rangle$  be a suitable  $(\beta+1)$ -tower for  $\mathbb{P}(T)$ , and fix  $p_0 \in M_0 \cap \mathbb{P}(T)$ . We will assume that  $\beta$  is a limit ordinal, and hence  $M_{\beta} = \bigcup_{\alpha < \beta} M_{\alpha}$ ; the case in which  $\beta$  is a successor is easier and proven similarly. Note also that if a condition p is totally  $(M_{\alpha}, \mathbb{P}(T))$ -generic for all  $\alpha < \beta$ , then it is also totally  $(M_{\beta}, \mathbb{P}(T))$ -generic.

Let  $\delta := M_{\beta} \cap \omega_1$ . Assume that  $\delta \in \mathcal{L}$ ; the case in which  $\delta \notin \mathcal{L}$  is similar and easier. There are now two cases to consider; we will deal with them in parallel:

- Case 1: There is  $\xi^* < \delta$  such that  $b_{\delta} \upharpoonright \xi^* \notin M_{\beta}$ ;
- Case 2: For every  $\xi < \delta$ ,  $b_{\delta} \upharpoonright \xi \in M_{\beta}$ .

If we are in Case 2, let

$$\xi := \sup\{\eta < \beta \mid \exists \sigma \in S_{p_0} \ [f_{p_0}(\sigma) = b_\beta \upharpoonright \eta]\}.$$

There are now two subcases to consider:

- Case 2a:  $b_{\delta} \upharpoonright \xi \in M_0$ ;
- Case 2b:  $b_{\delta} \upharpoonright \xi \notin M_0$ .

If we are in Case 2a, begin by applying Lemma 6.11 to find a  $p'_0 \leq p_0$  and a  $\sigma_0 \in S_{p'_0}$ such that

- $p'_0 \in M_0$ ;  $f_{p'_0}(\sigma_0) \sqsubseteq b_{\delta}$ ; for all i < 2,  $\sigma_0 \frown i \in S_{p'_0}$  and  $f_{p'_0}(\sigma_0 \frown i) \perp b_{\delta}$ .

If we are in Case 1 or 2b, let  $p'_0 = p_0$  and leave  $\sigma_0$  undefined for now.

Let  $\langle \alpha_n \mid n < \omega \rangle$  be a strictly increasing sequence of ordinals that is cofinal in  $\beta$ . If we are in Case 2b, additionally choose  $\alpha_0$  so that  $\alpha_0 + 1$  is the least ordinal  $\varepsilon < \beta$  such that  $b_{\delta} \upharpoonright \xi \in M_{\varepsilon}$  (note that this  $\varepsilon$  must be a successor ordinal by the continuity of  $\langle M_{\eta} \mid \eta < \beta \rangle$ ).

We will build a decreasing sequence  $\langle p_n \mid n < \omega \rangle$  of conditions in  $\mathbb{P}(T)$  such that, for all  $n < \omega$ , we have

- $p_{n+1} \in M_{\alpha_n+1};$
- $p_{n+1}$  is totally  $(M_{\eta}, \mathbb{P}(T))$ -generic for all  $\eta \leq \alpha_n$ .

We will also arrange so that  $\langle p_n \mid n < \omega \rangle$  will have a lower bound

$$p_{\infty} = \left(\bigcup_{n < \omega} S_{p_n}, \bigcup_{n < \omega} f_{p_n}\right)$$

in  $\mathbb{P}(T)$ . This lower bound will then be the desired condition that is totally  $(M_{\eta}, \mathbb{P}(T))$ -generic for all  $\eta \leq \beta$ . Note that, by construction, we will have  $S_{p_{\infty}} \subseteq$  $M_{\beta}$ .

Let  $\langle \gamma_n \mid 0 < n < \omega \rangle$  enumerate  $\mathcal{L} \cap \delta$  (with repetitions, if necessary) in such a way that  $\gamma_n \in M_{\alpha_n+1}$  for all  $0 < n < \omega$ . We will arrange so that, for all  $0 < n < \omega$ , there is  $\sigma_n \in S_{p_n}$  such that

- $f_{p_n}(\sigma_n) \sqsubseteq b_{\gamma_n};$  for all i < 2,  $\sigma_n \frown i \in S_{p_n}$  and  $f_{p_n}(\sigma_n \frown i) \perp b_{\gamma_n}.$

We now describe the rest of the construction. The first step will be different from the others due to the need to take care of Case 2b. We begin by applying the inductive hypothesis in  $M_{\alpha_0+1}$  to find  $p'_1 \leq p'_0$  such that

- $p'_1 \in M_{\alpha_0+1}$  and  $S_{p'_1}, f_{p'_1} \subseteq M_{\alpha_0}$ ;  $p'_1$  is totally  $(M_\eta, \mathbb{P}(T))$ -generic for all  $\eta \leq \alpha_0$ .

Suppose first that we are in Case 2b. By our choice of  $\alpha_0$  and the fact that  $S_{p'_1} \subseteq$  $M_{\alpha_0}$ , we know that  $b_{\delta} \upharpoonright \xi \notin S_{p'_1}$ . Thus, we still have

$$\xi = \sup\{\eta < \beta \mid \exists \sigma \in S_{p'_1} \ [f_{p'_1}(\sigma) = b_\beta \upharpoonright \eta]\}.$$

We can therefore apply Lemma 6.11 to find  $p_1^* \leq p_1'$  and  $\sigma_0 \in S_{p_1^*}$  such that

- p<sub>1</sub><sup>\*</sup> ∈ M<sub>α₀+1</sub>;
   f<sub>p1</sub><sup>\*</sup>(σ₀) ⊑ b<sub>δ</sub>;
   for all i < 2, σ₀<sup>¬</sup>⟨i⟩ ∈ S<sub>p1</sub><sup>\*</sup> and f<sub>p1</sub><sup>\*</sup>(σ₀<sup>¬</sup>⟨i⟩) ⊥ b<sub>δ</sub>.

If we are in Case 1 or 2a, let  $p_1^* = p_1'$ ; in Case 1, leave  $\sigma_0$  permanently undefined. Now, regardless of the case we are in, apply Lemma 6.11 once again to find  $p_1 \leq p_1^*$ and  $\sigma_1 \in S_{p_1}$  such that

• 
$$p_1 \in M_{\alpha_0+1}$$

- $f_{p_1}(\sigma_1) \sqsubseteq b_{\gamma_1};$  for all  $i < 2, \sigma_1 \frown i \in S_{p_1}$  and  $f_{p_1}(\sigma_1 \frown i) \perp b_{\gamma_1}.$

The construction is now uniform across all cases. Suppose that  $0 < n < \omega$  and we have constructed  $p_n$ . Apply the inductive hypothesis to the tower  $\langle M_\eta \mid \alpha_{n-1} < \beta_n \rangle$  $\eta \leq \alpha_n$  inside  $M_{\alpha_n+1}$  to find  $p'_{n+1} \leq p_n$  that is totally  $(M_\eta, \mathbb{P}(T))$ -generic for all  $\eta \in (\alpha_{n-1}, \alpha_n]$ . Since  $p'_{n+1} \leq p_n$  and  $p_n$  is totally  $(M_\eta, \mathbb{P}(T))$ -generic for all  $\eta \leq \alpha_{n-1}$ , it follows that  $p'_{n+1}$  is in fact totally  $(M_{\eta}, \mathbb{P}(T))$ -generic for all  $\eta \leq \alpha_n$ . Now apply Lemma 6.11 to find  $p_{n+1} \leq p'_{n+1}$  and  $\sigma_{n+1} \in S_{p_{n+1}}$  such that

- $p_{n+1} \in M_{\alpha_n+1}$ ;  $f_{p_{n+1}}(\sigma_{n+1}) \sqsubseteq b_{\gamma_{n+1}}$ ; for all i < 2,  $\sigma_{n+1} \frown i \in S_{p_{n+1}}$  and  $f_{p_{n+1}}(\sigma_{n+1} \frown i) \perp b_{\gamma_{n+1}}$ .

This completes the construction. It remains to verify that  $p_{\infty} \in \mathbb{P}(T)$ . Since  $S_{p_{\infty}} \subseteq M_{\beta}$ , the only way that  $p_{\infty}$  could fail to be in  $\mathbb{P}(T)$  is if there is  $\gamma \in \mathcal{L} \cap (\delta+1)$ and  $\sigma \in \partial S_{\infty}$  such that  $\bigcup \{ f_{p_{\infty}}(\sigma \upharpoonright \eta) \mid \eta < \operatorname{dom}(\sigma) \} = b_{\gamma}$ . Suppose there are such a  $\gamma$  and  $\sigma$ .

Assume first that  $\gamma = \delta$ . If we are in Case 1, then there is  $\xi^* < \delta$  such that  $b_{\delta} \upharpoonright \xi^* \notin M_{\beta}$ . Since  $S_{p_{\infty}}, f_{p_{\infty}} \subseteq M_{\beta}$ , there can be no  $\eta < \operatorname{dom}(\sigma)$  such that  $f_{p_{\infty}}(\sigma \upharpoonright \eta) \sqsupseteq b_{\delta} \upharpoonright \xi^*$ , which is a contradiction. If we are in Case 2, then recall our choice of  $\sigma_0$ , and let  $\xi' < \delta$  be such that  $f_{p_1}(\sigma_0 \widehat{\phantom{\alpha}} i) \perp b_{\delta} \upharpoonright \xi'$  for all i < 2. Then, by Lemma 6.10, there is no  $\eta < \operatorname{dom}(\sigma)$  such that  $f_{p_{\infty}}(\sigma \upharpoonright \eta) \sqsupseteq b_{\delta} \upharpoonright \xi'$ , which is again a contradiction.

Finally, assume that  $\gamma \in \mathcal{L} \cap \delta$ . Then there is  $0 < n < \omega$  such that  $\gamma = \gamma_n$ . Let  $\xi' < \gamma$  be such that  $f_{p_n}(\sigma_n \cap i) \perp b_{\gamma} \upharpoonright \xi'$  for all i < 2. Then, again by Lemma 6.10, there is no  $\eta < \operatorname{dom}(\sigma)$  such that  $f_{p_{\infty}}(\sigma \upharpoonright \eta) \supseteq b_{\gamma} \upharpoonright \xi'$ , yielding the final contradiction and completing the proof. 

