FORCING A $\square(\kappa)$-LIKE PRINCIPLE TO HOLD AT A WEAKLY COMPACT CARDINAL

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ABSTRACT. Hellsten [Hel03a] proved that when κ is Π^1_n-indescribable, the n-*club* subsets of κ provide a filter base for the Π^1_n-indescribability ideal, and hence can also be used to give a characterization of Π^1_n-indescribable sets which resembles the definition of stationarity: a set $S \subseteq \kappa$ is Π^1_n-indescribable if and only if $S \cap C \neq \varnothing$ for every n-club $C \subseteq \kappa$. By replacing clubs with n-clubs in the definition of $\square(\kappa)$, one obtains a $\square(\kappa)$-like principle $\square_n(\kappa)$, a version of which was first considered by Brickhill and Welch [BW]. The principle $\square_n(\kappa)$ is consistent with the Π^1_n-indescribability of κ but inconsistent with the Π^1_{n+1}-indescribability of κ. By generalizing the standard forcing to add a $\square(\kappa)$-sequence, we show that if κ is κ^+-weakly compact and GCH holds then there is a cofinality-preserving forcing extension in which κ remains κ^+-weakly compact and $\square_1(\kappa)$ holds. If κ is Π^1_2-indescribable and GCH holds then there is a cofinality-preserving forcing extension in which κ is κ^+-weakly compact, $\square_1(\kappa)$ holds and every weakly compact subset of κ has a weakly compact proper initial segment. As an application, we prove that, relative to a Π^1_2-indescribable cardinal, it is consistent that κ is κ^+-weakly compact, every weakly compact subset of κ has a weakly compact proper initial segment, and there exist two weakly compact subsets S^0 and S^1 of κ such that there is no $\beta < \kappa$ for which both $S^0 \cap \beta$ and $S^1 \cap \beta$ are weakly compact.

1. INTRODUCTION

In this paper, we investigate an incompactness principle $\square_1(\kappa)$, which is closely related to $\square(\kappa)$ but is consistent with weak compactness. Let us begin by recalling the basic facts about $\square(\kappa)$.

The principle $\square(\kappa)$ asserts that there is a κ-length coherent sequence of clubs $\vec{C} = \langle C_\alpha : \alpha \in \lim(\kappa)\rangle$ that cannot be threaded. For an uncountable cardinal κ, a sequence $\vec{C} = \langle C_\alpha : \alpha \in \lim(\kappa)\rangle$ of clubs $C_\alpha \subseteq \alpha$ is called *coherent* if whenever β is a limit point of C_α we have $C_\beta = C_\alpha \cap \beta$. Given a coherent sequence $\vec{C}$, we say that C is a *thread* through $\vec{C}$ if C is a club subset of κ and $C \cap \alpha = C_\alpha$ for every limit point α of C. A coherent sequence $\vec{C}$ is called a $\square(\kappa)$-sequence if it cannot be threaded, and $\square(\kappa)$ holds if there is a $\square(\kappa)$-sequence. It is easy to see that $\square(\kappa)$ implies that κ is not weakly compact, and thus $\square(\kappa)$ can be viewed as asserting that κ exhibits a certain amount of incompactness. The principle $\square(\kappa)$ was isolated by Todorčević [Tod87], building on work of Jensen [Jen72], who showed that, if $V = L$, then $\square(\kappa)$ holds for every regular uncountable κ that is not weakly compact.

The natural $\leq\kappa$-strategically closed forcing to add a $\square(\kappa)$-sequence [LH14, Lemma 35] preserves the inaccessibility as well as the Mahloness of κ, but kills the weak

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compactness of κ and indeed adds a non-reflecting stationary set. However, if κ is weakly compact, there is a forcing [HLH17] which adds a $\square(\kappa)$-sequence and also preserves the fact that every stationary subset of κ reflects. Thus, relative to the existence of a weakly compact cardinal, $\square(\kappa)$ is consistent with $\mathrm{Refl}(\kappa)$, the principle that every stationary set reflects. However, $\square(\kappa)$ implies the failure of the simultaneous stationary reflection principle $\mathrm{Refl}(\kappa, 2)$ which states that if S and T are any two stationary subsets of κ, then there is some $\alpha < \kappa$ with $\mathrm{cf}(\alpha) > \omega$ such that $S \cap \alpha$ and $T \cap \alpha$ are both stationary in α. In fact, $\square(\kappa)$ implies that every stationary subset of κ can be partitioned into two stationary sets that do not simultaneously reflect [HLH17, Theorem 2.1].

If κ is a weakly compact cardinal, then the collection of non–Π^1_1-indescribable subsets of κ forms a natural normal ideal called the Π^1_1-indescribability ideal:

$$\Pi^1_1(\kappa) = \{X \subseteq \kappa : X \text{ is not } \Pi^1_1\text{-indescribable}\}.$$

A set $S \subseteq \kappa$ is Π^1_1-*indescribable* if for every $A \subseteq V_\kappa$ and every Π^1_1-sentence φ, whenever $(V_\kappa, \in, A) \models \varphi$ there is an $\alpha \in S$ such that $(V_\alpha, \in, A \cap V_\alpha) \models \varphi$. More generally, a Π^1_n-indescribable cardinal κ carries the analogously defined Π^1_n-indescribability ideal. It is natural to ask the question: which results concerning the nonstationary ideal can be generalized to the various ideals associated to large cardinals, such as the Π^1_n-indescribability ideals? It follows from the work of Sun [Sun93] and Hellsten [Hel03a] that when κ is Π^1_n-indescribable the collection of n-*club* subsets of κ (see the next section for definitions) is a filter-base for the filter $\Pi^1_n(\kappa)^*$ dual to the Π^1_n-indescribability ideal, yielding a characterization of Π^1_n-indescribable sets that resembles the definition of stationarity: when κ is Π^1_n-indescribable, a set $S \subseteq \kappa$ is Π^1_n-indescribable if and only if $S \cap C \neq \varnothing$ for every n-club $C \subseteq \kappa$. Several recent results have used this characterization ([Hel06], [Hel10], [Cod19] and [CS20]) to generalize theorems concerning the nonstationary ideal to the Π^1_1-indescribability ideal. For technical reasons discussed below in Section 8, there has been less success with the Π^1_n-indescribability ideals for $n > 1$. In this article we continue this line of research: by replacing "clubs" with "1-clubs" we obtain a $\square(\kappa)$-like principle $\square_1(\kappa)$ (see Definition 2.1) that is consistent with weak compactness but not with Π^1_2-indescribability. Brickhill and Welch [BW] showed that a slightly different version of $\square_1(\kappa)$, which they call $\square^1(\kappa)$, can hold at a weakly compact cardinal in L. See Remark 2.3 for a discussion of the relationship between $\square^1(\kappa)$ and $\square_1(\kappa)$. In this article we consider the extent to which principles such as $\square_1(\kappa)$ can be forced to hold at large cardinals.

We will see that the principle $\square_1(\kappa)$ holds trivially at weakly compact cardinals κ below which stationary reflection fails. (This is analogous to the fact that $\square(\kappa)$ holds trivially for every κ of cofinality ω_1.) Thus, the task at hand is not just to force $\square_1(\kappa)$ to hold at a weakly compact cardinal, but to show that one can force $\square_1(\kappa)$ to hold at a weakly compact cardinal κ even when stationary reflection holds at many cardinals below κ, so that nontrivial coherence of the sequence is obtained. Recall that when κ is κ^+-weakly compact (see Definition 5.8 below), the set of weakly compact cardinals below κ is weakly compact and much more, so, in particular, the set of inaccessible $\alpha < \kappa$ at which stationary reflection holds is weakly compact. By [BW, Theorem 3.24], assuming $V = L$, if κ is κ^+-weakly compact and κ is not Π^1_2-indescribable then $\square_1(\kappa)$ holds. We show that the same can be forced.

Theorem 1.1. *If κ is κ^+-weakly compact and the* GCH *holds, then there is a cofinality-preserving forcing extension in which*

(1) *κ remains κ^+-weakly compact and*
(2) *$\square_1(\kappa)$ holds.*

We will also investigate the relationship between $\square_1(\kappa)$ and weakly compact reflection principles. The *weakly compact reflection principle* $\text{Refl}_1(\kappa)$ states that κ is weakly compact and for every weakly compact $S \subseteq \kappa$ there is an $\alpha < \kappa$ such that $S \cap \alpha$ is weakly compact. It is straightforward to see that if κ is Π^1_2-indescribable, then $\text{Refl}_1(\kappa)$ holds, and if $\text{Refl}_1(\kappa)$ holds, then κ is ω-weakly compact (see [Cod19, Section 2]). However, the following results show that neither of these implications can be reversed. The first author [Cod19] showed that if $\text{Refl}_1(\kappa)$ holds then there is a forcing which adds a non-reflecting weakly compact subset of κ and preserves the ω-weak compactness of κ, hence the ω-weak compactness of κ does not imply $\text{Refl}_1(\kappa)$. The first author and Hiroshi Sakai [CS20] showed that $\text{Refl}_1(\kappa)$ can hold at the least ω-weakly compact cardinal, and hence $\text{Refl}_1(\kappa)$ does not imply the Π^1_2-indescribability of κ. Just as $\square(\kappa)$ and $\text{Refl}(\kappa)$ can hold simultaneously relative to a weakly compact cardinal, we will prove that $\square_1(\kappa)$ and $\text{Refl}_1(\kappa)$ can hold simultaneously relative to a Π^1_2-indescribable cardinal; this provides a new consistency result which does not follow from the results in [BW] due to the fact that, if $V = L$, then $\text{Refl}_1(\kappa)$ holds at a weakly compact cardinal if and only if κ is Π^1_2-indescribable.

Theorem 1.2. *Suppose that κ is Π^1_2-indescribable and the* GCH *holds. Then there is a cofinality-preserving forcing extension in which*

(1) *$\square_1(\kappa)$ holds,*
(2) *$\text{Refl}_1(\kappa)$ holds and*
(3) *κ is κ^+-weakly compact.*

In Section 2, using n-club subsets of κ, we formulate a generalization of $\square_1(\kappa)$ to higher degrees of indescribability. It is easily seen that $\square_n(\kappa)$ implies that κ is not Π^1_{n+1}-indescribable (see Proposition 2.9 below). However, for technical reasons outlined in Section 8, our methods do not seem to show that $\square_n(\kappa)$ can hold nontrivially (see Definition 2.10) when κ is Π^1_n-indescribable. Our methods do allow for a generalization of Hellsten's 1-club shooting forcing to n-club shooting, and we also show that, if S is a Π^1_n-indescribable set, a 1-club can be shot through S while preserving the Π^1_n-indescribability of all Π^1_n-indescribable subsets of S.

Finally, we consider the influence of $\square_n(\kappa)$ on *simultaneous* reflection of Π^1_n-indescribable sets. We let $\text{Refl}_n(\kappa, \mu)$ denote the following simultaneous reflection principle: κ is Π^1_n-indescribable and whenever $\{S_\alpha : \alpha < \mu\}$ is a collection of Π^1_n-indescribable sets, there is a $\beta < \kappa$ such that $S_\alpha \cap \beta$ is Π^1_n-indescribable for all $\alpha < \mu$. In Section 7, we show that for $n \geq 1$, if $\square_n(\kappa)$ holds at a Π^1_n-indescribable cardinal, then the simultaneous reflection principle $\text{Refl}_n(\kappa, 2)$ fails (see Theorem 7.1). As a consequence, we show that relative to a Π^1_2-indescribable cardinal, it is consistent that $\text{Refl}_1(\kappa)$ holds and $\text{Refl}_1(\kappa, 2)$ fails (see Corollary 7.4).

2. The principles $\square_n(\kappa)$

Suppose that κ is a cardinal. A set $S \subseteq \kappa$ is Π^1_n-*indescribable* if for every $A \subseteq V_\kappa$ and every Π^1_n-sentence φ, whenever $(V_\kappa, \in, A) \models \varphi$ there is an $\alpha \in S$ such that $(V_\alpha, \in, A \cap V_\alpha) \models \varphi$. The cardinal κ is said to be Π^1_n-*indescribable* if κ is a Π^1_n-indescribable subset of κ. The Π^1_0-indescribable cardinals are precisely the inaccessible cardinals, and, if κ is inaccessible, then $S \subseteq \kappa$ is Π^1_0-indescribable if and only if it is stationary. The Π^1_1-indescribable cardinals are precisely the weakly compact cardinals.

The Π^1_n-*indescribability ideal* on κ is

$$\Pi^1_n(\kappa) = \{X \subseteq \kappa : X \text{ is not } \Pi^1_n\text{-indescribable}\},$$

the corresponding collection of positive sets is

$$\Pi^1_n(\kappa)^+ = \{X \subseteq \kappa : X \text{ is } \Pi^1_n\text{-indescribable}\}$$

and the dual filter is

$$\Pi^1_n(\kappa)^* = \{\kappa \setminus X : X \in \Pi^1_n(\kappa)\}.$$

Clearly, if κ is not Π^1_n-indescribable, then $\Pi^1_n(\kappa) = P(\kappa)$. Lévy proved [Lév71] that if κ is Π^1_n-indescribable, then $\Pi^1_n(\kappa)$ is a nontrivial normal ideal on κ.

A set $C \subseteq \kappa$ is called 0-*club* if it is a club. A set $X \subseteq \kappa$ is said to be n-*closed* if it contains all of its Π^1_{n-1}-indescribable reflection points: whenever $\alpha < \kappa$ and $X \cap \alpha$ is Π^1_{n-1}-indescribable, then $\alpha \in X$ (note that such α must be Π^1_{n-1}-indescribable). If a set $C \subseteq \kappa$ is both n-closed and Π^1_{n-1}-indescribable, then C is said to be an n-*club* subset of κ. For example, $C \subseteq \kappa$ is 1-club if and only if it is stationary and contains all of its inaccessible stationary reflection points (a stationary reflection point that occurs at an inaccessible cardinal), and $C \subseteq \kappa$ is 2-club if and only if it is weakly compact and contains all of its weakly compact reflection points. Building on work of Sun [Sun93], Hellsten showed [Hel03b] that when κ is a Π^1_n-indescribable cardinal, a set $S \subseteq \kappa$ is Π^1_n-indescribable if and only if $S \cap C \neq \varnothing$ for every n-club $C \subseteq \kappa$. Thus, when κ is Π^1_n-indescribable, the collection of n-club subsets of κ generates the filter $\Pi^1_n(\kappa)^*$. In particular, this implies that n-club sets are themselves Π^1_n-indescribable.

For $n < \omega$ and $X \subseteq \kappa$, we define the n-*trace of* X to be

$$\mathrm{Tr}_n(X) = \{\alpha < \kappa : X \cap \alpha \in \Pi^1_n(\alpha)^+\}.$$

Notice that when $X = \kappa$, $\mathrm{Tr}_n(\kappa)$ is the set of Π^1_n-indescribable cardinals below κ, and in particular $\mathrm{Tr}_0(\kappa)$ is the set of inaccessible cardinals less than κ. For uniformity of notation, let us say that an ordinal α is Π^1_{-1}-*indescribable* if it is a limit ordinal, and if α is a limit ordinal, $S \subseteq \alpha$ is Π^1_{-1}-*indescribable* if it is unbounded in α. Thus, if $X \subseteq \kappa$, then $\mathrm{Tr}_{-1}(X) = \{\alpha < \kappa : \sup(X \cap \alpha) = \alpha\}$.

Definition 2.1. Suppose $n < \omega$ and $\mathrm{Tr}_{n-1}(\kappa)$ is cofinal in κ. A sequence $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Tr}_{n-1}(\kappa)\rangle$ is called a *coherent sequence of* n-*clubs* if

(1) for all $\alpha \in \mathrm{Tr}_{n-1}(\kappa)$, C_α is an n-club subset of α and
(2) for all $\alpha < \beta$ in $\mathrm{Tr}_{n-1}(\kappa)$, $C_\beta \cap \alpha \in \Pi^1_{n-1}(\alpha)^+$ implies $C_\alpha = C_\beta \cap \alpha$.

We say that a set $C \subseteq \kappa$ is a *thread* through a coherent sequence of n-clubs

$$\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Tr}_{n-1}(\kappa)\rangle$$

if C is n-club and for all $\alpha \in \mathrm{Tr}_{n-1}(\kappa)$, $C \cap \alpha \in \Pi^1_{n-1}(\alpha)^+$ implies $C_\alpha = C \cap \alpha$. A coherent sequence of n-clubs $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Tr}_{n-1}(\kappa)\rangle$ is called a $\square_n(\kappa)$-*sequence* if there is no thread through $\vec{C}$. We say that $\square_n(\kappa)$ holds if there is a $\square_n(\kappa)$-sequence $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Tr}_{n-1}(\kappa)\rangle$.

Remark 2.2. Note that $\square_0(\kappa)$ is simply $\square(\kappa)$. For $n = 1$, the principle $\square_1(\kappa)$ states that there is a coherent sequence of 1-clubs

$$\langle C_\alpha : \alpha < \kappa \text{ is inaccessible}\rangle$$

that cannot be threaded.

Remark 2.3. Before we prove some basic results about $\square_n(\kappa)$, let us consider a similar principle due to Brickhill and Welch [BW]. First, let us note that the notion of n-club, for all n, used in [BW] was first introduced by Beklemishev [Bek09] in 2009 (see also [BG14]) in the context of Generalized Provability Logics, and independently introduced and studied in full generality for all ordinals by Bagaria [Bag12] in 2012 (see [Bag19] for more information on the history of the notion).

We will consider the Brickhill-Welch principle in detail in the case $n = 1$. We will refer to the notion of 1-club used in [BW] as *strong* 1*-club* in order to avoid confusion. A set $C \subseteq \kappa$ is a *strong* 1*-club* if it is stationary in κ and whenever $C \cap \alpha$ is stationary in α then $\alpha \in C$. This notion of strong 1-club is precisely the same notion considered in [Sun93]. Thus, it follows from the results of [Sun93] that when κ is weakly Π^1_1-indescribable[1] the collection of strong 1-clubs generates the weak Π^1_1-indescribable filter. Moreover, when κ is weakly compact, the collection of strong 1-club subsets of κ generates the weakly compact filter $\Pi^1_1(\kappa)$. For $S \subseteq \kappa$, Brickhill and Welch [BW] define $\square^1(\kappa)$ to be the principle asserting the existence of a sequence $\langle C_\alpha : \mathrm{cf}(\alpha) > \omega\rangle$ such that

(1) C_α is a strong 1-club in α,
(2) whenever $C_\alpha \cap \beta$ is stationary in β we have $C_\beta = C_\alpha \cap \beta$ and
(3) there is no $C \subseteq \kappa$ that is strong 1-club in κ such that whenever $C \cap \alpha$ is stationary in α we have $C_\alpha = C \cap \alpha$.

Such a sequence is called a $\square^1(\kappa)$-sequence. Let us show that when κ is weakly compact, the Brickhill-Welch principle $\square^1(\kappa)$ implies our principle $\square_1(\kappa)$. Suppose $\langle C_\alpha : \mathrm{cf}(\alpha) > \omega\rangle$ is a $\square^1(\kappa)$-sequence. Clearly, $\langle C_\alpha : \alpha \in \mathrm{Tr}_0(\kappa)\rangle$ is a coherent sequence of 1-clubs. For the sake of contradiction, suppose $C \subseteq \kappa$ is a 1-club thread through $\langle C_\alpha : \alpha \in \mathrm{Tr}_0(\kappa)\rangle$. Using the coherence of $\langle C_\alpha : \mathrm{cf}(\alpha) > \omega\rangle$ it is straightforward to check that C is a strong 1-club and whenever $C \cap \alpha$ is stationary in α we have $C_\alpha = C \cap \alpha$. This contradicts $\square^1(\kappa)$. Thus $\square^1(\kappa)$ implies $\square_1(\kappa)$. It is not known whether $\square_1(\kappa)$ implies $\square^1(\kappa)$.

Brickhill and Welch also generalized their definition to obtain the principles $\square^n(\kappa)$, and again it is not difficult to see that $\square^n(\kappa)$ implies our principle $\square_n(\kappa)$.

Generalizing the fact that $\square(\kappa)$ implies κ is not weakly compact, let us show that $\square_n(\kappa)$ implies κ is not Π^1_{n+1}-indescribable. To do this, we first recall the Hauser characterization of Π^1_n-indescribability.

We say that a transitive model $\langle M, \in\rangle$ is a *κ-model* if $|M| = \kappa$, $\kappa \in M$, $M^{<\kappa} \subseteq M$, and $M \models \mathrm{ZFC}^-$ (ZFC without the power set axiom). It is not difficult to see that if κ is inaccessible, then V_κ is an element of every κ-model M.

Definition 2.4 (Hauser [Hau91])**.** Suppose κ is inaccessible. For $n \geq 0$, a κ-model N is *Π^1_n-correct at κ* if and only if

$$V_\kappa \models \varphi \iff (V_\kappa \models \varphi)^N$$

[1]Recall that a set $S \subseteq \kappa$ is *weakly Π^1_1-indescribable* if for all $A \subseteq \kappa$ and all Π^1_1 sentences φ, $(\kappa, \in, A) \models \varphi$ implies that there is and $\alpha \in S$ such that $(\alpha, \in, A \cap \alpha) \models \varphi$.

for all Π^1_n-formulas φ whose parameters are contained in $N \cap V_{\kappa+1}$.

Remark 2.5. Notice that every κ-model is Π^1_0-correct at κ.

Theorem 2.6 (Hauser [Hau91])**.** *The following statements are equivalent for every inaccessible cardinal κ, every subset $S \subseteq \kappa$, and all $0 < n < \omega$.*

(1) S is Π^1_n-indescribable.
(2) For every κ-model M with $S \in M$, there is a Π^1_{n-1}-correct κ-model N and an elementary embedding $j : M \to N$ with $\mathrm{crit}(j) = \kappa$ such that $\kappa \in j(S)$.
(3) For every $A \subseteq \kappa$ there is a κ-model M with $A, S \in M$ for which there is a Π^1_{n-1}-correct κ-model N and an elementary embedding $j : M \to N$ with $\mathrm{crit}(j) = \kappa$ such that $\kappa \in j(S)$.
(4) For every $A \subseteq \kappa$ there is a κ-model M with $A, S \in M$ for which there is a Π^1_{n-1}-correct κ-model N and an elementary embedding $j : M \to N$ with $\mathrm{crit}(j) = \kappa$ such that $\kappa \in j(S)$ and $j, M \in N$.

Lemma 2.7. *Suppose κ is a cardinal. If $S \in \Pi^1_n(\kappa)^+$ and $S_\alpha \in \Pi^1_n(\alpha)^+$ for each $\alpha \in S$, then $\bigcup_{\alpha \in S} S_\alpha \in \Pi^1_n(\kappa)^+$.*

Proof. Fix an n-club C in κ. The set $\mathrm{Tr}_{n-1}(C)$ is n-closed because if $\mathrm{Tr}_{n-1}(C) \cap \alpha \in \Pi^1_{n-1}(\alpha)^+$, then $C \cap \alpha \in \Pi^1_{n-1}(\alpha)^+$ since, by n-closure of C, $\mathrm{Tr}_{n-1}(C) \subseteq C$. Also, $\mathrm{Tr}_{n-1}(C)$ meets every n-club D because the intersection $C \cap D$ is an n-club. Thus, $\mathrm{Tr}_{n-1}(C)$ is an n-club. It follows that there is an $\alpha \in S \cap \mathrm{Tr}_{n-1}(C)$. Since S_α is Π^1_n-indescribable in α and $C \cap \alpha$ is an n-club in α, we have $S_\alpha \cap C \cap \alpha \neq \varnothing$, and hence $\left(\bigcup_{\alpha \in S} S_\alpha\right) \cap C \neq \varnothing$. $\square$

A simple complexity calculation shows that for every $n < \omega$, there is a Π^1_{n+1}-formula $\chi_n(X)$ such that for every κ and every $S \subseteq \kappa$, $(V_\kappa, \in) \models \chi_n(S)$ if and only if S is Π^1_n-indescribable (see [Kan03, Corollary 6.9]). It therefore follows that there is a Π^1_n-formula $\psi_n(X)$ such that for every κ and every $C \subseteq \kappa$, $(V_\kappa, \in) \models \psi_n(C)$ if and only if C is an n-club subset of κ. Thus, in particular, a Π^1_n-correct model N is going to be correct about Π^1_{n-1}-indescribable sets as well as n-clubs.

Corollary 2.8. *Suppose κ is Π^1_n-indescribable. If $S \in \Pi^1_n(\kappa)^+$, then*

$$\mathrm{Tr}_{n-1}(S) = \{\alpha < \kappa : S \cap \alpha \in \Pi^1_{n-1}(\alpha)^+\}$$

is an n-club.

Proof. Suppose S is Π^1_n-indescribable. First, let us argue that $\mathrm{Tr}_{n-1}(S)$ is Π^1_n-indescribable. Let M be a κ-model with $S, \mathrm{Tr}_{n-1}(S) \in M$ and let $j : M \to N$ be an elementary embedding with critical point κ such that N is Π^1_{n-1}-correct and $\kappa \in j(S)$. The Π^1_{n-1}-correctness of N implies that $j(S) \cap \kappa = S$ is a Π^1_{n-1}-indescribable subset of κ in N. Thus, $\kappa \in j(\mathrm{Tr}_{n-1}(S))$. Hence $\mathrm{Tr}_{n-1}(S)$ is Π^1_n-indescribable.

It remains to show that $\mathrm{Tr}_{n-1}(S)$ is n-closed, which is equivalent to showing that if $\mathrm{Tr}_{n-1}(S) \cap \alpha \in \Pi^1_{n-1}(\alpha)^+$, then $S \cap \alpha \in \Pi^1_{n-1}(\alpha)^+$. More generally, observe that if $X \subseteq \alpha$ and $\mathrm{Tr}_m(X)$ is Π^1_m-indescribable, then $X = \bigcup_{\beta \in \mathrm{Tr}_m(X)} X \cap \beta$ must be Π^1_m-indescribable by Lemma 2.7. $\square$

Proposition 2.9. *For every $n < \omega$, $\square_n(\kappa)$ implies that κ is not Π^1_{n+1}-indescribable.*

Proof. Suppose $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Tr}_n(\kappa)\rangle$ is a $\square_n(\kappa)$-sequence and κ is Π^1_{n+1}-indescribable. Let M be a κ-model with $\vec{C} \in M$. Since κ is Π^1_{n+1}-indescribable, we may let $j : M \to N$ be an elementary embedding with critical point κ and a Π^1_n-correct N as in Theorem 2.6 (2). By elementarity, it follows that $j(\vec{C}) = \langle \bar{C}_\alpha : \alpha \in \mathrm{Tr}^N_n(j(\kappa))\rangle$ is a $\square_n(j(\kappa))$-sequence in N. Since N is Π^1_n-correct, we know that $\kappa \in \mathrm{Tr}^N_n(j(\kappa))$ and $\bar{C}_\kappa$ must also be n-club in V. Since $j(\vec{C})$ is a $\square_n(j(\kappa))$-sequence in N, it follows that for every Π^1_n-indescribable $\alpha < \kappa$ if $\bar{C}_\kappa \cap \alpha \in \Pi^1_n(\alpha)^+$, then $\bar{C}_\kappa \cap \alpha = C_\alpha$, and hence $\bar{C}_\kappa$ is a thread through $\vec{C}$, a contradiction. $\square$

Let us now describe the sense in which $\square_n(\kappa)$ can hold trivially when κ is Π^1_n-indescribable and certain reflection principles fail often below κ.

Definition 2.10. Suppose $n < \omega$ and $\mathrm{Tr}_{n-1}(\kappa)$ is cofinal in κ. We say that $\square_n(\kappa)$ *holds trivially* if there is a $\square_n(\kappa)$-sequence $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Tr}_{n-1}(\kappa)\rangle$ and a club $E \subseteq \kappa$ such that for all $\alpha \in \mathrm{Tr}_{n-1}(\kappa) \cap E$, C_α is trivially an n-club subset of α in the sense that C_α is a Π^1_{n-1}-indescribable subset of α and has no Π^1_{n-1}-indescribable proper initial segment (see Remark 2.12 below for the reason for adopting this notion of triviality).

Notice that $\square(\kappa)$ holds trivially if $\mathrm{cf}(\kappa) = \omega_1$. In this case we can find a club $E \subseteq \kappa$ consisting of ordinals of countable cofinality. For all $\alpha \in E$, we can let C_α be a cofinal subset of α of order type ω. Then, for every limit ordinal $\beta \in \kappa \setminus E$, we can let $\alpha_\beta = 0$ if $\beta < \min(E)$ and $\alpha_\beta = \max(E \cap \beta)$ otherwise, and set C_β to be the interval (α_β, β). It is easily verified that a sequence thus defined is a $\square(\kappa)$-sequence.

Recall that the principle $\mathrm{Refl}_n(\kappa)$ holds if and only if κ is Π^1_n-indescribable and for every Π^1_n-indescribable subset X of κ, there is an $\alpha < \kappa$ such that $X \cap \alpha$ is Π^1_n-indescribable (see [Cod19] and [CS20] for more details).

Proposition 2.11. *Suppose $1 \leq n < \omega$ and κ is Π^1_n-indescribable. Then $\square_n(\kappa)$ holds trivially if and only if there is a club $E \subseteq \kappa$ such that $\neg\mathrm{Refl}_{n-1}(\alpha)$ holds for every $\alpha \in \mathrm{Tr}_{n-1}(\kappa) \cap E$.*

Proof. If $\square_n(\kappa)$ holds trivially, then there is a $\square_n(\kappa)$-sequence $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Tr}_{n-1}(\kappa)\rangle$ and a club $E \subseteq \kappa$ such that for $\alpha \in \mathrm{Tr}_{n-1}(\kappa) \cap E$, C_α is a Π^1_{n-1}-indescribable set with no Π^1_{n-1}-indescribable initial segment, in which case C_α is a witness to the fact that $\mathrm{Refl}_{n-1}(\alpha)$ fails.