#### **Lemma 6.13.** $\mathbb{P}(T)$ is completely proper.

*Proof.* Suppose that M is suitable for  $\mathbb{Q}$  and we are given  $M \to N_i$  for i < 2. Let  $\varepsilon_i: (M, \in) \to (N_i, \in)$  witness this. Let  $\delta := M \cap \omega_1$ . Note that  $\delta \in N_0 \cap N_1$ . By elementarity, for i < 2, the restriction  $\varepsilon_i \upharpoonright \delta$  must be the identity map on  $\delta$ , and hence  $\pi \upharpoonright (T \cap M)$  is the identity map on  $T \cap M$ . Without loss of generality, assume that  $\delta \in \mathcal{L}^{N_0} \cap \mathcal{L}^{N_1}$ ; the other cases are similar but easier. For each i < 2, let  $b^i_{\delta}$  be such that

 $N_i \models "b^i_{\delta}$  is the unique element of  $[T^{N_i}_{\leq \delta}] \setminus T^{N_i}_{\delta}$ .

Let  $p \in M \cap \mathbb{P}(T)$  be arbitrary. We will now build a decreasing sequence  $\langle p_n |$  $n < \omega$  from  $M \cap \mathbb{P}(T)$  below p such that the upward closure of  $\{p_n \mid n < \omega\}$  in  $M \cap \mathbb{P}(T)$  is an  $(M, \mathbb{P}(T))$ -generic filter. Begin by letting  $p_0 = p$ . Now define  $p_1$  as follows. If there is  $\xi < \delta$  such that  $b_{\delta}^0 \upharpoonright \xi \notin M$ , then simply let  $p_1 = p_0$ . Otherwise, apply Lemma 6.11 to find  $p_1 \leq p_0$  and  $\sigma_1 \in S_{p_1}$  such that

- $p_1 \in M$ ;  $f_{p_1}(\sigma_1) \sqsubseteq b_{\delta}^0$ ; for all j < 2,  $\sigma_1 \widehat{\phantom{a}} j \in S_{p_1}$  and  $f_{p_1}(\sigma_1 \widehat{\phantom{a}} j) \perp b_{\delta}^0$ .

Next, find  $p_2 \leq p_1$  in the same way, but replacing  $b^0_{\delta}$  by  $b^1_{\delta}$ . Finally, let  $\langle D_n \mid n < \omega \rangle$ enumerate all dense open subsets of  $\mathbb{P}(T)$  that are elements of M, and recursively define a decreasing sequence  $\langle p_n \mid 2 < n < \omega \rangle$  from  $M \cap \mathbb{P}(T)$  such that  $p_n \in D_{n-3}$ for all  $n < \omega$ . Notice that, for all  $\gamma \in \mathcal{L} \cap \delta$ , the set  $D^*_{\gamma}$  of all  $q \in \mathbb{P}(T)$  for which there is  $\sigma \in S_q$  such that

- $f_q(\sigma) \sqsubseteq b_{\gamma}$ ; and for all  $j < 2 \ \sigma^{\frown} j \in S_q$  and  $f_q(\sigma^{\frown} j) \perp b_{\gamma}$

is a dense open subset of  $\mathbb{P}(T)$  that is in M; there is therefore  $n < \omega$  such that  $p_n \in D^*_{\gamma}.$ 

Let G be the upward closure of  $\{p_n \mid n < \omega\}$  in  $M \cap \mathbb{P}(T)$ . By construction, it is clear that G is an  $(M, \mathbb{P}(T))$ -generic filter. It remains to show that it is  $\overline{MN_i}$ prebounded for all i < 2. To this end, fix i < 2 and an arrow  $N_i \rightarrow P$ , witnessed by a map  $\epsilon : (N_i, \epsilon) \to (P, \epsilon)$ , such that  $G \in P$ . Note that, for each  $n < \omega$ , we have  $\epsilon \circ \varepsilon_i(p_n) = p_n$ , and hence  $G^P = G$ . We will therefore be done if we show that

$$p_{\infty} = \left(\bigcup_{n < \omega} S_{p_n}, \bigcup_{n < \omega} f_{p_n}\right) = \left(\bigcup_{q \in G} S_q, \bigcup_{q \in G} f_q\right)$$

is in  $\mathbb{P}(T)^P$ .

First note that  $p_{\infty} \in P$ , since  $G \in P$ . Also, the range of  $f_{p_{\infty}}$  is contained in  $T^P_{<\delta} = T_{<\delta}$ . Therefore, the only way that  $p_{\infty}$  could fail to be in  $\mathbb{P}(T)^P$  is if there is

 $\gamma \in \mathcal{L}^P \cap (\delta + 1)$  and  $\sigma \in \partial S_\infty$  such that  $\bigcup \{ f_{p_\infty}(\sigma \upharpoonright \eta) \mid \eta < \operatorname{dom}(\sigma) \} = b_\gamma^P$ , where  $b_\gamma^P$  is such that

 $P \models "b_{\gamma}^{P}$  is the unique element of  $[T_{<\gamma}^{P}] \setminus T_{\gamma}^{P}$ ".

Suppose that there are such a  $\gamma$  and  $\sigma$ .

Note that  $\delta < N_i \cap \omega_1$ , so  $\epsilon \upharpoonright (\delta + 1)$  is the identity map. It follows that  $\mathcal{L}^P \cap (\delta + 1) = \mathcal{L}^{N_i} \cap (\delta + 1)$ , and, moreover,  $\mathcal{L}^P \cap \delta = \mathcal{L} \cap \delta$ . Suppose first that  $\gamma = \delta$ , in which case we have  $b_{\delta}^P = b_{\delta}^i$ . If there is  $\xi < \delta$  such that  $b_{\delta}^i \upharpoonright \xi \notin M$ , then, since  $S_{p_{\infty}}, f_{p_{\infty}} \subseteq M$ , there can be no  $\eta < \operatorname{dom}(\sigma)$  such that  $f_{p_{\infty}}(\sigma \upharpoonright \eta) \sqsupseteq b_{\delta}^i \upharpoonright \xi$ , which is a contradiction. Otherwise, when constructing  $p_{i+1}$ , we fixed a  $\sigma_{i+1} \in S_{p_{i+1}}$  such that  $f_{p_{i+1}}(\sigma_{i+1}) \sqsubseteq b_{\delta}^i$  and for which there exists some  $\xi' < \delta$  such that  $f_{p_{i+1}}(\sigma_{i+1} \frown j) \perp b_{\delta}^i \upharpoonright \xi'$  for all j < 2. Then, by Lemma 6.10, there is no  $\eta < \operatorname{dom}(\sigma)$  such that  $f_{p_{\infty}}(\sigma \upharpoonright \eta) \sqsupseteq b_{\delta}^i \upharpoonright \xi'$ , which is again a contradiction.

Suppose finally that  $\gamma < \delta$ , in which case  $b_{\gamma}^{P} = b_{\gamma}$ . Then there is  $n < \omega$  such that  $p_n \in D_{\gamma}^*$ , and we reach a contradiction exactly as in the second case in the previous paragraph. Thus, G is in fact  $\overrightarrow{MN_i}$ -prebounded, and hence  $\mathbb{P}(T)$  is completely proper.

By a standard genericity argument, if G is  $\mathbb{P}(T)$ -generic over V, then  $\bigcup \{f_p \mid p \in G\}$  witnesses that, in V[G], T contains a copy of  ${}^{<\omega_1}2$  (which, since  $\mathbb{P}(T)$  is totally proper, is the same when calculated in V or in V[G]).

We now define a countable support iteration  $\langle \mathbb{P}_{\eta}, \hat{\mathbb{Q}}_{\xi} \mid \eta \leq \omega_2, \xi < \omega_2 \rangle$  such that, for each  $\eta < \omega_2$ , there is a  $\mathbb{P}_{\eta}$ -name  $\dot{T}_{\eta}$  for a full, splitting subtree of  $\langle \omega_1 \omega_1 \rangle$  of height and size  $\omega_1$  such that

$$\Vdash_{\mathbb{P}_n} \dot{\mathbb{Q}}_n = \mathbb{P}(\dot{T}_n).$$

By Theorem 6.8,  $\mathbb{P}_{\eta}$  will be totally proper for each  $\eta \leq \omega_2$ , and hence

- CH will hold in  $V^{\mathbb{P}_{\eta}}$ ; and
- $({}^{<\omega_1}\omega_1)^{V^{\mathbb{P}\eta}} = ({}^{<\omega_1}\omega_1)^V.$

It follows that, for each  $\eta < \omega_2$ , we have

$$\Vdash_{\mathbb{P}_{\eta}} "\mathbb{Q}_{\eta} \text{ is a proper forcing of size } \omega_1".$$

Therefore, by [1, Theorem 2.10],  $\mathbb{P}_{\eta}$  has the  $\omega_2$ -cc and is of size  $\leq 2^{\omega_1} = \omega_2$  for each  $\eta \leq \omega_2$ . Thus, by a standard bookkeeping argument, we can arrange so that, for every  $\mathbb{P}_{\omega_2}$ -name  $\dot{T}$  for a full, splitting subtree of  $\langle \omega_1 \omega_1 \rangle$  of height and size  $\omega_1$ , there is  $\eta < \omega_2$  such that  $\Vdash_{\mathbb{P}_{\omega_2}} \dot{T} = \dot{T}_{\eta}$ . By genericity,

$$\Vdash_{\mathbb{P}_{n+1}}$$
 " $\dot{T}_{\eta}$  contains a copy of  ${}^{<\omega_1}2$ ".

Since  ${}^{<\omega_1}2$  is the same when calculated in V or in  $V^{\mathbb{P}_{\eta}}$  for any  $\eta \leq \omega_2$ , it follows that

$$\Vdash_{\mathbb{P}_{\omega_2}}$$
 "*T* contains a copy of  ${}^{<\omega_1}2$ ".

Hence,  $V^{\mathbb{P}_{\omega_2}}$  is our desired model.