Conversely, suppose that $E \subseteq \kappa$ is a club and $\neg\mathrm{Refl}_{n-1}(\alpha)$ holds for every $\alpha \in \mathrm{Tr}_{n-1}(\kappa) \cap E$. For each $\alpha \in \mathrm{Tr}_{n-1}(\kappa) \cap E$, let C_α be a Π^1_{n-1}-indescribable subset of α which has no Π^1_{n-1}-indescribable proper initial segment. Then each C_α is trivially n-club in α. For all $\beta \in \mathrm{Tr}_{n-1}(\kappa) \setminus E$, let $\alpha_\beta = \max(E \cap \beta)$, and let C_β be the interval (α_β, β). Then $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Tr}_{n-1}(\kappa)\rangle$ is easily seen to be a coherent sequence of n-clubs, since there are no points at which coherence needs to be checked for indices in E and coherence is easily checked for indices outside of E because of the uniformity of the definition. We must argue that $\vec{C}$ has no thread. Suppose there is a thread $C \subseteq \kappa$ through $\vec{C}$. Since κ is Π^1_n-indescribable and C is an n-club subset of κ it follows, by Corollary 2.8, that $\mathrm{Tr}_{n-1}(C)$ is an n-club in κ. Thus we can choose $\alpha, \beta \in \mathrm{Tr}_{n-1}(C) \cap E$ with $\alpha < \beta$. Since C is a thread we have $C_\alpha = C_\beta \cap \alpha = C \cap \alpha$, which contradicts the fact that C_β has no Π^1_{n-1}-indescribable proper initial segment. This shows that $\square_n(\kappa)$ holds trivially. $\square$

Remark 2.12. It seems like it might be more optimal to change Definition 2.10 to instead say that $\Box_n(\kappa)$ holds trivially if there is a $\Box_n(\kappa)$-sequence $\vec{C}$ and an *n-club* $E \subseteq \kappa$ such that for all $\alpha \in \mathrm{Tr}_{n-1}(\kappa) \cap E$, C_α is trivially an n-club subset of α. However, we were not able to prove the analogue of Proposition 2.11 corresponding to this alternative definition, namely that $\Box_n(\kappa)$ holds trivially if and only if there is an *n-club* $E \subseteq \kappa$ such that $\neg\mathrm{Refl}_{n-1}(\alpha)$ holds for every $\alpha \in \mathrm{Tr}_{n-1}(\kappa) \cap E$.

Corollary 2.13. *In L, if κ is the least Π^1_n-indescribable cardinal, then $\Box_n(\kappa)$ holds trivially.*

Proof. Generalizing a result of Jensen [Jen72], Bagaria, Magidor and Sakai proved [BMS15] that in L a cardinal κ is Π^1_n-indescribable if and only if $\mathrm{Refl}_{n-1}(\kappa)$ holds. Suppose $V = L$ and κ is the least Π^1_n-indescribable cardinal. Then $\mathrm{Refl}_{n-1}(\alpha)$ fails for all $\alpha < \kappa$. Hence by Proposition 2.11, $\Box_n(\kappa)$ holds trivially. $\Box$

Brickhill and Welch showed more generally that in L, if κ is Π^1_n-indescribable and not Π^1_{n+1}-indescribable, then their principle $\Box^n(\kappa)$ holds. Since L can have Π^1_n-indescribable, but not Π^1_{n+1}-indescribable, cardinals below which, for instance, the set of Π^1_{n-1}-indescribable cardinals is Π^1_n-indescribable, it follows from reasonable assumptions that in L our principle $\Box_n(\kappa)$ can hold nontrivially at a Π^1_n-indescribable cardinal. We do not know how to force $\Box_n(\kappa)$ to hold non-trivially at a Π^1_n-indescribable cardinal.

Another consequence of Proposition 2.11 is that we can force $\Box_1(\kappa)$ to hold *trivially* at a Π^1_1-indescribable cardinal by killing certain stationary reflection principles below κ.

Recall that a partial order $\mathbb{P}$ is said to be *α-strategically closed*, for an ordinal α, if Player II has a winning strategy in the following two-player game $\mathcal{G}_\alpha(\mathbb{P})$ of perfect information. In a run of $\mathcal{G}_\alpha(\mathbb{P})$, the two players take turns playing elements of a decreasing sequence $\langle p_\beta : \beta < \alpha\rangle$ of conditions from $\mathbb{P}$. Player I plays at all odd ordinal stages, and Player II plays at all even ordinal stages (in particular, at limits). Player II goes first and must play $1_{\mathbb{P}}$ on their first move. Player I wins if there is a limit ordinal $\gamma < \alpha$ such that $\langle p_\beta : \beta < \gamma\rangle$ has no lower bound (i.e., if Player II is unable to play at stage γ). If the game continues successfully for α-many moves, then Player II wins. Clearly, for a cardinal α, if $\mathbb{P}$ is α-strategically closed, then $\mathbb{P}$ is $<\alpha$-distributive, and hence adds no new $<\alpha$-sequences of ground model sets.

We will use the following general proposition about indestructibility of weakly compact cardinals.

Definition 2.14. Suppose κ is an inaccessible cardinal. We say that a forcing iteration

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \kappa,\ \beta < \kappa\rangle$$

is *good* if it has Easton support and, for all $\alpha < \kappa$, if α is inaccessible, then $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a poset such that $1_{\mathbb{P}_\alpha} \Vdash \dot{\mathbb{Q}}_\alpha \in \dot{V}_\kappa$, where $\dot{V}_\kappa$ is a $\mathbb{P}_\alpha$-name for $(V_\kappa)^{V^{\mathbb{P}_\alpha}}$ and, otherwise, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for trivial forcing.

If $\mathbb{P}_\kappa$ is a good iteration, then we can argue by induction on α that $\mathbb{P}_\alpha \in V_\kappa$ for every $\alpha < \kappa$. This is because, if $\mathbb{P}_\alpha \in V_\kappa$ and $1_{\mathbb{P}_\alpha} \Vdash \dot{\mathbb{Q}}_\alpha \in \dot{V}_\kappa$, then $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha \in V_\kappa$. The following standard proposition about good iterations can be found, for example, in [Cum10].

Proposition 2.15. *Suppose κ is a Mahlo cardinal. Then a good iteration $\mathbb{P}_\kappa$ has size κ and is κ-c.c.*

Lemma 2.16. *Suppose κ is weakly compact and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \kappa,\ \beta < \kappa\rangle$ is a good iteration which at non-trivial stages α has $1_{\mathbb{P}_\alpha} \Vdash$ "$\dot{\mathbb{Q}}_\alpha$ is α-strategically closed", and let G be $\mathbb{P}$-generic over V. Then κ remains weakly compact in $V[G]$.*

Proof. By Proposition 2.15, we can assume without loss that $\mathbb{P} \subseteq V_\kappa$. Since κ is weakly compact, there are κ-models M and N with $A \in M$ for which there is an elementary embedding $j : M \to N$ with critical point κ. A nice-name counting argument, using the κ-c.c. and the fact that the tails of the forcing iteration are eventually α-distributive for every $\alpha < \kappa$, shows that κ is inaccessible in $V[G]$.

Suppose $A \in P(\kappa)^{V[G]}$ and let $\dot{A} \in H(\kappa^+)^V$ be a $\mathbb{P}_\kappa$-name such that $\dot{A}_G = A$. Let M be a κ-model with $\mathbb{P}_\kappa, \dot{A} \in M$ for which there are a κ-model N and an elementary embedding $j : M \to N$ with critical point κ. Since $N^{<\kappa} \cap V \subseteq N$, we have $j(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa * \dot{\mathbb{P}}_{\kappa,j(\kappa)}$, where N believes that $1_{\mathbb{P}_\kappa} \Vdash$ "$\dot{\mathbb{Q}}_\kappa$ is κ-strategically closed", and $\dot{\mathbb{P}}_{\kappa,j(\kappa)}$ is a $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$-name for N's version of the tail of the iteration $j(\mathbb{P}_\kappa)$ of length $j(\kappa)$. By the generic closure criterion (Lemma 3.2), since $\mathbb{P}_\kappa$ has the κ-c.c., $N[G]$ is a κ-model in $V[G]$. The poset $(\dot{\mathbb{Q}}_\kappa * \dot{\mathbb{P}}_{\kappa,j(\kappa)})_G$ is κ-strategically closed in $N[G]$, so, by diagonalizing, we can build an $N[G]$-generic filter $H * G' \in V[G]$ for $(\dot{\mathbb{Q}}_\kappa * \dot{\mathbb{P}}_{\kappa,j(\kappa)})_G$. Since conditions in $\mathbb{P}_\kappa$ have supports of size less than the critical point of j we have $j\,"\,G \subseteq \hat{G} =_{\mathrm{def}} G * H * G'$. Thus j lifts to $j : M[G] \to N[\hat{G}]$. Since $A = \dot{A}_G \in M[G]$, this shows that κ remains weakly compact in $V[G]$. $\square$

Proposition 2.17. *If κ is Π^1_1-indescribable (weakly compact), then there is a forcing extension in which $\square_1(\kappa)$ holds trivially and κ remains Π^1_1-indescribable.*

Proof. For regular $\alpha > \omega$, let $\mathbb{S}_\alpha$ denote the usual forcing to add a nonreflecting stationary subset of $\alpha \cap \mathrm{cof}(\omega)$ (see Example 6.5 in [Cum10]). Recall that conditions in $\mathbb{S}_\alpha$ are bounded subsets p of $\alpha \cap \mathrm{cof}(\omega)$ such that for every $\beta \leq \sup(p)$ with $\mathrm{cf}(\beta) > \omega$, the set $p \cap \beta$ is nonstationary in β. It is not difficult to see that the poset $\mathbb{S}_\alpha$ is α-strategically closed.

Now we let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \kappa,\ \beta < \kappa\rangle$ be an Easton-support iteration of length κ such that if $\alpha < \kappa$ is inaccessible, then $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for $\mathbb{S}_\alpha^{V^{\mathbb{P}_\alpha}}$, and otherwise $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for trivial forcing.

Suppose G is generic for $\mathbb{P}_\kappa$ over V. By Lemma 2.16, since $\mathbb{P}_\kappa$ has all the right properties, κ remains weakly compact in $V[G]$. Also, in $V[G]$, for each inaccessible $\alpha < \kappa$, by a routine genericity argument and the fact that the tail of the forcing iteration from stage $\alpha + 1$ to κ is α^+-strategically closed, the stage α generic H_α obtained from G yields a nonreflecting stationary subset of α: $S_\alpha = \bigcup H_\alpha$. Thus in $V[G]$, $\mathrm{Refl}_0(\alpha)$ fails for all inaccessible $\alpha < \kappa$, and hence $\square_1(\kappa)$ holds trivially by Proposition 2.11. $\square$

In Section 5 we will show that $\square_1(\kappa)$ can hold non-trivially at a weakly compact cardinal.

3. Preserving Π^1_n-indescribability by forcing

In this section, we will provide some results to be used in indestructibility arguments for Π^1_n-indescribable cardinals in later sections.

The following two folklore lemmas (and their variants) are often used in indestructibility arguments for large cardinals characterized by the existence of elementary embeddings between κ-models (see [Cum10, Proposition 8.3–8.4] for more details).

Lemma 3.1 (Ground closure criterion)**.** *Suppose κ is a cardinal, M is a κ-model, $\mathbb{P} \in M$ is a forcing notion, and $G \in V$ is generic for $\mathbb{P}$ over M. Then $M[G]$ is a κ-model.*

Lemma 3.2 (Generic closure criterion)**.** *Suppose κ is a cardinal, M is a κ-model, $\mathbb{P} \in M$ is a forcing notion with the κ-c.c., and G is generic for $\mathbb{P}$ over V. Then $M[G]$ is a κ-model in $V[G]$.*

Lemma 3.3. *Suppose κ is inaccessible, $\mathbb{P}$ is a κ-strategically closed forcing and G is generic for $\mathbb{P}$ over V. Then $(V_\kappa, \in, A) \models \forall X \psi(X, A)$ implies $((V_\kappa, \in, A) \models \forall X \psi(X, A))^{V[G]}$ for all $A \in V^V_{\kappa+1}$ and all first order ψ.*

Proof. First, observe that since $\mathbb{P}$ is $<\kappa$-distributive, κ remains inaccessible in $V[G]$ and $V_\kappa = V_\kappa^{V[G]}$. Suppose towards a contradiction that $(V_\kappa, \in, A) \models \forall X \psi(X, A)$, but for some $B \subseteq V_\kappa$ in $V[G]$, $(V_\kappa, \in, A) \models \neg\psi(B, A)$. Let $\dot{B}$ be a $\mathbb{P}$-name for B. Since κ is inaccessible in $V[G]$, the set

$$C = \{\alpha < \kappa : (V_\alpha, \in, A \cap \alpha, B \cap \alpha) \models \neg\psi(B \cap \alpha, A \cap \alpha))\}$$

contains a club in $V[G]$. Let $\dot{C}$ be a $\mathbb{P}$-name for such a club. In V, we can use Player II's winning strategy in $\mathcal{G}_\kappa(\mathbb{P})$ together with the names $\dot{B}$ and $\dot{C}$ to build $\hat{B}$ and $\hat{C}$ such that $\hat{C}$ is club in κ and for each $\alpha \in \hat{C}$ we have

$$(V_\alpha, \in, A \cap V_\alpha, \hat{B} \cap V_\alpha) \models \neg\psi(\hat{B} \cap V_\alpha, A \cap V_\alpha).$$

Since $(V_\kappa, \in, A) \models \forall X \psi(X, A)$, we have $(V_\kappa, \in, A, \hat{B}) \models \psi(\hat{B}, A)$, and since κ is inaccessible, the set

$$\{\alpha < \kappa : (V_\alpha, \in, A \cap V_\alpha, \hat{B} \cap V_\alpha) \models \psi(\hat{B} \cap \alpha, A \cap \alpha)\}$$

contains a club. Thus, there is an $\alpha \in \hat{C}$ such that

$$(V_\alpha, \in, A \cap V_\alpha, \hat{B} \cap V_\alpha) \models \psi(\hat{B} \cap V_\alpha, A \cap V_\alpha),$$

a contradiction. □

Corollary 3.4. *Suppose κ is inaccessible, $\mathbb{P}$ is a κ-strategically closed forcing notion and G is generic for $\mathbb{P}$ over V. If N is a Π^1_1-correct κ-model in V, then N remains a Π^1_1-correct κ-model in $V[G]$.*

Proof. Clearly N remains a κ-model because $\mathbb{P}$ is $<\kappa$-distributive. Let φ be a Π^1_1-statement, and suppose first that $(V_\kappa \models \varphi)^N$. By Π^1_1-correctness, $V_\kappa \models \varphi$, and so by Lemma 3.3, $(V_\kappa \models \varphi)^{V[G]}$. On the other hand, if $(V_\kappa \models \neg\varphi)^N$, then there is a $B \subseteq V_\kappa$ in N witnessing this failure. Since N, V, and $V[G]$ all have the same V_κ, B witnesses the failure of φ in both V and $V[G]$ as well, so $(V_\kappa \models \neg\varphi)^{V[G]}$. □

Proposition 3.5. *Suppose κ is inaccessible, $\mathbb{P}$ is κ-strategically closed, and G is generic for $\mathbb{P}$ over V. If $S \in P(\kappa)^V$ is Π^1_1-indescribable in $V[G]$, then S is Π^1_1-indescribable in V.*

Proof. Suppose towards a contradiction that there is $S \in P(\kappa)^V$ that is Π^1_1-indescribable in $V[G]$ but not Π^1_1-indescribable in V. In V, find a subset $A \subseteq V_\kappa$ and a Π^1_1 statement $\varphi = \forall X \psi(X, A)$ such that $(V_\kappa, \in, A) \models \varphi$ and for all $\alpha \in S$ we have $(V_\alpha, \in, A \cap V_\alpha) \models \neg\varphi$. Since $\mathbb{P}$ is $<\kappa$-distributive, V and $V[G]$ have the same V_κ, so it follows that in $V[G]$, by the Π^1_1-indescribability of S, it must be the case that $(V_\kappa, \in, A) \models \exists X \neg\psi(X, A)$. Working in $V[G]$, we fix $B \subseteq V_\kappa$ such that $(V_\kappa, \in, A) \models \neg\psi(B, A)$ and observe that the set

$$C = \{\alpha < \kappa : V_\alpha \models \neg\psi(B \cap V_\alpha, A \cap V_\alpha)\}$$

contains a club. Let $\dot{C}$ be a $\mathbb{P}$-name for such a club, and let $\dot{B}$ be a $\mathbb{P}$-name for B. In V, we can use Player II's winning strategy in $\mathcal{G}_\kappa(\mathbb{P})$ together with $\dot{B}$ and $\dot{C}$ to build $\hat{B}$ and $\hat{C}$ such that $\hat{C} \subseteq \kappa$ is club and $\forall \alpha \in \hat{C}$, $V_\alpha \models \neg\psi(\hat{B} \cap V_\alpha, A \cap V_\alpha)$. But this implies that $V_\kappa \models \neg\psi(\hat{B}, A)$, a contradiction. $\square$

Hamkins showed [Ham98] that if κ is weakly compact, any forcing of size less than κ will produce an extension in which κ is *superdestructible* in the sense that any further $<\kappa$-closed forcing which adds a subset of κ will kill the weak compactness of κ. Thus, the converse of Proposition 3.5 is clearly false because the forcing $\mathrm{Add}(\kappa, 1)$ to add a Cohen subset to κ with bounded conditions can destroy the weak compactness of κ and it is $<\kappa$-closed and therefore κ-strategically closed. We will see in Section 6 (Remark 6.4) that Proposition 3.5 can fail for Π^1_2-indescribable sets.