If  $\kappa$  is a regular uncountable cardinal, then we follow [8] and say that a  $\kappa$ -tree T is superthin if  $|[T_{<\delta}]| < \kappa$  for all limit ordinals  $\delta < \kappa$ . Note that cofinally splitting  $\omega_1$ -trees cannot be superthin, so the notion is primarily of interest for  $\kappa > \omega_1$ . In [8], Lücke and Schlicht prove that if  $\kappa$  is a regular uncountable cardinal and there exists a superthin  $\kappa$ -Kurepa tree, then there exists a superthin  $\kappa$ -Kurepa tree  $T \subseteq {}^{<\kappa}\kappa$  such that [T] is a retract of  ${}^{\kappa}\kappa$ . They moreover show that such trees consistently

exist; for example, if V = L, then superthin  $\kappa$ -Kurepa trees exist whenever  $\kappa$  is the successor of a cardinal of uncountable cofinality. Superthinness seems to be in tension with fullness, but we show now that, for regular  $\kappa > \omega_1$ , we can consistently have full, superthin,  $\kappa$ -Kurepa trees. For concreteness, we focus on the case  $\kappa = \omega_2$ .

**Theorem 6.14.** Suppose that  $\diamondsuit + (2^{\omega_1} = \omega_2)$  holds. Then there is a cardinalityand cofinality-preserving poset  $\mathbb{P}$  such that, in  $V^{\mathbb{P}}$ , there is a normal, splitting, full, superthin  $\omega_2$ -Kurepa tree.

*Proof.* The poset  $\mathbb{P}$  consists of all pairs  $p = (T_p, f_p)$  such that

- there is an  $\eta_p < \omega_2$  such that  $T_p$  is a normal, splitting, full subtree of  $<\eta_p+1_{\omega_2}$ :
- for all  $\xi \leq \eta_p$ , we have  $|T_p \cap {}^{\xi}\omega_2| \leq \omega_1$ ;
- for every limit ordinal  $\xi \leq \eta_p$ , we have  $|[(T_p)_{<\xi}]| \leq \omega_1$ ;
- $f_p$  is a partial function of size  $\omega_1$  from  $\omega_3$  to  $T_p \cap {}^{\eta_q}\omega_2$ .

If  $p_0, p_1 \in \mathbb{P}$ , then  $p_1 \leq p_0$  if

- $\eta_{p_1} \ge \eta_{p_1};$   $T_{p_1} \cap {}^{<\eta_{p_0}+1}\omega_2 = T_{p_0};$   $\operatorname{dom}(f_{p_1}) \supseteq \operatorname{dom}(f_{p_0});$  for all  $\alpha \in \operatorname{dom}(f_{p_0})$ , we have  $f_{p_1}(\alpha) \sqsupseteq f_{p_0}(\alpha).$

By a standard  $\Delta$ -system argument, and using the assumption that  $2^{\omega_1} = \omega_2$ , it follows that  $\mathbb{P}$  has the  $\omega_3$ -cc.

Claim 6.15.  $\mathbb{P}$  is  $(\omega_1 + 1)$ -strategically closed.

*Proof.* We describe a winning strategy for Player II in  $\partial_{\omega_1+1}(\mathbb{P})$ . Given a (partial) play  $\langle p_{\alpha} \mid \alpha < \gamma \rangle$  of  $\partial_{\omega_1+1}(\mathbb{P})$ , for all  $\alpha < \gamma$ , we let  $T^{\alpha}$ ,  $f^{\alpha}$ , and  $\eta_{\alpha}$  denote  $T_{p_{\alpha}}$ ,  $f_{p_{\alpha}}$ , and  $\eta_{p_{\alpha}}$ , respectively. As part of Player II's winning strategy, they also fix, as they play a round  $\langle p_{\alpha} \mid \alpha \leq \omega_1 \rangle$ , arbitrary surjections  $\pi_{\alpha} : \omega_1 \to T^{\alpha}_{\eta_{\alpha}}$  for each ordinal  $\alpha < \omega_1$ . We will also ensure that, for every even ordinal  $\alpha < \omega_1$ , the map  $f^{\alpha}$  is a bijection between its domain and  $T^{\alpha}_{\eta_{\alpha}}$ .

Using  $\Diamond$ , fix a sequence  $\langle a_{\alpha} : \alpha \to \alpha \mid \alpha < \omega_1 \rangle$  such that, for all  $f : \omega_1 \to \omega_1$ , the set

$$\{\alpha < \omega_1 \mid f \restriction \alpha = a_\alpha\}$$

is stationary in  $\omega_1$ .

Suppose now that  $\beta \leq \omega_1$  is a nonzero even ordinal and  $\langle p_\alpha \mid \alpha < \beta \rangle$  is a partial run of  $\partial_{\omega_1+1}(\mathbb{P})$ , with Player II playing thus far according to their winning strategy. Suppose also that Player II has fixed surjections  $\pi_{\alpha}: \omega_1 \to T^{\alpha}_{\eta_{\alpha}}$  for each  $\alpha$  such that there exists an even ordinal  $\beta'$  with  $\alpha \leq \beta' < \beta$ . We now describe how Player II should select  $p_{\beta}$ .

Suppose first that  $\beta = \alpha + 1$  is a successor ordinal. In this case, simply let  $\eta_{\beta} = \eta_{\alpha} + 1$  and

$$T^{\beta}_{\eta_{\beta}} = \{ \sigma^{\frown} i \mid \sigma \in T^{\alpha}_{\eta_{\alpha}} \text{ and } i < \omega_1 \}.$$

We now describe how to choose  $f^{\beta}$ . We are going to let  $\operatorname{dom}(f^{\beta}) = \operatorname{dom}(f^{\alpha}) \cup e_{\beta}$ for some (possibly empty)  $e_{\beta} \in [\omega_3]^{\leq \omega_1}$  disjoint from dom $(f^{\alpha})$ . First, for all  $\gamma \in$ dom $(f^{\alpha})$ , let  $f^{\beta}(\gamma) = f^{\alpha}(\gamma)^{\frown} i$  for some  $i < \omega_1$  in such a way that  $f^{\beta} \upharpoonright \text{dom}(f^{\alpha})$  is injective (this is possible, since each element of  $T^{\alpha}_{\eta_{\alpha}}$  has  $\omega_1$ -many successors in  $T^{\beta}_{\eta_{\beta}}$ ). The resulting function may not be surjective, though, so choose some  $e_{\beta}$  as above of the appropriate cardinality and define  $f^{\beta}$  on  $e_{\beta}$  in such a way that the resulting

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function is a bijection. Finally, let  $\pi_{\alpha}$  and  $\pi_{\beta}$  be arbitrary surjections from  $\omega_1$  onto  $T^{\alpha}_{\eta_{\alpha}}$  and  $T^{\beta}_{\eta_{\beta}}$ , respectively.

Suppose next that  $\beta$  is a limit ordinal of countable cofinality. Let  $\eta_{\beta} = \sup\{\eta_{\alpha} \mid \alpha < \beta\}$ . Note that, by the specification of Player II's strategy at successor stages, we know that  $\eta_{\beta} > \eta_{\alpha}$  for all  $\alpha < \beta$ ; in particular,  $\eta_{\beta}$  is a limit ordinal of countable cofinality. Let  $T^{<\beta} = \bigcup\{T^{\alpha} \mid \alpha < \beta\}$ , so  $T^{<\beta}$  is a normal, splitting, full subtree of  ${}^{<\eta_{\beta}}\omega_{2}$ . Let  $d_{\beta} = \bigcup\{\dim(f^{\alpha}) \mid \alpha < \beta\}$  and, for each  $\gamma \in d_{\beta}$ , let

$$b_{\gamma}^{\beta} = \bigcup \{ f^{\alpha}(\gamma) \mid \alpha < \beta \text{ and } \gamma \in \operatorname{dom}(f^{\alpha}) \}.$$

Note that  $b_{\gamma}^{\beta}$  is in  $[T^{<\beta}]$ . Our  $\diamond$  sequence gives us a function  $a_{\beta} : \beta \to \beta$ . Consider the subset  $A^{\beta} = \{\pi_{\alpha}(a_{\beta}(\alpha)) \mid \alpha < \beta\}$  of  $T^{<\beta}$ . If  $A^{\beta}$  is linearly ordered by  $\subseteq$ , then let  $b_{*}^{\beta} = \bigcup A^{\beta}$ , and note that  $b_{*}^{\beta} \in [T^{<\beta}]$ . There are now two cases to consider.

**Case 1.** Suppose first that  $A^{\beta}$  is linearly ordered and, for every  $\gamma \in d_{\beta}$ , we have  $b_*^{\beta} \neq b_{\gamma}^{\beta}$ . In this case, let  $T_{\eta_{\beta}}^{\beta} = [T^{<\beta}] \setminus \{b_*^{\beta}\}$ . Note that, since  $cf(\beta) = \omega$ , CH holds, and every level of  $T^{<\beta}$  has size at most  $\omega_1$ , we have  $|T_{\eta_{\beta}}^{\beta}| = \omega_1$ . To define  $f^{\beta}$ , we will let dom $(f^{\beta}) = d_{\beta} \cup e_{\beta}$  for some  $e_{\beta} \in [\omega_3]^{\leq \omega_1}$  disjoint from  $d_{\beta}$ . For  $\gamma \in d_{\beta}$ , let  $f^{\beta}(\gamma) = b_{\gamma}^{\beta}$ . Note that  $f^{\beta} \upharpoonright d_{\beta}$  is injective since, for all even  $\alpha < \beta$ ,  $f^{\alpha}$  is injective. As in the successor case, now choose a set  $e_{\beta}$  of the appropriate cardinality and define  $f^{\beta}$  on  $e_{\beta}$  so that the resulting function is a bijection. Finally, let  $\pi_{\beta}$  be an arbitrary surjection from  $\omega_1$  onto  $T_{\eta_{\beta}}^{\beta}$ .

**Case 2.** If we were not in Case 1, i.e., if either  $A^{\beta}$  is not linearly ordered or  $b_*^{\beta} = b_{\gamma}^{\beta}$  for some  $\gamma \in d^*$ , then proceed exactly as in Case 1, except let  $T_{\eta_{\beta}}^{\beta} = [T^{<\beta}]$ .

Finally, suppose that  $\beta = \omega_1$ . In this case, we just need to show that  $\langle p_\alpha \mid \alpha < \omega_1 \rangle$  has a lower bound. As above, let  $\eta_{\omega_1} = \sup\{\eta_\alpha \mid \alpha < \omega_1\}$ , let  $T^{<\omega_1} = \bigcup\{T^\alpha \mid \alpha < \omega_1\}$ , and let  $d_{\omega_1} = \bigcup\{\operatorname{dom}(f^\alpha) \mid \alpha < \omega_1\}$ . For each  $\gamma \in d_{\omega_1}$ , let

$$b_{\gamma}^{\omega_1} = \bigcup \{ f^{\alpha}(\gamma) \mid \alpha < \omega_1 \text{ and } \gamma \in \operatorname{dom}(f^{\alpha}) \}.$$

For all  $\gamma \in d_{\omega_1}$ , we have  $b_{\gamma}^{\omega_1} \in [T^{<\omega_1}]$ .