A good iteration $\mathbb{P}_\kappa$ of length κ is said to be *progressively closed* if for every $\alpha < \kappa$, there is $\alpha \leq \beta_\alpha < \kappa$ such that every stage after β_α is forced to be α-strategically closed. In this case, it is not difficult to see that $\mathbb{P}_{\beta_\alpha}$ forces that the tail of the iteration is α-strategically closed. Next, we will show that good progressively closed κ-length iterations preserve Π^1_n-correctness of κ-models.

Let $\mathbb{P}$ be a forcing notion and suppose σ is a $\mathbb{P}$-name. Recall that τ is a *nice name for a subset of* σ if

$$\tau = \bigcup(\{\pi\} \times A_\pi : \pi \in \mathrm{dom}(\sigma)\},$$

where each A_π is an antichain of $\mathbb{P}$. It is well known and easy to verify that for every $\mathbb{P}$-name μ, there is a nice name τ for a subset of σ such that $1_{\mathbb{P}} \Vdash \mu \subseteq \sigma \rightarrow \mu = \tau$. We call such τ the *nice replacement for* μ.

Lemma 3.6. *Suppose σ is a $\mathbb{P}$-name and $n \geq 0$. Let X_σ be the set of nice names for subsets of σ, let p be a condition in $\mathbb{P}$ and let φ be any Π_n-assertion in the forcing language of the form*

$$(\forall x_1 \subseteq \sigma)(\exists x_2 \subseteq \sigma) \cdots \psi(x_1, \ldots, x_n).$$

Then $p \Vdash \varphi$ if and only if

$$(\forall \tau_1 \in X_\sigma)(\exists \tau_2 \in X_\sigma) \cdots p \Vdash \psi(\tau_1, \ldots, \tau_n).$$

The analogous statement holds for Σ_n-assertions in the forcing language.

Proof. We will prove the lemma simultaneously for Π_n and Σ_n statements by induction on n. Clearly the lemma holds for $n = 0$. Assume inductively that the lemma holds for some n, and suppose φ is an assertion in the forcing language of complexity Π_{n+1}. Let

$$\varphi = (\forall x_1 \subseteq \sigma)(\exists x_2 \subseteq \sigma) \cdots \psi(x_1, \ldots, x_{n+1}) = (\forall x_1 \subseteq \sigma)\bar{\varphi}(x_1),$$

where ψ is Δ_0 and $\bar{\varphi}(x)$ is Σ_n. For the forward direction, clearly $p \Vdash \varphi$ implies $(\forall \tau_1 \in X_\sigma)(p \Vdash \bar{\varphi}(\tau_1))$. By the inductive hypothesis applied to $p \Vdash \bar{\varphi}(\tau_1)$, we conclude that $(\forall \tau_1 \in X_\sigma)(\exists \tau_2 \in X_\sigma) \cdots p \Vdash \psi(\tau_1, \ldots, \tau_n)$. For the converse, suppose $(\forall \tau_1 \in X_\sigma)(\exists \tau_2 \in X_\sigma) \cdots p \Vdash \psi(\tau_1, \ldots, \tau_n)$ holds. Let us argue that $p \Vdash (\forall x_1 \subseteq \sigma)(\exists x_2 \subseteq \sigma) \cdots \psi(x_1, \ldots, x_n)$. If not, there is some $q \leq p$ and some $\mathbb{P}$-name μ for a subset of σ such that $q \Vdash (\forall x_2 \subseteq \sigma) \cdots \neg\psi(\mu, x_2, \ldots, x_n)$. Let τ be a nice replacement for μ so that $q \Vdash (\forall x_2 \subseteq \sigma) \cdots \neg\psi(\tau, x_2, \ldots, x_n)$, or in other words, $q \Vdash \neg\bar{\varphi}(\tau)$. By assumption $(\exists \tau_2 \in X_\sigma) \cdots p \Vdash \psi(\tau, \tau_2, \ldots, \tau_n)$, so applying the inductive hypothesis, we obtain $p \Vdash (\exists x_2 \subseteq \sigma) \cdots \psi(\tau, x_2, \ldots, x_n)$ and hence $p \Vdash \bar{\varphi}(\tau)$, a contradiction. The proof of the lemma for Σ_{n+1} statements is similar. □

Theorem 3.7. *Suppose κ is a Mahlo cardinal, N is a Π^1_n-correct κ-model and $\mathbb{P} \in N$ is a progressively closed good Easton-support iteration of length κ. If $G \subseteq \mathbb{P}$ is generic over V, then $N[G]$ is a Π^1_n-correct κ-model in $V[G]$.*

Proof. By Proposition 2.15, $\mathbb{P}$ has the κ-c.c. and without loss of generality $\mathbb{P} \subseteq V_\kappa$. Thus, by the generic closure criterion Lemma 3.2, $N[G]$ remains a κ-model in $V[G]$. By the progressive closure of the iteration, $V_\kappa^{V[G]} = V_\kappa[G]$. Thus, $V_\kappa^{N[G]} = V_\kappa^{V[G]}$. Let $\sigma \in N$ be a $\mathbb{P}$-name such that $\sigma_G = V_\kappa^{N[G]} = V_\kappa^{V[G]}$ and $\mathrm{dom}(\sigma) \subseteq V_\kappa$.

Let us argue that $N[G]$ is Π^1_n-correct. Suppose $(V_\kappa^{N[G]}, \in, A) \models \varphi$ in $N[G]$, where

$$\varphi = \forall X_1 \exists X_2 \cdots \psi(X_1, \ldots, X_n, A)$$

is Π^1_n and all quantifiers appearing in ψ are first-order over $V_\kappa^{N[G]}$. Let $\dot{A}$ be a $\mathbb{P}$-name for A such that $\mathrm{dom}(\dot{A}) \subseteq V_\kappa$. Let $\bar{\psi}(x_1, \ldots, x_n, \dot{A})$ be a formula in the forcing language obtained from ψ by replacing all parameters with $\mathbb{P}$-names and all first-order quantifiers "Qx" with "$Qx \in \sigma$" for $Q = \forall, \exists$. Let $\bar{\varphi}(\sigma, \dot{A})$ denote the following formula in the forcing language:

$$(\forall x_1 \subseteq \sigma)(\exists x_2 \subseteq \sigma) \cdots \bar{\psi}(x_1, \ldots, x_n, \dot{A}).$$

Since $(V_\kappa^{N[G]}, \in, A) \models \varphi$ holds in $N[G]$, it follows that $N[G] \models \bar{\varphi}(\sigma_G, \dot{A}_G)$. Thus, we may choose $p \in G$ with $(p \Vdash \bar{\varphi}(\sigma, \dot{A}))^N$. By Lemma 3.6,

$$(\forall \tau_1 \in X_\sigma)(\exists \tau_2 \in X_\sigma) \cdots p \Vdash \psi(\tau_1, \ldots, \tau_n, \dot{A}) \tag{3.1}$$

holds in N. The statement $p \Vdash \psi(\tau_1, \ldots, \tau_n, \dot{A})$ is first-order in the structure $(V_\kappa, \in, \tau_1, \ldots, \tau_n, \sigma, \dot{A}, \mathbb{P})$. This requires that the forcing relation for $\mathbb{P}$ is definable over V_κ, and in general would not hold for all class partial orders, but does hold for pretame partial orders (see [Fri00] for definition and proof) and progressively closed Easton-support iterations are pretame (see for example, section"Reverse Easton Forcing" in Chapter 2 of [Fri00]). Furthermore, since "$\tau \in X_\sigma$" can be expressed by a first-order formula $\chi(\tau, \sigma)$ over $(V_\kappa, \in, \sigma, \tau, \mathbb{P})$, it follows that the statement in (3.1) is Π^1_n over $(V_\kappa, \in, \sigma, \dot{A})$. Since $N \models$ "(3.1) holds in $(V_\kappa, \in, \sigma, \dot{A})$" and N is Π^1_n-correct at κ, it follows that (3.1) holds in $(V_\kappa, \in, \sigma, \dot{A})$. Hence by Lemma 3.6, $p \Vdash \bar{\varphi}(\sigma, \dot{A})$ over V, and since $p \in G$, we conclude that $V[G] \models \bar{\varphi}(\sigma, \dot{A}_G)$, which implies $(V_\kappa^{V[G]}, \in, A) \models \varphi$ in $V[G]$.

An analogous argument establishes the converse, verifying that, if

$$(V_\kappa^{V[G]}, \in, A) \models \varphi$$

for a Π^1_n-assertion φ and $A \in N[G]$, then the same assertion holds in $N[G]$. □

A similar argument yields the following result.

Corollary 3.8. *Suppose κ is an inaccessible cardinal, N is a Π^1_n-correct κ-model and $\mathbb{P} \in N$ is a $<\kappa$-distributive forcing notion of size κ. If $G \subseteq \mathbb{P}$ is generic over V, then $N[G]$ remains a Π^1_n-correct κ-model in $V[G]$.*

Proof. Without loss of generality we assume that $\mathbb{P} \subseteq V_\kappa$. Since $\mathbb{P}$ is $<\kappa$-distributive, N remains a κ-model in $V[G]$, and, since $G \in V[G]$, it follows that $N[G]$ is a κ-model in $V[G]$ by the ground closure criterion, Lemma 3.1, applied in $V[G]$. The $<\kappa$-distributivity of $\mathbb{P}$ entails that $V_\kappa^{N[G]} = V_\kappa^{V[G]} = V_\kappa$ and that $\mathbb{P}$ is a pretame class forcing over V_κ (see [Fri00] for proof that $<$Ord-distributivity implies pretameness) ensuring the definability of the forcing relation. Since the statement "τ is a nice name for a subset of $\check{V}_\kappa$" is first-order over the structure $(V_\kappa, \in, \tau, \mathbb{P})$, the rest of the argument can be carried out as in the proof of Theorem 3.7. $\square$

The conclusion of Corollary 3.8 need not hold if the N-generic filter G is not fully V-generic (see Remark 5.7).

4. Shooting n-clubs

Hellsten [Hel10] showed that if $W \subseteq \kappa$ is any Π^1_1-indescribable (i.e., weakly compact) subset of κ, then there is a forcing extension in which W contains a 1-club and all weakly compact subsets of W remain weakly compact. We will define a generalization of Hellsten's forcing to shoot an n-club through a Π^1_n-indescribable subset of a cardinal κ while preserving the Π^1_n-indescribability of all its subsets, so that, in particular, κ remains Π^1_n-indescribable in the forcing extension.

Suppose γ is an inaccessible cardinal and $A \subseteq \gamma$ is cofinal. For $n \geq 1$, we define a poset $T^n(A)$ consisting of all bounded n-closed $c \subseteq A$ ordered by end extension: $c \leq d$ if and only if $d = c \cap \sup_{\alpha \in d}(\alpha + 1)$.

We now argue that, if $n \geq 1$, γ is inaccessible, and $A \subseteq \gamma$ is cofinal, then $T^n(A)$ is γ-strategically closed. We will actually prove a stronger statement, for which we need to introduce some new terminology. Given an ordinal γ, a subset $X \subseteq \gamma$, and a poset $\mathbb{P}$, let $\mathcal{G}_{\gamma,X}(\mathbb{P})$ be the modification of $\mathcal{G}_\gamma(\mathbb{P})$ in which Player I plays at all stages indexed by an ordinal in X and Player II plays elsewhere, and it is still the case that Player I wins if and only if there is a limit ordinal $\beta < \gamma$ such that $\langle p_\alpha : \alpha < \beta \rangle$ has no lower bound in $\mathbb{P}$. So $\mathcal{G}_\gamma(\mathbb{P})$ is precisely the game $\mathcal{G}_{\gamma,X}(\mathbb{P})$, where X is the set of odd ordinals less than γ.

Lemma 4.1. *Suppose that $n \geq 1$, γ is inaccessible, $A \subseteq \gamma$ is cofinal, and $X \subseteq \gamma$ is such that, for all $\beta < \gamma$, $X \cap \beta$ is not a Π^1_{n-1}-indescribable subset of β. Then Player II has a winning strategy in the game $\mathcal{G}_{\gamma,X}(T^n(A))$.*

Proof. We describe Player II's winning strategy in $\mathcal{G}_{\gamma,X}(T^n(A))$. Suppose that $\alpha \in \gamma \setminus X$ and $\langle c_\beta : \beta < \alpha \rangle$ is a partial run of the game in which Player II has thus far played according to the prescribed winning strategy. We describe how Player II chooses her next play, c_α.

If $\alpha = \alpha_0 + 1$ is a successor ordinal, then Player II simply plays any condition $c_\alpha \lneq c_{\alpha_0}$. If α is a limit ordinal, then Player II records an ordinal $\gamma_\alpha = \bigcup_{\beta<\alpha} c_\beta$, chooses an element $\eta_\alpha \in A \setminus (\gamma_\alpha + 1)$, and plays $c_\alpha = \left(\bigcup_{\beta<\alpha} c_\beta\right) \cup \{\eta_\alpha\}$. Notice, in particular, that $\gamma_\alpha \notin c_\alpha$. In order to argue that c_α is a condition in $T^n(A)$, we need to verify, letting $c = \bigcup_{\beta<\alpha} c_\beta$, that c is not a Π^1_{n-1}-indescribable subset of γ_α.