**Subclaim 6.16.** For every  $b \in [T^{<\omega_1}]$ , there is  $\gamma \in d_{\omega_1}$  such that  $b = b_{\gamma}^{\omega_1}$ .

*Proof.* Suppose for the sake of contradiction that  $b \in [T^{<\omega_1}]$  and, for all  $\gamma \in d_{\omega_1}$ , we have  $b \neq b_{\gamma}^{\omega_1}$ . Recall that, for each  $\alpha \in \lim(\omega_1)$ , the function  $f^{\alpha}$  is bijective; therefore, there is a unique  $\gamma_{\alpha} \in \operatorname{dom}(f^{\alpha})$  such that  $f^{\alpha}(\gamma_{\alpha}) \sqsubseteq b$ . By assumption, for each  $\alpha \in \lim(\omega_1)$ , we can find  $\alpha^{\dagger} \in \lim(\omega_1)$  such that  $b \upharpoonright \alpha^{\dagger} \neq b_{\gamma_{\alpha}}^{\omega_1} \upharpoonright \alpha^{\dagger}$ . Let

$$C = \{ \alpha' \in \lim(\omega_1) \mid \forall \alpha \in \lim(\alpha') \ \alpha^{\dagger} < \alpha' \}.$$

Then C is a club in  $\omega_1$  and, for all  $\alpha' \in C$  and all  $\gamma \in d_{\alpha'}$ , we have  $b \upharpoonright \eta_{\alpha'} \neq b_{\gamma'}^{\alpha'}$ .

Let  $f: \omega_1 \to \omega_1$  be such that, for each  $\alpha < \omega_1$ , we have  $\pi_{\alpha}(f(\alpha)) = b \upharpoonright \eta_{\alpha}$ . We can then find  $\alpha' \in C$  such that  $f \upharpoonright \alpha' = a_{\alpha'}$ . Now recall the specification of Player II's strategy at stage  $\alpha'$  of this run of the game. Unraveling the definitions, we have  $b_{\alpha}^{\alpha'} = b \upharpoonright \eta_{\alpha'}$ , and, for every  $\gamma \in d_{\alpha'}$ , we have  $b_{\alpha}^{\alpha'} \neq b_{\gamma}^{\alpha'}$ . Therefore, Player II played  $T_{\eta_{\alpha'}}^{\alpha'} = [T^{<\alpha'}] \setminus \{b_{\alpha}^{\alpha'}\}$ . In particular,  $b \upharpoonright \eta_{\alpha'} \notin T^{<\omega_1}$ , contradicting the assumption that  $b \in [T^{<\omega_1}]$ .

Since  $|d_{\omega_1}| = \omega_1$ , it follows that  $|[T^{<\omega_1}]| = \omega_1$ . Define a subtree  $T^{\omega_1}$  of  ${}^{<\eta_{\omega_1}+1}\omega_2$ by letting  $T^{\omega_1} \cap {}^{<\eta_{\omega_1}}\omega_2 = T^{<\omega_1}$  and  $T^{\omega_1}_{\eta_{\omega_1}} = [T^{<\omega_1}]$ . The normality of  $T^{\omega_1}$  follows from the fact that  $f^{\alpha}$  is surjective onto  $T^{\alpha}_{\eta_{\alpha}}$  for all even ordinals  $\alpha < \omega_1$ , and hence each element of  $T^{<\omega_1}$  is an initial segment of  $b_{\gamma}^{\omega_1}$  for some  $\gamma \in d_{\omega_1}$ . Let  $\operatorname{dom}(f^{\omega_1}) = d_{\omega_1}$  and, for each  $\gamma \in d_{\omega_1}$ , let  $f^{\omega_1}(\gamma) = b_{\gamma}^{\omega_1}$ . Then  $(T^{\omega_1}, f^{\omega_1})$  is a lower bound for  $\langle p_{\alpha} \mid \alpha < \omega_1 \rangle$  in  $\mathbb{P}$ .

Since  $\mathbb{P}$  has the  $\omega_3$ -cc and is  $(\omega_1 + 1)$ -strategically closed, it follows that it preserves all cardinalities and cofinalities. Let G be  $\mathbb{P}$ -generic over V, let  $T = \bigcup\{T_p \mid p \in G\}$ , let  $d = \bigcup\{\operatorname{dom}(f_p) \mid p \in G\}$ , and define a function f with domain d by letting  $f(\gamma) = \bigcup\{f_p(\gamma) \mid p \in G \text{ and } \gamma \in \operatorname{dom}(p)\}$ . Then standard genericity arguments, combined with the arguments of the proof of Claim 6.15, show that

- T is a normal, splitting, full, superthin  $\omega_2$ -tree;
- $d = \omega_3;$
- f is an injective function from  $\omega_3$  to [T].

Thus, T is the desired tree as in the statement of the theorem.

# 7. Adding superthin subtrees

In this section, we prove Theorem D. Fix for the remainder of the section a regular uncountable cardinal  $\mu$  such that  $\mu^{<\mu} = \mu$ ,  $2^{\mu} = \mu^+$ , and  $2^{\mu^+} = \mu^{++}$ , and let  $\kappa = \mu^+$ . Fix for now a closed set  $E \subseteq {}^{<\kappa}\kappa$  such that  $|E| > \kappa$ .

We introduce a forcing notion  $\mathbb{P}(E)$  that will add a superthin  $\kappa$ -Kurepa subtree to T(E). We first thin E out to a subset such that all nonempty neighborhoods are large. Let  $\Sigma = \{\sigma \in {}^{<\kappa}\kappa \mid |E \cap N_{\sigma}| \leq \kappa\}$ , and let

$$E' = E \setminus \bigcup \{ N_{\sigma} \mid \sigma \in \Sigma \}.$$

By replacing E with E', assume from now on that E has the property that, for all  $\sigma \in {}^{<\kappa}\kappa$ , either  $E \cap N_{\sigma} = \emptyset$  or  $|E \cap N_{\sigma}| > \kappa$ .

Conditions of  $\mathbb{P}(E)$  are all triples of the form  $p = (B^p, \gamma^p, t^p)$  such that

- $B^p \in [E]^{\leq \mu}$  is nonempty;
- $\gamma^p < \kappa$  is such that, for all distinct  $b, b' \in B^p$ , we have  $b \upharpoonright \gamma^p \neq b' \upharpoonright \gamma^p$ ;
- $t^p = \{b \upharpoonright \alpha \mid b \in B^p, \ \alpha \le \gamma^p\};$
- $t^p$  looks like an initial segment of a normal superthin subtree of T(E), i.e.,
  - $-t^p$  is a normal subtree of  $\leq \gamma^p \kappa$ ;
    - all levels of  $t^p$  have cardinality at most  $\mu$ ;
    - for all limit ordinals  $\gamma \leq \gamma^p$ , we have  $|[t_{\leq \gamma}^p]| \leq \mu$ .

We note that, for a condition  $p \in \mathbb{P}(E)$ ,  $t^p$  is uniquely determined by  $B^p$  and  $\gamma^p$ ; we include it in the notation for convenience.

Given  $p, q \in \mathbb{P}(E)$ , we set  $q \leq p$  if and only if

- $B^q \supset B^p$ ;
- $\gamma^q \ge \gamma^p;$
- $t^q$  end-extends  $t^p$ , i.e.,  $t^q \cap \leq \gamma^p \kappa = t^p$ .

We also include  $(\emptyset, \emptyset, \emptyset)$  as  $1_{\mathbb{P}(E)}$ .

# **Proposition 7.1.** $\mathbb{P}(E)$ is $\kappa^+$ -Knaster.

*Proof.* Let  $\langle p_{\delta} | \delta < \kappa^{+} \rangle$  be a sequence of conditions from  $\mathbb{P}(E)$ , with each  $p_{\delta}$  of the form  $(B^{\delta}, \gamma^{\delta}, t^{\delta})$ . Since  $\kappa^{<\kappa} = \kappa$ , by thinning out our sequence if necessary we can assume that there are fixed  $\gamma < \kappa$  and  $t \subseteq {}^{\leq \gamma}\kappa$  such that, for all  $\delta < \kappa^{+}$ , we have  $\gamma^{\delta} = \gamma$  and  $t^{\delta} = t$ .

We claim that, for all  $\delta_0 < \delta_1 < \kappa^+$ , the conditions  $p_{\delta_0}$  and  $p_{\delta_1}$  are compatible. To this end, fix such  $\delta_0 < \delta_1$ . For each  $\sigma \in t_{\gamma}$  and each i < 2, there is a unique

 $b_{\sigma,i} \in B^{\delta_i}$  such that  $b_{\sigma,i} \upharpoonright \gamma = \sigma$ . Choose  $\gamma' \in [\gamma, \kappa)$  large enough so that, for all  $\sigma \in t_{\gamma}$ , either  $b_{\sigma,0} = b_{\sigma,i}$  or  $b_{\sigma,0} \upharpoonright \gamma' \neq b_{\sigma,1} \upharpoonright \gamma'$ . Let  $B' = B^{\delta_0} \cup B^{\delta_1}$  and  $t' = \{b \upharpoonright \alpha \mid b \in B', \ \alpha \leq \gamma'\}$ . Then it is readily verified that  $p' = (B', \gamma', t')$  is a common extension of  $p^{\delta_0}$  and  $p^{\delta_1}$  in  $\mathbb{P}(E)$ . For example, to verify that  $|[t_{<\beta}]| \leq \mu$  for all limit ordinals  $\beta \leq \gamma'$ , fix such a  $\beta$ . If  $\beta \leq \gamma$ , then the desired conclusion follows from the fact that  $p^{\delta_0}$  and  $p^{\delta_1}$  are in  $\mathbb{P}(E)$ . If  $\beta \in (\gamma, \gamma']$ , then our construction immediately implies that every element of the form  $[t_{<\beta}]$  is of the form  $b \upharpoonright \beta$  for some  $b \in B'$ .