We can assume that γ_α is Π^1_{n-1}-indescribable, as otherwise c is clearly not Π^1_{n-1}-indescribable in γ_α. In particular, we have $\alpha \geq \gamma_\alpha$, since $\{\sup(c_\beta) : \beta < \alpha\}$ is cofinal in γ_α. But, since $X \cap \gamma_\alpha$ is not Π^1_{n-1}-indescribable by assumption, we know that $\gamma_\alpha \setminus X$ is cofinal in γ_α, so, by the definition of our strategy for Player II, $\langle \sup(c_\beta) : \beta \in \gamma_\alpha \setminus X\rangle$ is strictly increasing and thus cofinal in γ_α. Therefore, if $\alpha > \gamma_\alpha$, then it must be the case that $c = c_{\gamma_\alpha}$, and hence c is not Π^1_{n-1}-indescribable in γ_α due to the fact that $c_{\gamma_\alpha} \in T^n(A)$ is n-closed in γ and $\gamma_\alpha \notin c_{\gamma_\alpha}$.

We may therefore assume that $\alpha = \gamma_\alpha$. For every limit ordinal $\xi < \alpha$, let $\gamma_\xi = \bigcup_{\beta<\xi} c_\beta$. Then $\{\gamma_\xi : \xi < \alpha \text{ is a limit ordinal}\}$ is a club in γ_α. Moreover, if ξ is a limit ordinal in $\alpha \setminus X$, then, since Player II played according to their winning strategy at stage ξ, we have $\gamma_\xi \notin c$.

Let $D = \{\xi < \alpha : \xi \text{ is a limit ordinal and } \xi = \gamma_\xi\}$. Then D is a club (and hence an $(n-1)$-club) in $\alpha = \gamma_\alpha$. Since γ_α is Π^1_{n-1}-indescribable and, by assumption, $X \cap \gamma_\alpha$ is not Π^1_{n-1}-indescribable, we can fix an $(n-1)$-club E in γ_α such that $E \cap X = \varnothing$. Then $E \cap D$ is an $(n-1)$-club in γ_α and $E \cap D \cap c = \varnothing$, so c is not a Π^1_{n-1}-indescribable subset of γ_α. Thus, c_α is a valid play by Player II, and we have described a winning strategy in $\mathcal{G}_\kappa(T^n(A))$. □

Remark 4.2. The set of odd ordinals less than γ clearly satisfies the hypothesis of Lemma 4.1, so the lemma indeed implies that $T^n(A)$ is γ-strategically closed. But the lemma also applies to larger sets X, such as the set of all $\alpha < \gamma$ such that α is not Π^1_{n-2}-indescribable.

Theorem 4.3. *Suppose that $n \geq 1$ and $S \subseteq \kappa$ is Π^1_n-indescribable. Then there is a forcing extension in which S contains a 1-club and all Π^1_n-indescribable subsets of S from V remain Π^1_n-indescribable.*

Proof. Let $\mathbb{P}_{\kappa+1} = \langle(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta) : \alpha \leq \kappa+1,\ \beta \leq \kappa\rangle$ be an Easton-support iteration such that

- if $\gamma \leq \kappa$ is inaccessible and $S \cap \gamma$ is cofinal in γ, then $\dot{\mathbb{Q}}_\gamma = (T^1(S\cap\gamma))^{V^{\mathbb{P}_\gamma}}$;
- otherwise, $\dot{\mathbb{Q}}_\gamma$ is a $\mathbb{P}_\gamma$-name for trivial forcing.

Since κ is Π^1_n-indescribable, Proposition 2.15 implies that $\mathbb{P}_\kappa$ has size κ and the κ-c.c.. Forcing with $\mathbb{P}_{\kappa+1}$ therefore preserves the inaccessibility of κ because $\mathbb{P}_\kappa$ has the κ-c.c. and is progressively closed and $\dot{\mathbb{Q}}_\kappa$ is forced to be $<\kappa$-distributive.

Suppose $G * H \subseteq \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$ is generic over V. Let $C(\kappa) =_{\text{def}} \bigcup H$. We will show that, in $V[G*H]$, $C(\kappa)$ is a 1-club subset of κ; notice that it suffices to show that $C(\kappa)$ is a stationary subset of κ in $V[G*H]$ because then $C(\kappa)$ is clearly 1-closed since it is a union of a filter contained in $\mathbb{Q}_\kappa = T^1(S)^{V^{\mathbb{P}_\kappa}}$.

Suppose $T \subseteq S$ is Π^1_n-indescribable in V. We will simultaneously show that in $V[G*H]$, $C(\kappa)$ has nonempty intersection with every club subset of κ and T remains Π^1_n-indescribable (in particular, κ remains Π^1_n-indescribable). Fix $A, C \in P(\kappa)^{V[G*H]}$ such that C is a club subset of κ in $V[G*H]$. Let $\dot{A}, \dot{C}, \dot{C}(\kappa) \in H(\kappa^+)$ be $\mathbb{P}_{\kappa+1}$-names such $\dot{A}_{G*H} = A$, $\dot{C}_{G*H} = C$ and $\dot{C}(\kappa)_{G*H} = C(\kappa)$. In V, let M be a κ-model with $\dot{A}, \dot{C}, \dot{C}(\kappa), \mathbb{P}_{\kappa+1}, T, S \in M$. Since T is Π^1_n-indescribable in V, it follows by Theorem 2.6 that there is a Π^1_{n-1}-correct κ-model N and an elementary embedding $j : M \to N$ with critical point κ such that $\kappa \in j(T)$.

Since $N^{<\kappa} \cap V \subseteq N$ and $j(S) \cap \kappa = S$, it follows that $j(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * \dot{T}^1(S) * \dot{\mathbb{P}}_{\kappa,j(\kappa)}$, where $\dot{\mathbb{P}}_{\kappa,j(\kappa)}$ is a $\mathbb{P}_{\kappa+1}$-name for the tail of the iteration $j(\mathbb{P}_\kappa)$. Since $\mathbb{P}_\kappa$ has the κ-c.c., by the generic closure criterion (Lemma 3.2), $N[G]$ is a κ-model in $V[G]$. Since

$T^1(S)$ is κ-strategically closed, $N[G]$ remains a κ-model in $V[G*H]$, and hence by the ground closure criterion (Lemma 3.1), $N[G*H]$ is a κ-model in $V[G*H]$. Since $\mathbb{P}_{\kappa,j(\kappa)} = (\dot{\mathbb{P}}_{\kappa,j(\kappa)})_{G*H}$ is κ-strategically closed in $N[G*H]$ and $N[G*H]$ is a κ-model in $V[G*H]$, it follows that there is a filter $G' \in V[G*H]$ which is generic for $\mathbb{P}_{\kappa,j(\kappa)}$ over $N[G*H]$ and the embedding j lifts to $j : M[G] \to N[\hat{G}]$, where $\hat{G} \cong G * H * G'$.

Notice that $p = C(\kappa) \cup \{\kappa\} = \bigcup H \cup \{\kappa\} \in N[\hat{G}]$. Since $\kappa \in j(T) \subseteq j(S)$, we see that $N[\hat{G}] \models$ "p is a closed subset of $j(S)$". Thus, $p \in j(T^1(S))$. Since $j(T^1(S))$ is $j(\kappa)$-strategically closed in $N[\hat{G}]$ and $N[\hat{G}]$ is a κ-model in $V[G*H]$ by the ground closure criterion, there is a filter $\hat{H} \in V[G*H]$ generic for $j(T^1(S))$ over $N[\hat{G}]$ with $p \in \hat{H}$. Since p is below every condition in $j\,``\,H$, we have $j\,``\,H \subseteq \hat{H}$, and thus j lifts to $j : M[G*H] \to N[\hat{G}*\hat{H}]$, where $\kappa \in j(C(\kappa))$. By Theorem 3.7 and Corollary 3.8, $N[G*H]$ is a Π^1_{n-1}-correct κ-model in $V[G*H]$. Since $\mathbb{P}_{\kappa,j(\kappa)}$ and $j(T^1(S))$ are $(\kappa+1)$-strategically closed in $N[G*H]$, it follows that $N[G*H]$ and $N[\hat{G}*\hat{H}]$ have the same subsets of V_κ, so, in particular, $N[\hat{G}*\hat{H}]$ is a Π^1_{n-1}-correct κ-model in $V[G*H]$. Thus, by Theorem 2.6, we have verified that T remains Π^1_n-indescribable in $V[G*H]$.

It remains to show that $C(\kappa) \cap C \neq \varnothing$. Recall that C is a club subset of κ in $V[G*H]$, so $j(C)$ is a club subset of $j(\kappa)$ in $N[\hat{G}*\hat{H}]$. Since $j(C) \cap \kappa = C$, it follows that $\kappa \in j(C)$, and hence $\kappa \in j(C(\kappa) \cap C)$. By elementarity, $C(\kappa) \cap C \neq \varnothing$, so $C(\kappa)$ is a stationary and hence 1-club subset of κ in $V[G*H]$. $\square$

Remark 4.4. In the proof of Theorem 4.3, for any $m \leq n$ we can force with $T^m(S \cap \gamma)$ at every relevant $\gamma \leq \kappa$ instead of $T^1(S \cap \gamma)$. This iteration will still preserve the Π^1_n-indescribability of every subset of S that is Π^1_n-indescribable in V, and it will shoot an m-club through S. If $m > 1$, then this forcing will have slightly better closure properties then $T^1(S \cap \gamma)$ (see Lemma 4.1), which could be useful for certain applications, though we have not found any such applications as of yet.

5. $\square_1(\kappa)$ CAN HOLD NONTRIVIALLY AT A WEAKLY COMPACT CARDINAL

In this section, we will prove Theorem 1.1, which states that if κ is κ^+-weakly compact then the principle $\square_1(\kappa)$ can be forced to hold at κ while preserving the κ^+-weak compactness of κ. Let us remind the reader that the corresponding relative consistency result was first obtained by Brickhill and Welch [BW] assuming $V = L$.

First, we define a forcing to add a generic coherent sequence of 1-clubs to a Mahlo cardinal κ.

Definition 5.1. Suppose κ is a Mahlo cardinal. We define a forcing $\mathbb{Q}(\kappa)$ such that q is a condition in $\mathbb{Q}(\kappa)$ if and only if

- q is a sequence with $\mathrm{dom}(q) = \mathrm{inacc}(\kappa) \cap (\gamma^q + 1)$ for some $\gamma^q < \kappa$,
- $q(\alpha) = C^q_\alpha$ is a 1-club subset of α for each $\alpha \in \mathrm{dom}(q)$ and
- for all $\alpha, \beta \in \mathrm{dom}(q)$, if $C^q_\beta \cap \alpha \in \Pi^1_0(\alpha)^+$, then $C^q_\alpha = C^q_\beta \cap \alpha$.[2]

The ordering on $\mathbb{Q}(\kappa)$ is defined by letting $p \leq q$ if and only if p is an end extension of q.

[2]Equivalently, for all $\alpha, \beta \in \mathrm{dom}(q)$, if α is inaccessible and $C^q_\beta \cap \alpha$ is stationary, then $C^q_\alpha = C^q_\beta \cap \alpha$.

Proposition 5.2. *Suppose κ is a Mahlo cardinal. The poset $\mathbb{Q}(\kappa)$ is κ-strategically closed.*

Proof. We describe a winning strategy for Player II in the game $\mathcal{G}_\kappa(\mathbb{Q}(\kappa))$. We will recursively arrange so that, if $\delta < \kappa$ and $\langle q_\alpha : \alpha < \delta\rangle$ is a partial play of the game with Player II playing according to her winning strategy, then, for all limit ordinals $\beta < \delta$, we have $\{\gamma^{q_\alpha} : \alpha < \beta,\ \alpha \text{ even}\}$ is a club in its supremum and, if γ^{q_β} is inaccessible, is a subset of $C^{q_\beta}_{\gamma^{q_\beta}}$. We will also arrange that, for all even successor ordinals $\alpha < \beta < \delta$, γ^{q_α} and γ^{q_β} are inaccessible cardinals and $C^{q_\beta}_{\gamma^{q_\beta}} \cap \gamma^{q_\alpha} = C^{q_\alpha}_{\gamma^{q_\alpha}}$.

We first deal with successor ordinals. Suppose that $\delta < \kappa$ is an even ordinal and $\langle q_\alpha : \alpha \leq \delta + 1\rangle$ has been played. Suppose first that γ^{q_δ} is an inaccessible cardinal (in particular, by our recursion hypotheses, this must be the case if δ is a successor ordinal). In this case, let $\gamma^{q_{\delta+2}}$ be the least inaccessible cardinal above $\gamma^{q_{\delta+1}}$ and let $q_{\delta+2}$ be the condition extending $q_{\delta+1}$ by setting

$$C^{q_{\delta+2}}_{\gamma^{q_{\delta+2}}} = C^{q_\delta}_{\gamma^{q_\delta}} \cup \{\gamma^{q_\delta}\} \cup [\gamma^{q_{\delta+1}}, \gamma^{q_{\delta+2}}).$$

The fact that $C^{q_\delta}_{\gamma^{q_\delta}} \cup \{\gamma^{q_\delta}\} \subseteq C^{q_{\delta+2}}_{\gamma^{q_{\delta+2}}}$ ensures that the recursion hypothesis is maintained. The set $C^{q_{\delta+2}}_{\gamma^{q_{\delta+2}}}$ is stationary in $\gamma^{q_{\delta+2}}$ because it contains a tail, and it has all of its inaccessible stationary reflection points because those are $\leq\gamma^{q_\delta}$. The coherence property holds because we have omitted the interval $(\gamma^{q_\delta}, \gamma^{q_{\delta+1}})$ from $C^{q_{\delta+2}}_{\gamma^{q_{\delta+2}}}$ ensuring that for no α in that interval is $C^{q_{\delta+2}}_{\gamma^{q_{\delta+2}}} \cap \alpha$ stationary. It follows that $q_{\delta+2}$ is a condition and a valid play for Player II.

If γ^{q_δ} is not inaccessible, then δ is a limit ordinal (by our recursion hypothesis). In this case, again let $\gamma^{q_{\delta+2}}$ be the least inaccessible cardinal above $\gamma^{q_{\delta+1}}$, and define $q_{\delta+2}$ by setting

$$C^{q_{\delta+2}}_{\gamma^{q_{\delta+2}}} = \bigcup_{\substack{\alpha<\delta \\ \alpha \text{ even}}} C^{q_\delta}_{\gamma^{q_{\alpha+2}}} \cup \{\gamma^{q_\delta}\} \cup [\gamma^{q_{\delta+1}}, \gamma^{q_{\delta+2}}).$$

A similar argument as above verifies that $q_{\delta+2}$ is a valid play in the game and maintains our recursion hypotheses.

Finally, suppose that $\delta < \kappa$ is a limit ordinal and $\langle q_\alpha : \alpha < \delta\rangle$ has been played. Let $\gamma^{q_\delta} = \sup\{\gamma^{q_\alpha} : \alpha < \delta\}$. If γ^{q_δ} is not inaccessible, then we can simply set $q_\delta = \bigcup_{\alpha<\delta} q_\alpha$. If γ^{q_δ} is inaccessible, then we must additionally define $C^{q_\delta}_{\gamma^{q_\delta}}$. We do this by setting

$$C^{q_\delta}_{\gamma^{q_\delta}} = \bigcup_{\substack{\alpha<\delta \\ \alpha \text{ even}}} C^{q_\delta}_{\gamma^{q_{\alpha+2}}}.$$

It is easy to verify that this is as desired. The fact that $C^{q_\delta}_{\gamma^{q_\delta}}$ is stationary in γ^{q_δ} follows from the fact that $\{\gamma^{q_\alpha} : \alpha < \delta,\ \alpha \text{ even}\} \subseteq C^{q_\delta}_{\gamma^{q_\delta}}$, so it in fact contains a club in γ^{q_δ}. □

It follows from Proposition 5.2 that $\mathbb{Q}(\kappa)$ is $<\kappa$-distributive. In particular, if $G \subseteq \mathbb{Q}(\kappa)$ is a generic filter, then $\bigcup G$ is a coherent sequence of 1-clubs of length κ, since V_κ remains unchanged.

Next, we define a forcing which will be used to generically thread a coherent sequence of 1-clubs.

Definition 5.3. Suppose that $\vec{C}(\kappa) = \langle C_\alpha(\kappa) : \alpha \in \text{inacc}(\kappa)\rangle$ is a coherent sequence of 1-clubs. The poset $\mathbb{T}(\vec{C}(\kappa))$ consists of all conditions t such that

- t is a 1-closed bounded subset of κ and
- for every $\alpha < \kappa$, if $t \cap \alpha \in \Pi^1_0(\alpha)^+$, then $C_\alpha(\kappa) = t \cap \alpha$.[3]

The ordering on $\mathbb{T}(\vec{C}(\kappa))$ is defined by letting $t \leq s$ if and only if t end-extends s.

Lemma 5.4. *Suppose κ is a regular cardinal and $\vec{C}(\kappa)$ is a coherent sequence of 1-clubs. Then the poset $\mathbb{T}(\vec{C}(\kappa))$ is κ-strategically closed.*[4]

Proof. We describe a winning strategy for player II in $\mathcal{G}_\kappa(\mathbb{T}(\vec{C}(\kappa)))$. Player II's strategy at successor ordinal stages can be arbitrary provided that Player II chooses conditions properly extending Player I's previous play.

So let δ be a limit stage and let $\langle t_\alpha : \alpha < \delta\rangle$ be the sequence of conditions played at previous stages of the game. Player II then plays $t_\delta = \left(\bigcup_{\alpha<\delta} t_\alpha\right) \cup \{\kappa_\delta + 1\}$, where $\kappa_\delta = \sup\left(\bigcup_{\alpha<\delta} t_\alpha\right)$. We will also assume recursively that Player II has played according to this strategy successfully at previous limit stages of the game, so that, if $\lambda < \delta$ is a limit ordinal, then $\kappa_\lambda \notin t_\delta$. It remains to show that $t_\delta \in \mathbb{T}(\vec{C}(\kappa))$.

To argue that t_δ is a 1-closed subset of κ, it suffices to see that $t_\delta \cap \kappa_\delta$ is not stationary in κ_δ. By our recursive assumption, $\{\kappa_\lambda : \lambda < \delta\}$ is a club subset of κ_δ disjoint from t_δ, and hence $t_\delta \cap \kappa_\delta$ is not stationary in κ_δ. The coherence condition follows easily. $\square$

Lemma 5.5. *Suppose κ is a regular cardinal and $\vec{C}(\kappa) = \langle C_\alpha(\kappa) : \alpha \in \mathrm{inacc}(\kappa)\rangle$ is a coherent sequence of 1-clubs. If $G \subseteq \mathbb{T}(\vec{C}(\kappa))$ is generic over V, then $C_\kappa = \bigcup G$ threads $\vec{C}(\kappa)$ in $V[G]$.*

Proof. By the $<\kappa$-distributivity of $\mathbb{T}(\vec{C}(\kappa))$ and the definition of its conditions, C_κ meets the coherence requirements and contains all of its inaccessible stationary reflection points. So it remains to check that C_κ is stationary.

Fix a club $C \subseteq \kappa$ in $V[G]$ and let $\dot{C}$ be a $\mathbb{T}(\vec{C}(\kappa))$-name for C. Assume towards a contradiction that $C \cap C_\kappa = \varnothing$. Fix $t_0 \in \mathbb{T}(\vec{C}(\kappa))$ forcing that $\dot{C}$ is a club and $\dot{C} \cap \dot{C}_\kappa = \varnothing$, where $\dot{C}_\kappa$ is the canonical $\mathbb{T}(\vec{C}(\kappa))$-name for C_κ, and let β_0 be the supremum of t_0. Recursively define a decreasing sequence $\langle t_n : n < \omega\rangle$ of conditions from $\mathbb{T}(\vec{C}(\kappa))$ as follows, letting β_n denote $\sup(t_n)$. Given $n < \omega$, if t_n is defined, find an ordinal α_n with $\beta_n < \alpha_n < \kappa$ and a condition $t_{n+1} \leq t_n$ such that $t_{n+1} \Vdash \check{\alpha}_n \in \dot{C}$. Let $\alpha = \bigcup_{n<\omega} \alpha_n = \bigcup_{n<\omega} \beta_n$, and let $t = \bigcup_{n<\omega} t_n \cup \{\alpha\}$. Clearly t is a condition in $\mathbb{T}(\vec{C}(\kappa))$ and $t \Vdash \alpha \in \dot{C} \cap \dot{C}_\kappa$, which is the desired contradiction. $\square$

Theorem 5.6. *Suppose κ is weakly compact and the* GCH *holds. There is a cofinality-preserving forcing extension in which*

1. *for all $\gamma \leq \kappa$, every set $W \in P(\gamma)^V$ which is weakly compact in V remains weakly compact and*
2. *$\square_1(\kappa)$ holds.*

Proof. Define an Easton-support iteration $\langle(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta) : \alpha \leq \kappa+1,\ \beta \leq \kappa\rangle$ as follows.

- If $\gamma < \kappa$ is Mahlo, let $\dot{\mathbb{Q}}_\gamma = (\mathbb{Q}(\gamma) * \dot{\mathbb{T}}(\vec{C}(\gamma)))^{V^{\mathbb{P}_\gamma}}$, where $\vec{C}(\gamma)$ is the generic coherent sequence of 1-clubs of length γ added by $\mathbb{Q}(\gamma)$.
- If $\gamma = \kappa$, let $\dot{\mathbb{Q}}_\kappa = (\mathbb{Q}(\gamma))^{V^{\mathbb{P}_\kappa}}$.

[3]Equivalently, for every inaccessible cardinal $\alpha < \kappa$, if $t \cap \alpha$ is stationary in α then $C_\alpha(\kappa) = t \cap \alpha$.
[4]Note that the forcing to thread a $\square(\kappa)$-sequence is never κ-strategically closed.

- Otherwise, let $\dot{\mathbb{Q}}_\gamma$ be a $\mathbb{P}_\gamma$-name for trivial forcing.

Let $G * H \subseteq \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$ be generic over V. $V[G*H]$ is our desired model. Standard arguments using progressive closure of the iteration $\mathbb{P}_\kappa$ together with the GCH (see [Ham94]) show that cofinalities are preserved in $V[G*H]$.

The argument for the preservation of weakly compact subsets of $\gamma < \kappa$ is similar to and easier than the argument for the preservation of weakly compact subsets of κ, which is given next.

Recall that $\vec{C}(\kappa) = \bigcup H$ is a coherent sequence of 1-clubs of length κ. Fix $W \in P(\kappa)^V$ which is weakly compact in V. It remains to argue that in $V[G*H]$, W is weakly compact and $\vec{C}(\kappa)$ has no thread.

Fix a set $C \in P(\kappa)^{V[G*H]}$ which is a 1-club subset of κ in $V[G*H]$. We will simultaneously show that C is not a thread through $\vec{C}(\kappa)$ and that W remains weakly compact in $V[G*H]$. Fix $A \in P(\kappa)^{V[G*H]}$ and let $\dot{C}, \dot{A}, \tau \in H(\kappa^+)^V$ be $\mathbb{P}_{\kappa+1}$-names with $\dot{C}_{G*H} = C$, $\dot{A}_{G*H} = A$ and $\tau_{G*H} = \vec{C}(\kappa)$. Let M be a κ-model with $W, \dot{C}, \dot{A}, \tau, \mathbb{P}_{\kappa+1} \in M$. Since W is weakly compact in V, there is a κ-model N and an elementary embedding $j : M \to N$ such that $\mathrm{crit}(j) = \kappa$ and $\kappa \in j(W)$.

Since $N^{<\kappa} \cap V \subseteq N$, we have, in N,

$$j(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * (\dot{\mathbb{Q}}(\kappa) * \dot{\mathbb{T}}(\vec{C}(\kappa))) * \dot{\mathbb{P}}_{\kappa, j(\kappa)},$$

where $\dot{\mathbb{P}}_{\kappa,j(\kappa)}$ is a $\mathbb{P}_{\kappa+1} * \dot{\mathbb{T}}(\vec{C}(\kappa))$-name for the iteration from $\kappa+1$ to $j(\kappa)$. By Lemma 5.4, $\mathbb{T}(\vec{C}(\kappa))$ is κ-strategically closed in $N[G*H]$, and hence, using standard arguments, we can build a filter $h \in V[G*H]$ for $\mathbb{T}(\vec{C}(\kappa))$ which is generic over $N[G*H]$. Let $C_\kappa = \bigcup h$ and notice that $C_\kappa \neq C$ because $C \in N[G*H]$ and C_κ is generic over $N[G*H]$. Similarly, we can build a filter $G' \in V[G*H]$ which is generic for $\mathbb{P}_{\kappa,j(\kappa)} = (\dot{\mathbb{P}}_{\kappa,j(\kappa)})_{G*H*h}$ over $N[G*H*h]$. Since $j \,``\, G \subseteq G*H*h*G'$, the embedding can be extended to $j : M[G] \to N[\hat{G}]$, where $\hat{G} = G*H*h*G'$.

Let $\mathbb{Q}(\kappa) = (\dot{\mathbb{Q}}(\kappa))_G$. Working in $N[\hat{G}]$, since

$$\vec{C}(\kappa) = \bigcup H = \langle C_\alpha(\kappa) : \alpha \in \mathrm{inacc}(\kappa)\rangle$$

is a coherent sequence of 1-clubs and C_κ is a thread through $\vec{C}(\kappa)$ by Lemma 5.5, it follows that the function

$$q = \langle C_\alpha(\kappa) : \alpha \in \mathrm{inacc}(\kappa)\rangle \cup \{(\kappa, C_\kappa)\}$$

is a condition in $j(\mathbb{Q}(\kappa))$ below every element of $j \,``\, H$. We may build a filter $\hat{H} \in V[G*H]$ which is generic for $j(\mathbb{Q}(\kappa))$ over $N[\hat{G}]$ with $q \in \hat{H}$. Since $j \,``\, H \subseteq \hat{H}$, it follows that j extends to $j : M[G*H] \to N[\hat{G} * \hat{H}]$. Now $A \in M[G*H]$ and $\kappa \in j(W)$, so W is weakly compact in $V[G*H]$.

It remains to show that C is not a thread through $\vec{C}(\kappa)$. For the sake of contradiction, assume C is a thread through $\vec{C}(\kappa)$. By elementarity we see that in $N[\hat{G} * \hat{H}]$,

$$j(\vec{C}(\kappa)) = \langle \bar{C}_\alpha(j(\kappa)) : \alpha \in \mathrm{inacc}(j(\kappa))\rangle$$

is a coherent sequence of 1-clubs. Since $q = \vec{C}(\kappa) ^\frown \langle C_\kappa\rangle \in \hat{H}$ we have $\bar{C}_\kappa(j(\kappa)) = C_\kappa$. Now since C is a thread for $\vec{C}(\kappa)$ in $M[G*H]$, by elementarity, $j(C)$ is a thread for $j(\vec{C}(\kappa)) = \langle \bar{C}_\alpha(j(\kappa)) : \alpha \in \mathrm{inacc}(j(\kappa))\rangle$. Since κ is inaccessible in $N[\hat{G} * \hat{H}]$ and $\kappa \in \mathrm{Tr}_0(j(C))$, it follows that $C_\kappa = \bar{C}_\kappa(j(\kappa)) = j(C) \cap \kappa = C$, a contradiction. $\square$

Remark 5.7. Observe that in the proof of Theorem 5.6, if we assume that κ is Π^1_2-indescribable and that the target N of the embedding $j : M \to N$ we start with is Π^1_1-correct, then the κ-model $N[G*H]$ from the proof of Theorem 5.6 is Π^1_1-correct by Theorem 3.7 and Corollary 3.8. However, the κ-model $N[G*H*h]$ cannot be Π^1_1-correct because otherwise we would have shown that, in the extension $V[G*H]$, κ is Π^1_2-indescribable, contradicting Proposition 2.9. Thus, a forcing extension of a Π^1_1-correct κ-model, even by a κ-strategically closed forcing notion, need not be Π^1_1-correct if the generic filter is not fully V-generic.

For the next theorem, let us recall what it means for a cardinal κ to be α-weakly compact, where $\alpha \leq \kappa^+$. Suppose κ is a weakly compact cardinal. It is not difficult to see that if sets $X, Y \in P(\kappa)$ are equivalent modulo the ideal $\Pi^1_1(\kappa)$, then their traces $\mathrm{Tr}_1(X)$ and $\mathrm{Tr}_1(Y)$ are equivalent as well. Thus, the trace operation $\mathrm{Tr}_1 : P(\kappa) \to P(\kappa)$ leads to a well defined operation $\mathrm{Tr}_1 : P(\kappa)/\Pi^1_1(\kappa) \to P(\kappa)/\Pi^1_1(\kappa)$ on the collection $P(\kappa)/\Pi^1_1(\kappa)$ of equivalence classes of subsets of κ modulo the ideal $\Pi^1_1(\kappa)$. By taking diagonal intersections at limit ordinals, we can iterate the trace operation on the equivalence classes κ^+-many times (see [BTW77, Section 4] for a similar construction related to Mahloness). To be more precise, fix a sequence $\langle e_\beta \mid \kappa \leq \beta < \kappa^+,\ \beta \text{ limit}\rangle$, where $e_\beta : \kappa \to \beta$ is a bijection for all relevant β. To start, let $\mathrm{Tr}^0_1([S]) = [S]$ for all $[S] \in P(\kappa)/\Pi^1_1(\kappa)$. Given $\alpha < \kappa^+$, if $\mathrm{Tr}^\alpha_1 : P(\kappa)/\Pi^1_1(\kappa) \to P(\kappa)/\Pi^1_1(\kappa)$ has been defined, let $\mathrm{Tr}^{\alpha+1}_1 = \mathrm{Tr}_1 \circ \mathrm{Tr}^\alpha_1$. If $\beta < \kappa$ is a limit ordinal and Tr^α_1 has been defined for all $\alpha < \beta$, then define Tr^β_1 by letting $\mathrm{Tr}^\beta_1([S]) = [\bigcap_{\alpha<\beta} S_\alpha]$, where S_α is a representative element of $\mathrm{Tr}^\alpha_1([S])$ for all $\alpha < \beta$. Finally, if β is a limit ordinal and $\kappa \leq \beta < \kappa^+$, then let $\mathrm{Tr}^\beta_1([S]) = [\triangle_{\eta<\kappa} S_{e_\beta(\eta)}]$.