# **Proposition 7.2.** $\mathbb{P}(E)$ is $\kappa$ -strategically closed.

*Proof.* We describe a winning strategy for Player II in  $\partial_{\kappa}(\mathbb{P}(E))$ . The winning strategy is very simple: at every even stage of the game, Player II does the minimal amount of work necessary. More precisely, suppose that  $\beta < \kappa$  is an even ordinal and  $\langle p_{\alpha} \mid \alpha < \beta \rangle$  is a partial play of the game, with Player II playing so far according to the strategy being described here. For each  $\alpha < \beta$ , let  $p_{\alpha} = (B^{\alpha}, \gamma^{\alpha}, t^{\alpha})$ .

If  $\beta$  is a successor ordinal, then Player II can play arbitrarily; for instance, they can simply play  $p_{\beta} = p_{\beta-1}$ . If  $\beta$  is a limit ordinal, then they specify  $p_{\beta}$  by setting  $B^{\beta} = \bigcup \{B^{\alpha} \mid \alpha < \beta\}$  and  $\gamma^{\beta} = \sup \{\gamma^{\alpha} \mid \alpha < \beta\}$  (this uniquely determines  $t^{\beta}$ ).

In order to prove that this is a winning strategy, we must only verify that  $p_{\beta}$  thus defined is indeed a condition in  $\mathbb{P}(E)$ . Let  $\gamma = \gamma^{\beta}$  and  $t = t^{\beta}$ . The only nontrivial condition to verify is the requirement that  $|[t_{<\gamma}]| \leq \mu$ . If  $cf(\beta) < \mu$ , then this follows immediately from the fact that  $\mu^{<\mu} = \mu$  and all levels of t have cardinality at most  $\mu$ .

Thus, assume that  $\operatorname{cf}(\beta) = \mu$ . We claim that, in this case, every element of  $[t_{<\gamma}]$  is of the form  $b \upharpoonright \gamma$  for some  $b \in B$ . Since  $|B| \leq \mu$ , this will suffice. To this end, fix  $d \in [t_{<\gamma}]$ . Let  $\langle \alpha_i \mid i < \mu \rangle$  be a strictly increasing enumeration of a club in  $\beta$ . By construction, for each  $i < \mu$ , there is a unique  $b_i \in B^{\alpha_i}$  such that  $d \upharpoonright \gamma^{\alpha_i} = b_i \upharpoonright \gamma^{\alpha_i}$ . If  $j < \mu$  is a limit ordinal, then the assumption that Player II has played so far according to the described strategy implies that  $B^{\alpha_j} = \bigcup \{B^{\alpha_i} \mid i < j\}$  and  $\gamma^{\alpha_j} = \sup\{\gamma^{\alpha_i} \mid i < j\}$ . Therefore, for each limit ordinal  $j < \mu$ , we can find i(j) < j such that  $b_j \in B^{\alpha_{i(j)}}$ . By Fodor's Lemma, we can find a stationary  $S \subseteq \mu$  and a fixed  $i < \mu$  such that i(j) = i for all  $j \in S$ . Since distinct elements of  $B^{\alpha_i}$  have distinct restrictions to  $\gamma^{\alpha_i}$ , it follows that there must be a fixed  $b \in B^{\alpha_i}$  such that  $b_j = b$  for all  $j \in S$ . But then  $d \upharpoonright \gamma^{\alpha_j} = b \upharpoonright \gamma^{\alpha_j}$  for all  $j \in S$ , and hence  $d = b \upharpoonright \gamma$ .

By Propositions 7.1 and 7.2, forcing with  $\mathbb{P}(E)$  preserves all cardinalities and cofinalities. We now show that it adds a normal, cofinally splitting, superthin  $\kappa$ -Kurepa subtree of T(E).

**Theorem 7.3.** Suppose that G is  $\mathbb{P}(E)$ -generic over V, and let  $T = \bigcup \{t^p \mid p \in G\}$ . Then in V[G], T is a normal, cofinally splitting, superthin  $\kappa$ -Kurepa subtree of T(E).

*Proof.* In V, let  $\dot{G}$  be the canonical  $\mathbb{P}(E)$ -name for the generic filter, and let  $\dot{T}$  be a  $\mathbb{P}(E)$ -name for  $\bigcup \{t^p \mid p \in \dot{G}\}$ . It is immediate that, for every  $\gamma < \kappa$ , the set  $D_{\gamma} = \{p \in \mathbb{P}(E) \mid \gamma^p \geq \gamma\}$  is dense in  $\mathbb{P}(E)$ . As a result, the definition of  $\mathbb{P}(E)$  implies that  $\dot{T}$  is forced to be a superthin normal subtree of T(E).

We now show that  $\hat{T}$  is forced to be cofinally splitting. To this end, fix a condition  $p \in \mathbb{P}(E)$  and a  $\sigma \in {}^{<\kappa}\kappa$  such that  $p \Vdash \sigma \in \hat{T}$ . Note that there must be  $b \in B^p$ 

such that  $\sigma \sqsubseteq b$ ; otherwise, if  $\gamma \in [\gamma^p, \kappa)$  is such that  $\sigma \in {}^{\leq \gamma}\kappa$ , then the condition q with  $B^q = B^p$  and  $\gamma^q = \gamma$  would extend p and force  $\sigma \notin T$ . By increasing  $\gamma^p$  if necessary, we can assume that  $\sigma \in {}^{\leq \gamma^p}\kappa$ . Let  $\sigma' \in t^p_{\gamma^p}$  be such that  $\sigma \sqsubseteq \sigma'$ , and let b be the unique element of  $B^p$  such that  $\sigma' \sqsubseteq b$ . By our assumption about E, we know that  $|E \cap N_{\sigma'}| > \kappa$ , so we can fix  $b' \in E \cap N_{\sigma'}$  such that  $b' \neq b$ . Define a condition q by letting  $B^q = B^p \cup \{b'\}$  and letting  $\gamma^q \in (\gamma^p, \kappa)$  be large enough so that  $b \upharpoonright \gamma^q \neq b' \upharpoonright \gamma^q$ . Then q extends p and forces that  $b \upharpoonright \gamma^q$  and  $b' \upharpoonright \gamma^q$  are incomparable elements of T extending  $\sigma$ .

We finally show that  $\dot{T}$  is forced to have at least  $\kappa^+$ -many cofinal branches. Let  $\hat{B}$  be a  $\mathbb{P}(E)$ -name for  $\bigcup \{B^p \mid p \in \hat{G}\}$ . It will suffice to show that  $\hat{B}$  is forced to have cardinality greater than  $\kappa$ . Suppose for the sake of contradiction that  $p \in \mathbb{P}(E)$ and  $p \Vdash |B| \leq \kappa$ . Since  $\mathbb{P}(E)$  has the  $\kappa^+$ -c.c., we can find  $q \leq p$  and  $A \in [E]^{\kappa}$  such that  $q \Vdash B \subseteq A$ . Let  $\sigma$  be an arbitrary element of  $t^q_{\gamma^q}$ . Since  $|E \cap N_{\sigma}| > \kappa$ , we can find  $b \in (E \cap N_{\sigma}) \setminus (A \cup B^q)$ . Precisely as in the previous paragraph, we can find  $r \leq q$  such that  $b \in B^r$ , contradicting the fact that  $r \notin A$  and  $q \Vdash B \subseteq A$ . 

We now show that appropriate iterations of forcings of the form  $\mathbb{P}(E)$  are wellbehaved.

**Theorem 7.4.** Suppose that  $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j \mid i \leq \varepsilon, j < \varepsilon \rangle$  is a  $(\leq \mu)$ -support iteration such that, for all  $i < \varepsilon$ , there is a  $\mathbb{P}_i$ -name  $\dot{E}_i$  such that

•  $\dot{E}_i$  is forced to be a nonempty closed subset of  $\kappa \kappa$  such that, for all  $\sigma \in$  $T(\dot{E}_i)$ , we have  $|\dot{E}_i \cap N_{\sigma}| > \kappa$ ;

• 
$$\Vdash_{\mathbb{P}_i} \mathbb{Q}_i = \mathbb{P}(E).$$

Then  $\mathbb{P}_{\varepsilon}$  is  $\kappa^+$ -Knaster and  $\kappa$ -strategically closed.

*Proof.* Since  $\kappa = \mu^+$ , the iteration is taken with supports of size  $\leq \mu$ , and each iterand is forced to be  $\kappa$ -strategically closed, standard arguments imply that  $\mathbb{P}_{\varepsilon}$  is  $\kappa$ -strategically closed (roughly speaking, Player II simply plays according to their winning strategy on each coordinate).

To show that  $\mathbb{P}_{\varepsilon}$  is  $\kappa^+$ -Knaster, we first isolate a well-behaved dense subset of  $\mathbb{P}_{\varepsilon}$ . For concreteness, for all  $j \leq \varepsilon$ , we will think of conditions of  $\mathbb{P}_j$  as being functions whose domains are subsets of j of cardinality  $\leq \mu$ .

**Claim 7.5.** For all  $j \leq \varepsilon$ , let  $\mathbb{P}_j^*$  be the set of  $p \in \mathbb{P}_j$  such that, for all  $i \in \text{dom}(p)$ , there are  $\gamma^{p,i} < \kappa$ ,  $t^{p,i} \subseteq \leq \gamma^{p,i}\kappa$ , a collection  $B^{p,i}$  of  $\mathbb{P}_i$ -names, and a bijection  $\pi^{p,i}: B^{p,i} \to t^{p,i}_{\gamma^{p,i}}$  such that

- $p \upharpoonright i \Vdash_{\mathbb{P}_i} p(i) = (B^{p,i}, \gamma^{p,i}, t^{p,i});$  for all  $\dot{b} \in B^{p,i}$ ,  $p \upharpoonright i \Vdash_{\mathbb{P}_i} \dot{b} \upharpoonright \gamma^{p,i} = \pi^{p,i}(\dot{b}).$

Then  $\mathbb{P}_{i}^{*}$  is dense in  $\mathbb{P}_{j}$ .

Before we prove Claim 7.5, we establish the following useful fact.

**Claim 7.6.** Fix  $j \leq \varepsilon$ , let  $\beta < \kappa$  be a limit ordinal, and suppose that  $\vec{p} = \langle p_{\alpha} \mid \alpha < \varepsilon \rangle$  $|\beta\rangle$  is a decreasing sequence of conditions from  $\mathbb{P}_j$  such that

- for all  $\alpha < \beta$ , we have  $p_{\alpha} \in \mathbb{P}_{j}^{*}$ , as witnessed by  $\langle (B^{p_{\alpha},i}, \gamma^{p_{\alpha},i}, t^{p_{\alpha},i}, \pi^{p_{\alpha},i}) \mid$  $i \in \operatorname{dom}(p_{\alpha})$ ;
- for all  $\alpha < \alpha' < \beta$  and all  $i \in \operatorname{dom}(p_{\alpha})$ , we have  $B^{p_{\alpha},i} \subseteq B^{p_{\alpha'},i}$ ;
- for all limit ordinals  $\alpha' < \beta$ , we have
  - $\operatorname{dom}(p_{\alpha'}) = \bigcup \{ \operatorname{dom}(p_{\alpha}) \mid \alpha < \alpha' \};$

- for all  $i \in \operatorname{dom}(p_{\alpha'})$ , we have  $\gamma^{p_{\alpha'},i} = \sup\{\gamma^{p_{\alpha},i} \mid \alpha < \alpha' \text{ and } i \in \operatorname{dom}(p_{\alpha})\}$  and  $B^{p_{\alpha'},i} = \bigcup\{B^{p_{\alpha},i} \mid \alpha < \alpha' \text{ and } i \in \operatorname{dom}(p_{\alpha})\}.$ 

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Then  $\vec{p}$  has a lower bound  $p_{\beta}$  that is in  $\mathbb{P}_{j}^{*}$ , as witnessed by  $\langle (B^{p_{\beta},i}, \gamma^{p_{\beta},i}, t^{p_{\beta},i}, \pi^{p_{\beta},i}) | i \in \text{dom}(p_{\beta}) \rangle$  satisfying:

- dom $(p_{\beta}) = \bigcup \{ \operatorname{dom}(p_{\alpha}) \mid \alpha < \beta \};$
- for all  $i \in \operatorname{dom}(p_{\beta})$ , we have  $\gamma^{p_{\beta},i} = \sup\{\gamma^{p_{\alpha},i} \mid \alpha < \beta \text{ and } i \in \operatorname{dom}(p_{\alpha})\}$ and  $B^{p_{\beta},i} = \bigcup\{B^{p_{\alpha},i} \mid \alpha < \beta \text{ and } i \in \operatorname{dom}(p_{\alpha})\}.$

*Proof.* We define  $p_{\beta}$  as follows. First, let  $\operatorname{dom}(p_{\beta}) = \bigcup \{ \operatorname{dom}(p_{\alpha}) \mid \alpha < \beta \}$ . Then, for each  $i \in \operatorname{dom}(p_{\beta})$ , we let  $p_{\beta}(i)$  be a  $\mathbb{P}_i$ -name for the triple  $(B^{p_{\beta},i}, \gamma^{p_{\beta},i}, t^{p_{\beta},i})$ , defined as follows. First, let  $B^{p_{\beta},i} = \bigcup \{ B^{p_{\alpha},i} \mid \alpha < \beta \text{ and } i \in B^{p_{\alpha}} \}$  and let  $\gamma^{p_{\beta},i} = \sup \{ \gamma^{p_{\alpha},i} \mid \alpha < \beta \text{ and } i \in \operatorname{dom}(p_{\alpha}) \}$ .

Let  $s^{p_{\beta},i} = \bigcup \{ t^{p_{\alpha},i} \mid \alpha < \beta \text{ and } i \in \operatorname{dom}(p_{\alpha}) \}$  and, for each  $\dot{b} \in B^{p_{\beta},i}$ , let

$$d_{\dot{b}} = \bigcup \{ \pi^{p_{\alpha}, i}(\dot{b}) \mid \alpha < \beta, \ i \in \operatorname{dom}(p_{\alpha}), \text{ and } \dot{b} \in B^{p_{\alpha}, i} \}.$$

Then  $d_{\dot{b}}$  is a cofinal branch through  $s^{p_{\beta},i}$  and, if  $\dot{b}$  and  $\dot{b}'$  are distinct elements of  $B^{p_{\beta},i}$ , then they both appear in  $B^{p_{\alpha},i}$  for some  $\alpha < \beta$ , and the fact that  $\pi^{p_{\alpha},i}$  is injective implies that  $d_{\dot{b}}$  and  $d_{\dot{b}'}$  are distinct. We end by setting  $t^{p_{\beta},i} = s^{p_{\beta},i} \cup \{d_{\dot{b}} \mid \dot{b} \in B^{p_{\beta},i}\}$  and, for each  $\dot{b} \in B^{p_{\beta},i}$ , setting  $\pi^{p_{\beta},i}(\dot{b}) = d_{\dot{b}}$ .

We claim that  $p_{\beta}$  thus defined is as desired. The only nontrivial requirement to verify is that, for all  $i \in \text{dom}(p_{\beta})$ , the tree  $t^{p_{\beta},i}$  looks like an initial segment of a normal superthin subtree of  $\langle \kappa \kappa$ . This amounts to verifying that  $|[s^{p_{\beta},i}]| \leq \mu$ . This is proven exactly as in the proof of Proposition 7.2, so we leave it to the reader.  $\Box$ 

*Proof of Claim 7.5.* The proof is by induction on j. We will actually establish the following technical strengthening of the density of  $\mathbb{P}_{i}^{*}$ :

Suppose that  $p_{00} \in \mathbb{P}_j^*$ , as witnessed by  $\langle (B^{p_{00},i}, \gamma^{p_{00},i}, t^{p_{00},i}, \pi^{p_{00},i}) | i \in \operatorname{dom}(p_{00}) \rangle$ , and  $p_0 \leq_{\mathbb{P}_j} p_{00}$ . Then there is  $q \leq p_0$  in  $\mathbb{P}_j^*$  witnessed by  $\langle (B^{q,i}, \gamma^{q,i}, t^{q,i}, \pi^{q,i}) | i \in \operatorname{dom}(q) \rangle$  such that, for all  $i \in \operatorname{dom}(p_{00})$ , we have  $B^{p_{00},i} \subseteq B^{q,i}$ .

To see that this does indeed yield the density of  $\mathbb{P}_{j}^{*}$ , note that in the above statement we can let  $p_{0} \in \mathbb{P}_{j}$  be arbitrary and take  $p_{00}$  to be such that  $\operatorname{dom}(p_{00}) = \emptyset$ . Then  $p_{0} \leq p_{00}$ , and  $p_{00}$  is trivially in  $\mathbb{P}_{j}^{*}$ .

Fix  $j \leq \varepsilon$ , and suppose that the inductive hypothesis has been established for all i < j. Fix  $p_0 \leq p_{00}$  as in the statement of the hypothesis; we will find  $q \leq p_0$  as desired.

If j = 0, then we can take  $q = p_0$ . Suppose next that j is a successor ordinal, say  $j = j_0 + 1$ . Assume that  $j_0 \in \text{dom}(p_0)$ ; otherwise,  $p \in \mathbb{P}_{j_0}^*$ , and we can apply the inductive hypothesis to obtain the desired conclusion. Suppose also that  $j_0 \in \text{dom}(p_{00})$ ; the argument is similar but easier in the other case.

Since  $\mathbb{P}_{j_0}$  is  $\kappa$ -strategically closed and hence  $(\langle \kappa \rangle)$ -distributive, we can find  $p_1 \leq p_0 \upharpoonright j_0$  in  $\mathbb{P}_{j_0}$ , an ordinal  $\gamma^{j_0} < \kappa$ , a tree  $t^{j_0} \subseteq \leq \gamma^{j_0} \kappa$ , a set  $B^{j_0}$  of  $\mathbb{P}_{j_0}$ -names, and a bijection  $\pi^{j_0} : B^{j_0} \to t^{j_0}_{\gamma^{j_0}}$  such that

- $p_1 \Vdash_{\mathbb{P}_{j_0}} p_0(j_0) = (B^{j_0}, \gamma^{j_0}, t^{j_0});$
- for all  $\dot{b} \in B^{j_0}$ ,  $p_1 \Vdash \dot{b} \upharpoonright \gamma^{j_0} = \pi^{j_0}(\dot{b})$ .

Since  $p_0 \upharpoonright j_0 \Vdash p_0(j_0) \leq p_{00}$ , by extending  $p_1$  further if necessary we can assume that, for each  $\dot{b} \in B^{p_{00},j_0}$ , there is a unique  $\dot{b}' \in B^{j_0}$  such that

$$p_1 \Vdash_{\mathbb{P}_{j_0}} `` \Vdash_{\dot{\mathbb{Q}}_{j_0}} \dot{b}' = \dot{b}"$$

By replacing each  $\dot{b}'$  with  $\dot{b}$  in  $B^{j_0}$  and redefining  $\pi^{j_0}$  in the obvious way, we may thus assume that  $B^{j_0} \supseteq B^{p_{00},j_0}$ .

Now apply the inductive hypothesis to find  $p_2 \leq_{\mathbb{P}_{j_0}} p_1$  such that  $p_2 \in \mathbb{P}_{j_0}^*$  witnessed by  $\langle (B^{p_2,i}, \gamma^{p_2,i}, t^{p_2,i}, \pi^{p_2,i}) \mid i \in \operatorname{dom}(p_2) \rangle$  such that, for all  $i \in \operatorname{dom}(p_{00}) \cap j_0$ , we have  $B^{p_{00},i} \subseteq B^{p_2,i}$ . Finally, define  $q \in \mathbb{P}_j$  by setting  $\operatorname{dom}(q) = \operatorname{dom}(p_2) \cup \{j_0\}$ ,  $q \upharpoonright j_0 = p_2$ , and  $q(j_0) = (B^{j_0}, \gamma^{j_0}, t^{j_0})$ . Then q is as desired.