It is straightforward to verify that each of these functions is well-defined and does not depend on our choice of e_β for limit β.

Definition 5.8. Suppose κ is a regular cardinal. For $0 < \alpha < \kappa^+$ we say that κ is *α-weakly compact* if $\mathrm{Tr}^\beta_1([\kappa]) \neq [\varnothing]$ for all $\beta < \alpha$, and κ is *κ^+-weakly compact* if it is α-weakly compact for all $\alpha \in \kappa^+ \setminus \{0\}$.

Notice that κ is 1-weakly compact if and only if it is weakly compact and κ is ω-weakly compact if and only if for all $n < \omega$ the set $\{\gamma < \kappa : \gamma \text{ is } n\text{-weakly compact}\}$ is weakly compact in κ. For more details regarding α-weak compactness the reader is referred to [Cod19].

If $0 < \alpha \leq \kappa^+$ and we start with an α-weakly compact cardinal κ in Theorem 5.6, then it will remain α-weakly compact in the extension $V[G*H]$. This is the case because weakly compact subsets of all cardinals $\gamma \leq \kappa$ are preserved to $V[G*H]$, so it is easy to show by induction on $0 < \beta \leq \alpha$ that if a set X is in the equivalence class $\mathrm{Tr}^\beta([\kappa])$ as computed in V, then the equivalence class $\mathrm{Tr}^\beta([\kappa])$ as computed in $V[G*H]$ contains some $Y \supseteq X$. Thus, we get the following, which immediately implies Theorem 1.1 in the special case $\alpha = \kappa^+$.

Theorem 5.9. *If $0 < \alpha \leq \kappa^+$, κ is α-weakly compact and* GCH *holds, then there is a cofinality preserving forcing extension in which*

(1) *κ remains α-weakly compact and*
(2) *$\square_1(\kappa)$ holds.*

Next we will show that, if κ is Mahlo, one can characterize precisely when $\square_1(\kappa)$ holds after forcing with $\mathbb{Q}(\kappa)$. Notice that, if there is a stationary subset of κ that does not reflect at an inaccessible cardinal (i.e., if $\mathrm{Refl}_0(\kappa)$ fails), then $\square_1(\kappa)$ must fail, since any such non-reflecting stationary set of κ would then trivially be a thread through any coherent sequence of 1-clubs of length κ. We will see in Theorem 5.11 that $\mathrm{Refl}_0(\kappa)$ holding in the extension by $\mathbb{Q}(\kappa)$ is in fact sufficient for $\square_1(\kappa)$ to hold. First, we need the following general proposition. Recall that $\mathrm{Refl}_n(\kappa)$ holds if and only if κ is Π^1_n-indescribable and, for every Π^1_n-indescribable subset S of κ, there is an $\alpha < \kappa$ such that $S \cap \alpha$ is Π^1_n-indescribable.

Proposition 5.10. *Fix $n < \omega$. If κ is a cardinal,* $\mathrm{Refl}_n(\kappa)$ *holds and $S \in \Pi^1_n(\kappa)^+$, then the set*

$$T = \{\alpha < \kappa : (S \cap \alpha \in \Pi^1_n(\alpha)^+) \wedge (\mathrm{Refl}_n(\alpha) \textit{ fails})\}$$

is Π^1_n-indescribable.

Proof. We proceed by induction on κ. Suppose the proposition holds for all cardinals $\alpha < \kappa$, $\mathrm{Refl}_n(\kappa)$ holds and $S \in \Pi^1_n(\kappa)^+$. It suffices to show that $T \cap C \neq \varnothing$ for every n-club subset C of κ. Fix an n-club set C and note that $S \cap C$ is Π^1_n-indescribable. Thus, by $\mathrm{Refl}_n(\kappa)$, there is some $\alpha_0 < \kappa$ such that $S \cap C \cap \alpha_0 \in \Pi^1_n(\alpha_0)^+$. It follows that $\alpha_0 \in \mathrm{Tr}_n(S)$, but also $\alpha_0 \in C$ because C contains all of its Π^1_{n-1}-reflection points. If $\alpha_0 \in T$, we have shown that $T \cap C \neq \varnothing$. So suppose that $\alpha_0 \notin T$, so $\mathrm{Refl}_n(\alpha_0)$ holds. We can now appeal to the inductive hypothesis at α_0, applied to the Π^1_n-indescribable set $S \cap \alpha_0$ and the n-club $C \cap \alpha_0$, to find a cardinal $\alpha_1 \in T \cap C$. $\square$

Theorem 5.11. *Suppose κ is Mahlo and $p \in \mathbb{Q}(\kappa)$. The following are equivalent:*

(1) $p \Vdash_{\mathbb{Q}(\kappa)} \mathrm{Refl}_0(\kappa)$
(2) $p \Vdash_{\mathbb{Q}(\kappa)} \square_1(\kappa)$

Proof. The implication (2) $\Rightarrow$ (1) follows immediately from the observation that a stationary subset of κ that does not reflect at any inaccessible cardinal is a thread through any putative $\square_1(\kappa)$-sequence.

We now show (1) $\Rightarrow$ (2). Suppose for the sake of contradiction that $p \Vdash_{\mathbb{Q}(\kappa)} \mathrm{Refl}_0(\kappa)$ and there is $p_1 \leq_{\mathbb{Q}(\kappa)} p$ such that $p_1 \Vdash_{\mathbb{Q}(\kappa)} \neg\square_1(\kappa)$. In particular, p_1 forces that $\bigcup \dot{G}$ is not a $\square_1(\kappa)$-sequence, so there is a $\mathbb{Q}(\kappa)$-name $\dot{C}$ that is forced by p_1 to be a thread through $\bigcup \dot{G}$.

Let G be $\mathbb{Q}(\kappa)$-generic over V with $p_1 \in G$, and move to $V[G]$. Let $C = \dot{C}_G$. Since C is stationary in κ and $\mathrm{Refl}_0(\kappa)$ holds, Proposition 5.10 implies that there are stationarily many inaccessible $\lambda < \kappa$ such that C reflects at λ and $\mathrm{Refl}_0(\lambda)$ fails.

Next, observe that every sequence of elements of G of size less than κ has a lower bound in G. Suppose that $\beta < \kappa$, and fix in $V[G]$ a sequence $\vec{p} = \langle p_\xi : \xi < \beta \rangle$ of elements of G. The sequence $\vec{p}$ must be in V by the $<\kappa$-distributivity of $\mathbb{Q}(\kappa)$, and so there is a condition $p \in G$ forcing that $\vec{p}$ is contained in G. But then p is a lower bound for $\vec{p}$. Observe also that, for all $\gamma < \kappa$, the initial segment $C^{(\gamma)} = C \cap \gamma$ of C is in V.

Now, in $V[G]$, we build a strictly decreasing sequence of conditions $\langle q_\alpha : \alpha < \kappa \rangle$ from G such that

(1) $q_0 = p_1$,
(2) $\{\gamma^{q_\alpha} : \alpha < \kappa\}$, the set of suprema of the domains of the conditions, is a club and
(3) for all $\alpha < \kappa$, $q_{\alpha+1} \Vdash_{\mathbb{Q}(\kappa)} \dot{C} \cap \gamma^{q_\alpha} = \check{C}^{(\alpha)}$.

We can ensure that (2) holds as follows. At a limit stage $\lambda < \kappa$, given that we have already constructed $\langle q_\alpha : \alpha < \lambda\rangle$, we know that there is some $q \in G$ below our sequence. So we let $\gamma_\lambda = \bigcup_{\alpha<\lambda} \gamma_\alpha$ and take $q_\lambda = q \restriction \gamma_\lambda + 1$.

Thus, we can find an inaccessible cardinal λ such that $\lambda = \gamma^{q_\lambda}$, C reflects at λ, and $\mathrm{Refl}_0(\lambda)$ fails. Since $\mathrm{Refl}_0(\lambda)$ fails (in $V[G]$ and hence also in V, since forcing with $\mathbb{Q}(\kappa)$ did not add any bounded subsets to κ), we can fix in V a stationary $C_\lambda \subseteq \lambda$ that is different from $C^{(\lambda)} = C \cap \lambda$ and that does not reflect at any inaccessible cardinal below λ. Now form a condition $q^*_\lambda \in \mathbb{Q}$ with $\gamma^{q^*_\lambda} = \gamma^{q_\lambda} = \lambda$ by letting $q^*_\lambda \restriction \lambda = \bigcup_{\alpha<\lambda} q_\alpha$ and $C^{q^*_\lambda}_\lambda = C_\lambda$. This is easily seen to be a valid condition, because everything needed to construct it is in V and since C_λ does not reflect at any inaccessible cardinal. Since $q^*_\lambda \leq_{\mathbb{Q}(\kappa)} q_\alpha$ for all $\alpha < \lambda$, we have

$$q^*_\lambda \Vdash_{\mathbb{Q}(\kappa)} \dot{C} \cap \lambda = \check{C}^{(\lambda)}.$$

In particular, since $C^{(\gamma)} = C \cap \lambda$ is stationary in λ, and since q^*_λ extends p_1 and thus forces that $\dot{C}$ is a thread through $\bigcup \dot{G}$, it must be the case that q^*_λ forces that the λ-th entry in $\bigcup \dot{G}$ is $C^{(\lambda)}$. However, q^*_λ forces the λ-th entry in $\bigcup \dot{G}$ to be C_λ, which is different from $C \cap \lambda$. This gives the desired contradiction. $\square$

Remark 5.12. Since the weak compactness of κ implies $\mathrm{Refl}_0(\kappa)$, by Theorem 5.11 it follows that in the proof of Theorem 5.6, in order to show that $\square_1(\kappa)$ holds in $V[G * H]$ it suffices to show that κ remains weakly compact.

6. Consistency of $\square_1(\kappa)$ with $\mathrm{Refl}_1(\kappa)$

In this section, we will show that the principle $\square_1(\kappa)$ is consistent with $\mathrm{Refl}_1(\kappa)$. First, we will need a lemma showing that we can force the existence of a fast function while preserving Π^1_2-indescribability.

The fast function forcing $\mathbb{F}_\kappa$, introduced by Woodin, consists of conditions that are partial functions $p : \kappa \to \kappa$ such that for every $\gamma \in \mathrm{dom}(p)$, the following statements hold:

- γ is inaccessible,
- $p``\gamma \subseteq \gamma$, and
- $|p \restriction \gamma| < \gamma$.

The union $f : \kappa \to \kappa$ of a generic filter G for $\mathbb{F}_\kappa$ is called a *fast function*. Since G can clearly be recovered from f, we will often conflate the two. For example, $V[f]$ and $V[G]$ are the same model, and, if $\tau \in V$ is an $\mathbb{F}_\kappa$-name, then we will sometimes write τ_f for the interpretation of τ in $V[G]$. Let $\mathbb{F}_{[\gamma,\kappa)}$ denote the subset of $\mathbb{F}_\kappa$ consisting of conditions p with $\mathrm{dom}(p) \subseteq [\gamma, \kappa)$ and observe that $\mathbb{F}_{[\gamma,\kappa)}$ is $\leq\gamma$-closed. It is not difficult to see that for any condition $p \in \mathbb{F}_\kappa$ and $\gamma \in \mathrm{dom}(p)$, the forcing $\mathbb{F}_\kappa$ factors below p as

$$\mathbb{F}_\gamma \restriction p \cong \mathbb{F}_\kappa \restriction (p \restriction \gamma) \times \mathbb{F}_{[\gamma,\kappa)} \restriction (p \restriction [\gamma, \kappa)).$$

Lemma 6.1. *Suppose κ is Π^1_n-indescribable. In a generic extension $V[f]$ by fast function forcing, κ remains Π^1_n-indescribable and the fast function f has the following property. For every $A \in H(\kappa^+)$ and $\alpha < \kappa^+$, there are a κ-model M with $f, A \in M$, a Π^1_{n-1}-correct κ-model N and an elementary embedding $j : M \to N$ with critical point κ such that $j(f)(\kappa) = \alpha$ and $j, M \in N$.*

Proof. The cardinal κ remains inaccessible in $V[f]$ because for unboundedly many inaccessible $\alpha < \kappa$, there is a condition $p \in G$ with $\alpha \in \operatorname{dom} p$, so $\mathbb{F}_\kappa$ below p factors with a first factor of size α and a second factor that is $\leq\alpha$-closed.

Fix $A \in H(\kappa^+)^{V[f]}$ and $\alpha < \kappa^+$ (note that V and $V[f]$ have the same κ^+). Let $\dot{A}$ be an $\mathbb{F}_\kappa$-name for A and let $B \subseteq \kappa$ code α. By Theorem 2.6 (4), there are a κ-model M with $\mathbb{F}_\kappa, \dot{A}, B \in M$, a Π^1_{n-1}-correct κ-model N and an elementary embedding $j : M \to N$ with critical point κ such that $j, M \in N$. We will lift j to $M[G]$. Let $p = \langle \kappa, \alpha \rangle$ be a condition in $j(\mathbb{F}_\kappa)$. Below p, $j(\mathbb{F}_\kappa)$ factors as $j(\mathbb{F}_\kappa) \restriction p \cong \mathbb{F}_\kappa \times \mathbb{F}_{[\kappa, j(\kappa))} \restriction p$, where the second factor is $\leq\kappa$-closed in N. In V, we can build an N-generic function f' for $\mathbb{F}_{[\kappa,j(\kappa))}$ containing p, and so $f \times f'$ is N-generic for $j(\mathbb{F}_\kappa)$. Thus, we can lift j to $j : M[f] \to N[f][f']$, and clearly $M[f]$ and j are in $N[f][f']$.

It remains to verify that $M[f]$ is a κ-model and $N[f][f']$ is a Π^1_{n-1}-correct κ-model. The argument to show that $M[f]$ is a κ-model in $V[f]$ will be more involved than usual because, as $\mathbb{F}_\kappa$ is not κ-c.c., we cannot apply the generic closure criterion. Fixing $\beta < \kappa$, we will show that $M[f]^\beta \subseteq M[f]$ in $V[f]$. By density, there is an inaccessible cardinal $\alpha > \beta$ and a condition $p = \langle \{\gamma, \delta\} \rangle \in G$ such that $\gamma < \alpha < \delta$. Below p, $\mathbb{F}_\kappa$ factors as $\mathbb{F}_\gamma \times \mathbb{F}_{(\delta,\kappa)}$ and f factors as $f_\gamma \times f_{(\delta,\kappa)}$. Since $\mathbb{F}_\gamma$ clearly has the α-c.c., by the generic closure criterion, $M[f_\gamma]^\beta \subseteq M[f_\gamma]$ in $V[f_\gamma]$. Also, since $\mathbb{F}_{(\delta,\kappa)}$ is $\leq\alpha$-closed, $M[f_\gamma]^\beta \subseteq M[f_\gamma]$ in $V[f]$. Finally, by the ground closure criterion, $M[f_\gamma][f_{(\delta,\kappa)}]^\beta \subseteq M[f_\gamma][f_{(\delta,\kappa)}]$ in $V[f]$. The same argument shows that $N[f]$ is a κ-model in $V[f]$, and therefore, $N[f][f']$ is a κ-model as well. To show that $N[f]$ is Π^1_{n-1}-correct, we argue essentially as in the proof of Theorem 3.7. Arguments from [Fri00] (section "Reverse Easton Forcing" in Chapter 2) show that since $\mathbb{F}_\kappa$ can be factored as a forcing with an arbitrarily highly closed tail, it is pretame over V_κ, giving the definability of the forcing relation. Note also that we have $V_\kappa^{V[f]} = V_\kappa^{N[f]} = V_\kappa[f]$. Finally, the model $N[f][f']$ must also be Π^1_{n-1}-correct because the tail forcing $\mathbb{F}_{[\kappa,j(\kappa))}$ does not add any subsets to $V_\kappa[f]$ by closure. $\square$

It is not difficult to see that once we have a fast function, we also get a weak Laver function [Ham02].