Suppose next that j > 0 is a limit ordinal. If  $cf(j) > \mu$ , then there is i < j such that  $p_0 \in \mathbb{P}_i$ , so we can apply the inductive hypothesis to obtain the desired conclusion. Thus, assume that  $cf(j) \leq \mu$ . Let  $\langle i_\eta \mid \eta < cf(j) \rangle$  be an increasing enumeration of a club in j. Using the inductive hypothesis and Claim 7.6, recursively construct a sequence  $\langle q_\eta \mid \eta < cf(j) \rangle$  such that, for all  $\eta < cf(j)$ , we have:

- $q_{\eta} \in \mathbb{P}^*_{i_{\eta}}$ , as witnessed by  $\langle (B^{q_{\eta},i}, \gamma^{q_{\eta},i}, t^{q_{\eta},i}, \pi^{q_{\eta},i}) \mid i \in \operatorname{dom}(q_{\eta}) \rangle$ ;
- $q_{\eta} \leq_{\mathbb{P}_{i_{\eta}}} p_0 \upharpoonright i_{\eta};$
- for all  $i \in \operatorname{dom}(p_{00}) \cap i_{\eta}, B^{p_{00},i} \subseteq B^{q_{\eta},i}$ ;
- for all  $\xi < \eta$ , we have
  - $q_{\eta} \upharpoonright i_{\xi} \leq_{\mathbb{P}_{i_{\xi}}} q_{\xi};$
  - for all  $i \in \text{dom}(q_{\mathcal{E}}), B^{q_{\mathcal{E}},i} \subseteq B^{q_{\eta},i}$ ;
- if  $\eta$  is a limit ordinal, then
  - $\operatorname{dom}(q_{\eta}) = \bigcup \{ \operatorname{dom}(q_{\xi}) \mid \xi < \eta \};$
  - for all  $i \in \operatorname{dom}(q_{\eta})$ , we have  $\gamma^{q_{\eta},i} = \sup\{\gamma^{q_{\xi},i} \mid \xi < \eta \text{ and } i \in \operatorname{dom}(q_{\xi})\}$ and  $B^{q_{\eta},i} = \bigcup\{B^{q_{\xi},i} \mid \xi < \eta \text{ and } i \in \operatorname{dom}(q_{\xi})\}.$

The construction is straightforward, so we leave it to the reader. At the end of the construction, another appeal to Claim 7.6 yields a condition  $q \in \mathbb{P}_{j}^{*}$ , as witnessed by  $\langle (B^{q,i}, \gamma^{q,i}, t^{q,i}, \pi^{q,i}) | i \in \operatorname{dom}(q) \rangle$  such that

- q is a lower bound of  $\langle q_{\eta} | \eta < cf(j) \rangle$ ;
- dom $(q) = \bigcup \{ \operatorname{dom}(q_\eta) \mid \eta < \operatorname{cf}(j) \};$
- for all  $i \in \text{dom}(q)$ , we have  $B^{q,i} = \bigcup \{ B^{q_{\eta},i} \mid \eta < \text{cf}(j) \text{ and } i \in \text{dom}(q_{\eta}) \}.$

Then q is as desired, completing the proof of the claim.

The following will be the key claim in establishing that  $\mathbb{P}_{\varepsilon}$  is  $\kappa^+$ -Knaster.

**Claim 7.7.** Suppose that  $j \leq \varepsilon$  and, for  $\ell < 2$ ,  $p_{\ell} \in \mathbb{P}_{j}^{*}$ , as witnessed by

$$\langle (B^{p_{\ell},i}, \gamma^{p_{\ell},i}, t^{p_{\ell},i}, \pi^{p_{\ell},i}) \mid i \in \operatorname{dom}(p_{\ell}) \rangle.$$

Suppose moreover that, for all  $i \in \text{dom}(p_0) \cap \text{dom}(p_1)$ , we have  $t^{p_0,i} = t^{p_1,i}$ . Then  $p_0 \parallel p_1$ .

*Proof.* The proof is by induction on j. If j = 0, the conclusion is trivial. Suppose next that j is a successor ordinal, say  $j = j_0 + 1$ . Assume that  $j_0 \in \operatorname{dom}(p_0) \cap \operatorname{dom}(p_1)$ , as otherwise we can simply apply the induction hypothesis. First, apply the induction hypothesis to find  $q_0$  such that  $q_0 \leq_{\mathbb{P}_{j_0}} p_\ell \upharpoonright j_0$  for  $\ell < 2$ . Next, let  $\gamma = \gamma^{p_0, j_0} = \gamma^{p_1, j_0}$  and  $t = t^{p_0, j_0} = t^{p_1, j_0}$ . For every  $u \in t_{\gamma}$  and  $\ell < 2$ , there is a unique  $\dot{b}_{u,\ell} \in B^{p_\ell, j_0}$  such that  $q_0 \Vdash_{\mathbb{P}_{j_0}} \dot{b}_{u,\ell} \upharpoonright \gamma = u$ . Since  $\mathbb{P}_{j_0}$  is  $(<\kappa)$ -distributive, we can find  $q_1 \leq_{\mathbb{P}_{j_0}} q_0$  such that, for all  $u \in t_{\gamma}$ ,

•  $q_1$  decides the statement " $\dot{b}_{u,0} = \dot{b}_{u,1}$ ";

• if  $q_1 \Vdash \dot{b}_{u,0} \neq \dot{b}_{u,1}$ , then there is  $\gamma_u < \kappa$  such that  $q_1 \Vdash \dot{b}_{u,0} \upharpoonright \gamma_u \neq \dot{b}_{u,1} \upharpoonright \gamma_u$ . Let  $s_0 = \{u \in t_\gamma \mid q_1 \Vdash \dot{b}_{u,0} \neq \dot{b}_{u,1}\}$  and  $s_1 = t_\gamma \setminus s_0$ . Let  $\gamma^* = \sup\{\gamma_u \mid u \in s_0\}$  (or  $\gamma^* = \gamma$  if  $s_0 = \emptyset$ ). Again using the  $(<\kappa)$ -distributivity of  $\mathbb{P}_{j_0}$ , find  $q_2 \leq_{\mathbb{P}_{j_0}} q_1$  such that, for all  $u \in t_\gamma$  and  $\ell < 2$ ,  $q_2$  decides the value of  $\dot{b}_{u,\ell} \upharpoonright \gamma^*$ , say as  $b^*_{u,\ell} \in \gamma^* \kappa$ . Let  $t^*$  be the downward closure of the set  $\{b^*_{u,\ell} \mid u \in t_\gamma, \ell < 2\}$  in  ${}^{<\kappa}\kappa$ , and let

$$B^* = \{ \dot{b}_{u,0} \mid u \in t_{\gamma} \} \cup \{ \dot{b}_{u,1} \mid u \in s_0 \}.$$

Define a condition  $q \in \mathbb{P}_j$  by letting dom $(q) = \text{dom}(q_2) \cup \{j_0\}, q \upharpoonright j_0 = q_2$ , and  $q(j_0)$  be such that  $q_2 \Vdash q(j_0) = (B^*, \gamma^*, t^*)$ .

Finally, suppose that j is a limit ordinal. If  $cf(j) > \mu$ , then there is  $j_0 < j$  such that  $p_0, p_1 \in \mathbb{P}_{j_0}$ , so we can simply apply the induction hypothesis. Thus, assume that  $cf(j) \leq \mu$ . Let  $\langle i_\eta \mid \eta < cf(j) \rangle$  be an increasing enumeration of a club in j. Recursively build a sequence  $\langle q_\eta \mid \eta < cf(j) \rangle$  such that, for all  $\eta < cf(j)$ , we have:

- $q_{\eta} \in \mathbb{P}_{i_{\eta}}^{*}$ , as witnessed by  $\langle (B^{q_{\eta},i}, \gamma^{q_{\eta},i}, t^{q_{\eta},i}, \pi^{q_{\eta},i}) \mid i \in \operatorname{dom}(q_{\eta}) \rangle;$
- $q_{\eta} \leq_{\mathbb{P}_{i_{\eta}}} p_0 \upharpoonright i_{\eta}, p_1 \upharpoonright i_{\eta};$
- for all  $\xi < \eta$ , we have
  - $-q_{\eta} \restriction i_{\xi} \leq_{\mathbb{P}_{i_{\xi}}} q_{\xi};$
  - for all  $i \in \operatorname{dom}(q_{\xi}), B^{q_{\xi},i} \subseteq B^{q_{\eta},i};$
- if  $\eta$  is a limit ordinal, then
  - $-\operatorname{dom}(q_{\eta}) = \bigcup \{\operatorname{dom}(q_{\xi}) \mid \xi < \eta \};$
  - for all  $i \in \operatorname{dom}(q_{\eta})$ , we have  $\gamma^{q_{\eta},i} = \sup\{\gamma^{q_{\xi},i} \mid \xi < \eta \text{ and } i \in \operatorname{dom}(q_{\xi})\}$ and  $B^{q_{\eta},i} = \bigcup\{B^{q_{\xi},i} \mid \xi < \eta \text{ and } i \in \operatorname{dom}(q_{\xi})\}.$

The construction is straightforward using the induction hypothesis and Claim 7.6, so we leave it to the reader. At the end, another appeal to Claim 7.6 yields  $q \in \mathbb{P}_j$  that is a lower bound for  $\langle q_\eta \mid \eta < \operatorname{cf}(j) \rangle$ . This condition q extends both  $p_0$  and  $p_1$ , thus establishing the claim.

We are now finally ready to prove that  $\mathbb{P}_{\varepsilon}$  is  $\kappa^+$ -Knaster. To this end, suppose that  $\langle p_{\eta} \mid \eta < \kappa^+ \rangle$  is a sequence of conditions in  $\mathbb{P}_{\varepsilon}$ . By extending the conditions if necessary, we may assume that each  $p_{\eta}$  is in  $\mathbb{P}_{\varepsilon}^*$ , as witnessed by  $\langle (B^{\eta,i}, \gamma^{\eta,i}, t^{\eta,i}, \pi^{\eta,i}) \mid i \in \operatorname{dom}(p_{\eta}) \rangle$ . Since  $2^{\mu} = \kappa$ , we can find an unbounded  $A \subseteq \kappa^+$  such that

- $\langle \operatorname{dom}(p_n) \mid \eta \in A \rangle$  forms a  $\Delta$ -system, with root r;
- there is a sequence  $\langle t^i \mid i \in r \rangle$  such that, for all  $\eta \in A$  and all  $i \in r$ , we have  $t^{\eta,i} = t^i$ .

Then, by Claim 7.7,  $\langle p_{\eta} \mid \eta \in A \rangle$  is a sequence of pairwise compatible conditions.

We can now easily prove the main consistency result of this section. To make its statement self-contained, we recall the standing cardinal assumptions of this section at the start of the theorem.

**Theorem 7.8.** Suppose that  $\mu$  is a regular uncountable cardinal such that  $\mu^{<\mu} = \mu$ ,  $2^{\mu} = \mu^+$ , and  $2^{\mu^+} = \mu^{++}$ , and let  $\kappa = \mu^+$ . Then there is a forcing extension in which all cardinalities and cofinalities are preserved and in which

- (1) there exists a  $\kappa$ -Kurepa tree;
- (2) every  $\kappa$ -Kurepa tree contains a normal superthin  $\kappa$ -Kurepa subtree;

(3) the cardinal arithmetic of the ground model is preserved; in particular, if GCH holds in the ground model, then it continues to hold in the extension.

*Proof.* We can assume that there is a  $\kappa$ -Kurepa tree in V; if not, then first force with one of the standard forcings to add a  $\kappa$ -Kurepa tree with  $\kappa^+$ -many cofinal branches (e.g., the analogue of the forcing  $\mathbb{Q}$  from the proof of Theorem 5.1).

We will force with a  $(\leq \mu)$ -support iteration  $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j \mid i \leq \kappa^+, j < \kappa^+ \rangle$  such that, for all  $i < \kappa^+, \dot{\mathbb{Q}}_i$  is forced by  $\mathbb{P}_i$  to be either trivial forcing or of the form  $\mathbb{P}([\dot{T}_i])$ , where  $\dot{T}_i$  is a  $\mathbb{P}_i$ -name for a  $\kappa$ -Kurepa tree such that

$$\Vdash_{\mathbb{P}_i} \forall \sigma \in \dot{T}_i \ |[\dot{T}_i] \cap N_{\sigma}| = \kappa^+.$$

By Theorem 7.4,  $\mathbb{P} = \mathbb{P}_{\kappa^+}$  will be  $\kappa^+$ -Knaster and  $\kappa$ -strategically closed. It will therefore preserve all cardinalities and cofinalities, and hence will preserve the  $\kappa$ -Kurepa tree that exists in the ground model. Moreover, since the length of the iteration is  $\kappa^+$ , it will not change the value of  $2^{\nu}$  for any cardinal  $\nu$ . Moreover, for every  $i < \kappa^+$ , the quotient forcing  $\mathbb{P}/\mathbb{P}_i$  is equivalent to an iteration of the same shape as  $\mathbb{P}$  and therefore shares the same properties.

We will recursively specify  $\mathbb{P}$  by concurrently specifying a sequence  $\langle T_i \mid i < \kappa^+ \rangle$  such that

- for all  $i < \kappa^+$ ,  $\dot{T}_i$  is a  $\mathbb{P}_i$ -name for a subtree of  ${}^{<\kappa}\kappa$ ;
- it is forced by  $\mathbb{P}_i$  that:
  - if  $\dot{T}_i$  is not a  $\kappa$ -Kurepa tree, then  $\dot{\mathbb{Q}}_i$  is trivial forcing;
  - if  $\dot{T}_i$  is a  $\kappa$ -Kurepa tree, then  $\dot{Q}_i$  is  $\mathbb{P}([\dot{T}_i^*])$ , where  $\dot{T}_i^*$  is the  $\mathbb{P}_i$ -name for the set of  $\sigma \in \dot{T}_i$  that are contained in  $\kappa^+$ -many distinct cofinal branches through  $\dot{T}_i$  (note that, in this case,  $\dot{T}_i^*$  is forced to be a  $\kappa$ -Kurepa tree itself).

By the cardinal arithmetic assumptions and the chain condition of  $\mathbb{P}$ , there will only be  $\kappa^+$ -many nice  $\mathbb{P}$ -names for subtrees of  $\langle \kappa \kappa$ . Moreover, also by the arithmetic and chain condition, each such name  $\dot{T}$  will in fact be a  $\mathbb{P}_i$ -name for some  $i < \kappa^+$ . Therefore, by a standard bookkeeping argument, we can construct the sequence  $\langle \dot{T}_i \mid i < \kappa^+ \rangle$  in such a way that, in  $V^{\mathbb{P}}$ , for every subtree T of  $\langle \kappa \kappa$ , there is  $i < \kappa^+$ such that T equals the interpretation of  $\dot{T}_i$  in  $V^{\mathbb{P}}$ .

Let G be  $\mathbb{P}$ -generic over V. For all  $i < \kappa^+$ , let  $G_i$  be the  $\mathbb{P}_i$ -generic filter induced by G. We claim that V[G] is the desired forcing extension. We have already argued that requirements (1) and (3) in the statement of the theorem hold. It remains to verify (2). To this end, let  $T \in V[G]$  be a  $\kappa$ -Kurepa tree. We may assume that it is a subtree of  ${}^{<\kappa}\kappa$ . By our construction of  $\mathbb{P}$ , we can find  $i < \kappa$  such that T equals the interpretation of  $\dot{T}_i$ . In particular,  $T \in V[G_i]$ . Moreover, since the quotient forcing  $\mathbb{P}/\mathbb{P}_i$  is  $\kappa$ -strategically closed, Lemma 2.11 implies that all of the cofinal branches through T in V[G] are already in  $V[G_i]$ . In particular, T is a  $\kappa$ -Kurepa tree in  $V[G_i]$ . Therefore,  $\mathbb{Q}_i$  is  $\mathbb{P}([T^*])$ , where  $T^*$  is the set of all  $\sigma \in T$  such that  $\kappa^+$ -many cofinal branches through T contain  $\sigma$ . Forcing with  $\mathbb{Q}_i$  therefore adds a normal superthin  $\kappa$ -Kurepa subtree to  $T^*$ , and hence to T. Since T was arbitrary, this completes the proof of the theorem.  $\Box$ 

#### 8. Open questions

In this final section, we present some closing remarks and survey some of the remaining open questions. We first make the following observations about Theorem 3.8.

- In Theorem 3.8, the assumption that E is a continuous image of  ${}^{\kappa}\kappa$  is slightly more than is needed. As the proof makes clear, it is enough to assume that E is the continuous image of [T] for some normal, splitting,  $(\langle \omega_1 \rangle)$ -closed tree  $T \subseteq {}^{\langle \kappa}\kappa$ . Note that, for such a tree T, the set [T] is necessarily  $(\omega + 1)$ -perfect.
- The conclusion of Theorem 3.8 falls slightly short of the assertion that  $E \setminus X$  is  $(\omega + 1)$ -perfect, since it only gives a winning strategy for games  $G_{\kappa}(E, x_0, \omega + 1)$  rather than  $G_{\kappa}(E \setminus X, x_0, \omega + 1)$ .

In light of these observations, the following questions are natural.

**Question 8.1.** Suppose that  $\kappa$  is a regular uncountable cardinal and  $E \subseteq {}^{\kappa}\kappa$  is a closed set.

- (1) Suppose that E is a continuous image of  $\kappa \kappa$ . Must there be  $X \subseteq E$  such that  $|X| \leq \kappa^{<\kappa}$  and  $E \setminus X$  is  $(\omega + 1)$ -perfect?
- (2) Suppose that there is an  $(\omega + 1)$ -perfect set  $D \subseteq {}^{\kappa}\kappa$  such that E is a continuous image of D. Must there be  $X \subseteq E$  such that  $|X| \leq {}^{\kappa < \kappa}$  and, for every  $x_0 \in E \setminus X$ , Player II has a winning strategy in  $G_{\kappa}(E, x_0, \omega + 1)$ ? Must there be  $X \subseteq E$  such that  $|X| \leq {}^{\kappa < \kappa}$  and  $E \setminus X$  is  $(\omega + 1)$ -perfect?

We now turn to some questions arising from Theorem 5.1. The first concerns whether an inaccessible cardinal is necessary to obtain the conclusion of the theorem.

**Question 8.2.** Suppose that GCH holds, there is an  $\omega_2$ -Kurepa tree and, for every  $\omega_2$ -Kurepa tree  $S \subseteq {}^{<\omega_2}\omega_2$ , [S] is not a continuous image of  ${}^{\omega_2}\omega_2$ . Must  $(\omega_3)^V$  be inaccessible in L?

We also ask whether the opposite extreme to that obtained in Theorem 5.1 can hold.

**Question 8.3.** Is it consistent with GCH that there are  $\omega_2$ -Kurepa trees and, for every  $\omega_2$ -Kurepa tree  $T \subseteq {}^{<\omega_2}\omega_2$ , [T] is a continuous image (or even a retract) of  ${}^{\omega_2}\omega_2$ ?

In all of the known consistent examples of  $\omega_2$ -Kurepa trees  $T \subseteq {}^{<\omega_2}\omega_2$ , there exist  $x \in [T]$  and  $\sigma \in T$  such that  $\sigma \sqsubseteq x$  and  $|[T] \cap N_{\sigma}| \le \omega_2$ . We conjecture that this is in necessarily the case.

**Conjecture 8.4.** Suppose that GCH holds, and let  $T \subseteq {}^{<\omega_2}\omega_2$  be an  $\omega_2$ -Kurepa tree such that, for every  $\sigma \in T$ , we have  $|[T] \cap N_{\sigma}| > \omega_2$ . Then [T] is not a continuous image of  ${}^{\omega_2}\omega_2$ 

Note that, if GCH holds and  $T \subseteq {}^{<\omega_2}\omega_2$  is an  $\omega_2$ -Kurepa tree, then  $T' := \{\sigma \in T \mid |[T] \cap N_{\sigma}| > \omega_2\}$  is an  $\omega_2$ -Kurepa tree satisfying the hypothesis of Conjecture 8.4. Therefore, a positive resolution of Conjecture 8.4 would entail a negative answer to Question 8.3.

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(Lambie-Hanson) INSTITUTE OF MATHEMATICS, CZECH ACADEMY OF SCIENCES, ŽITNÁ 25, PRAGUE 1, 115 67, CZECH REPUBLIC

*Email address*: lambiehanson@math.cas.cz *URL*: https://users.math.cas.cz/~lambiehanson/

(Stejskalová) Charles University, Department of Logic, Celetná 20, Prague 1, 11642, Czech Republic

Email address: sarka.stejskalova@ff.cuni.cz URL: logika.ff.cuni.cz/sarka

INSTITUTE OF MATHEMATICS, CZECH ACADEMY OF SCIENCES, ŽITNÁ 25, PRAGUE 1, 115 67, CZECH REPUBLIC