Lemma 6.2. *Suppose κ is Π^1_n-indescribable. In the generic extension $V[f]$ by fast function forcing, there is a function $\ell : \kappa \to V_\kappa$ satisfying the following property. For all $A, B \in H(\kappa^+)^{V[f]}$, there are a κ-model M with $\ell, A, B \in M$, a Π^1_{n-1}-correct κ-model N and an elementary embedding $j : M \to N$ with critical point κ such that $j(\ell)(\kappa) = B$ and $j, M \in N$.*

Proof. Fix any bijection $b : \kappa \to V_\kappa$ in V. In $V[f]$, define $\ell : \kappa \to V_\kappa$ by letting $\ell(\gamma) = b(f(\gamma))_{f\restriction\gamma}$ provided that $f \restriction \gamma$ is $\mathbb{F}_\gamma$-generic over V and $b(f(\gamma))$ is an $\mathbb{F}_\gamma$-name. By adapting the proof of Lemma 6.1, we verify that ℓ has the desired properties as follows. Working in $V[f]$, fix $A, B \in H(\kappa^+)^{V[f]}$ and let $\dot{\ell}, \dot{A}, \dot{B}$ be nice $\mathbb{F}_\kappa$-names for ℓ, A and B respectively. By Theorem 2.6 (4), there are a κ-model M with $\dot{\ell}, \dot{A}, \dot{B}, \mathbb{F}_\kappa, b \in M$, a Π^1_{n-1}-correct κ-model N and an elementary embedding $j : M \to N$ with critical point κ such that $j, M \in N$. Since $\mathbb{F}_\kappa$ is κ^+-c.c., we can assume without loss of generality that $\dot{B} \in j(V_\kappa)$. By elementarity $j(b) : j(\kappa) \to j(V_\kappa)$ is a bijection, and thus there is some ordinal $\alpha < j(\kappa)$ such that $j(b)(\alpha) = \dot{B}$. As in the proof of Lemma 6.1, we may lift j to $j : M[f] \to N[f][f']$ such that $j(f)(\kappa) = \alpha$. Now we have $j(\ell)(\kappa) = j(b)(j(f)(\kappa))_{j(f)\restriction\kappa} = j(b)(\alpha)_f = \dot{B}_f = B$.

Now one may prove that $M[f]$ is a κ-model and $N[f][f']$ is a Π^1_{n-1}-correct κ-model exactly as in the proof of Lemma 6.1. $\square$

Theorem 1.2. *Suppose κ is Π^1_2-indescribable and* GCH *holds. Then there is a cofinality-preserving forcing extension $V[G]$ in which*

(1) $\square_1(\kappa)$ *holds,*
(2) $\mathrm{Refl}_1(\kappa)$ *holds and*
(3) *κ is κ^+-weakly compact.*

Proof. By passing to an extension with a fast function, we can assume without loss of generality that there is a function $\ell{:}\kappa \to V_\kappa$ such that for any $A, B \in H(\kappa^+)$ there are a κ-model M with $\ell, A, B \in M$, a Π^1_1-correct κ-model N and an elementary embedding $j : M \to N$ with critical point κ such that $j(\ell)(\kappa) = B$.

Let $\mathbb{P}_\kappa = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \kappa,\ \beta < \kappa\rangle$ be the Easton-support iteration defined as follows.

- If $\alpha < \kappa$ is inaccessible and $\ell(\alpha)$ is a $\mathbb{P}_\alpha$-name for an α-strategically closed, α^+-c.c. forcing notion, then $\dot{\mathbb{Q}}_\alpha = \ell(\alpha)$.
- Otherwise, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for trivial forcing.

Let G be generic for $\mathbb{P}_\kappa$ over V. In $V[G]$, we define a 2-step iteration

$$\mathbb{Q}_\kappa = \mathbb{Q}_{\kappa,0} * (\dot{\mathbb{Q}}_{\kappa,1} \times \dot{\mathbb{Q}}_{\kappa,2})$$

as follows.

- $\mathbb{Q}_{\kappa,0}$ is the forcing to add a $\square_1(\kappa)$-sequence from Definition 5.1.
- $\dot{\mathbb{Q}}_{\kappa,2}$ is a $\mathbb{Q}_{\kappa,0}$-name for the forcing $\mathbb{T}(\vec{C}(\kappa))$ to thread the generic $\square_1(\kappa)$-sequence.
- $\dot{\mathbb{Q}}_{\kappa,1}$ is a $\mathbb{Q}_{\kappa,0}$-name for an iteration $\langle \mathbb{R}_\eta, \dot{\mathbb{S}}_\xi : \eta \leq \kappa^+,\ \xi < \kappa^+\rangle$ with supports of size $<\kappa$ defined as follows. For each $\eta < \kappa^+$, a $\mathbb{Q}_{\kappa,0} * \dot{\mathbb{R}}_\eta$-name $\dot{S}_\eta$ is chosen for a stationary subset of κ such that
$$\Vdash_{\mathbb{Q}_{\kappa,0}*(\dot{\mathbb{R}}_\eta \times \dot{\mathbb{Q}}_{\kappa,2})} \text{“there is a 1-club in } \kappa \text{ disjoint from } \dot{S}_\eta\text{”},$$
and then $\dot{\mathbb{S}}_\eta$ is a $\mathbb{Q}_{\kappa,0} * \dot{\mathbb{R}}_\eta$-name for the forcing $T^1(\kappa \setminus \dot{S}_\eta)$ to shoot a 1-club through the complement of $\dot{S}_\eta$.

Notice that $\mathbb{P}_\kappa$ is κ-c.c. and preserves GCH and, in $V[G]$, the forcing $\mathbb{Q}_\kappa$ is κ^+-c.c.. We also claim that $\mathbb{Q}_\kappa$ is κ-strategically closed in $V[G]$. To see this, first note that, in $V[G]$, $\mathbb{Q}_{\kappa,0}$ is κ-strategically closed by Proposition 5.2. Next, $\dot{\mathbb{Q}}_{\kappa,2}$ is a $\mathbb{Q}_{\kappa,0}$-name for a κ-strategically closed forcing, by Lemma 5.4. Finally, $\dot{\mathbb{Q}}_{\kappa,1}$ is a $\mathbb{Q}_{\kappa,0}$-name for a $({<}\kappa)$-support iteration, each iterand of which is, by Lemma 4.1, κ-strategically closed. A standard argument, in which Player II plays according to the respective winning strategy on each coordinate, shows that such an iteration must itself be κ-strategically closed. It then similarly follows that $\mathbb{Q}_\kappa = \mathbb{Q}_{\kappa,0} * (\dot{\mathbb{Q}}_{\kappa,1} \times \dot{\mathbb{Q}}_{\kappa,2})$ is κ-strategically closed in $V[G]$.

By standard chain condition arguments and bookkeeping, we can ensure that in $V^{\mathbb{P}_\kappa * \dot{\mathbb{Q}}_{\kappa,0} * \dot{\mathbb{Q}}_{\kappa,1}}$, if $S \subseteq \kappa$ is stationary and

$$\Vdash_{\mathbb{Q}_{\kappa,2}} \text{“there is a 1-club in } \kappa \text{ disjoint from } \check{S}\text{”},$$

then there is already a 1-club in κ disjoint from S.

Let $H = h_0 * (h_1 \times h_2)$ be generic for $\mathbb{Q}_\kappa$ over $V[G]$. Our desired model will be $V[G * h_0 * h_1]$. We must show that in $V[G * h_0 * h_1]$, κ is κ^+-weakly compact, $\mathrm{Refl}_1(\kappa)$ holds and $\Box_1(\kappa)$ holds.

In order to show that κ is κ^+-weakly compact in $V[G * h_0 * h_1]$, we will first prove the following.

Claim 6.3. *κ is Π^1_2-indescribable in $V[G * H]$.*

Proof. Fix $A \in P(\kappa)^{V[G*H]}$. We must find a κ-model M with $A \in M$, a Π^1_1-correct κ-model N and an elementary embedding $j : M \to N$ with critical point κ.

Let $\dot{A} \in V$ be a $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$-name for A. Since $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_{\kappa,0} * (\dot{\mathbb{Q}}_{\kappa,1} \times \dot{\mathbb{Q}}_{\kappa,2})$ has the κ^+-c.c., we can fix $\eta < \kappa^+$ such that $\dot{A}$ is a $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_{\kappa,0} * (\dot{\mathbb{R}}_\eta \times \dot{\mathbb{Q}}_{\kappa,2})$-name. Moreover, we can assume that $\dot{A}$, $\mathbb{P}_\kappa$ and $\dot{\mathbb{Q}}_{\kappa,0} * (\dot{\mathbb{R}}_\eta \times \dot{\mathbb{Q}}_{\kappa,2})$ are in $H(\kappa^+)$. For $\eta < \kappa^+$, let $h_1 \restriction \eta$ be the generic for $\mathbb{R}_\eta$ induced by h_1.

By Proposition 6.2, there are a κ-model M with $\ell, \mathbb{P}_\kappa, \dot{A}, \dot{\mathbb{Q}}_{\kappa,0} * (\dot{\mathbb{R}}_\eta \times \dot{\mathbb{Q}}_{\kappa,2}) \in M$, a Π^1_1-correct κ-model N and an elementary embedding $j : M \to N$ with critical point κ such that $j(\ell)(\kappa) = \dot{\mathbb{Q}}_{\kappa,0} * (\dot{\mathbb{R}}_\eta \times \dot{\mathbb{Q}}_{\kappa,2})$ and $j, M \in N$. Without loss of generality we may additionally assume that $M \models |\eta| = \kappa$ since a bijection witnessing this can easily be placed into such a κ-model.

Notice that $j(\mathbb{P}_\kappa)$ is an Easton-support iteration in N of length $j(\kappa)$ and

$$j(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * (\dot{\mathbb{Q}}_{\kappa,0} * (\dot{\mathbb{R}}_\eta \times \dot{\mathbb{Q}}_{\kappa,2})) * \dot{\mathbb{P}}_{\kappa,j(\kappa)}$$

by our choice of $j(l)(\kappa)$. By Theorem 3.7, $N[G]$ is Π^1_1-correct in $V[G]$, and by Corollary 3.8, $N[G * h_0 * (h_1 \restriction \eta \times h_2)]$ is Π^1_1-correct in $V[G * h_0 * (h_1 \restriction \eta \times h_2)]$. Hence $N[G * h_0 * (h_1 \restriction \eta \times h_2)]$ is Π^1_1-correct in $V[G * h_0 * (h_1 \times h_2)]$ by Corollary 3.4.

Since $(\dot{\mathbb{P}}_{\kappa,j(\kappa)})_{G*h_0*(h_1\restriction\eta\times h_2)} = \mathbb{P}_{\kappa,j(\kappa)}$ is κ-strategically closed in $N[G * h_0 * (h_1 \restriction \eta \times h_2)]$ and since $N[G * h_0 * (h_1 \restriction \eta \times h_2)]$ is a κ-model in $V[G * H]$, we can build a filter $G'_{\kappa,j(\kappa)}$ which is generic for $\mathbb{P}_{\kappa,j(\kappa)}$ over $N[G * h_0 * (h_1 \restriction \eta \times h_2)]$. Since $j \restriction G$ is the identity function, it follows that

$$j\,``\,G \subseteq \hat{G} =_{\mathrm{def}} G * h_0 * (h_1 \restriction \eta \times h_2) * G_{\kappa,j(\kappa)},$$

and thus j lifts to $j : M[G] \to N[\hat{G}]$.

Let $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{inacc}(\kappa)\rangle$ be the generic $\Box_1(\kappa)$-sequence added by h_0, and let T be the thread added by $h_2 \subseteq \mathbb{Q}_{\kappa,2} = \mathbb{T}(\vec{C}(\kappa))$. By Lemma 5.5, T is a 1-club in $N[G * h_0 * (h_1 \restriction \eta \times h_2)]$. Let $p_0 = \vec{C} \cup \{(\kappa, T)\}$. Then $p_0 \in N[\hat{G}]$ and $p_0 \in j(\mathbb{Q}_{\kappa,0})$. Moreover, $p_0 \leq_{j(\mathbb{Q}_{\kappa,0})} j(q)$ for all $q \in h_0$. By the strategic closure of $j(\mathbb{Q}_{\kappa,0})$ and the fact that $N[\hat{G}]$ is a κ-model in $V[G * H]$, we can build a filter $p_0 \in \hat{h}_0 \subseteq j(\mathbb{Q}_{\kappa,0})$ which is generic over $N[\hat{G}]$. Thus, j extends to $j : M[G * h_0] \to N[\hat{G} * \hat{h}_0]$.

Similarly, in $N[\hat{G} * \hat{h}_0]$, the set $p_2 = T \cup \{\kappa\}$ is a condition in $j(\mathbb{Q}_{\kappa,2})$ and $p_2 \leq_{j(\mathbb{Q}_{\kappa,2})} j(q)$ for all $q \in h_2$. Again, since $j(\mathbb{Q}_{\kappa,2})$ is κ-strategically closed in $N[\hat{G} * \hat{h}_0]$, which is a κ-model in $V[G * H]$, we can build a filter $p_2 \in \hat{h}_2 \subseteq j(\mathbb{Q}_{\kappa,2})$ which is generic for $j(\mathbb{Q}_{\kappa,2})$ over $N[\hat{G} * \hat{h}_0]$, and lift j to

$$j : M[G * h_0 * h_2] \to N[\hat{G} * \hat{h}_0 * \hat{h}_2].$$

Now we lift the embedding through $h_1 \restriction \eta$. Let $\mathbb{R}_\eta = (\dot{\mathbb{R}}_\eta)_{G*h_0}$. By elementarity, $j(\mathbb{R}_\eta)$ is an iteration of length $j(\eta)$ with supports of size less than $j(\kappa)$. For each $\xi < \eta$, $\dot{\mathbb{S}}_{j(\xi)} = j(\dot{\mathbb{S}}_\xi)$ is, in $N[\hat{G} * \hat{h}_0 * \hat{h}_2]$, a $j(\mathbb{R}_\xi) = \mathbb{R}_{j(\xi)}$-name for the forcing to shoot a 1-club disjoint from $j(\dot{S}_\xi)$. For all $\xi < \eta$, let

$$D_\xi = \bigcup\{(p(\xi))_{h_1\restriction\xi} : p \in h_1 \text{ and } \xi \in \mathrm{dom}(p)\}$$

and note that D_ξ is a 1-club subset of κ in $N[\hat{G} * \hat{h}_0 * \hat{h}_2]$ because the forcing after $\mathbb{R}_{\xi+1}$ is κ-strategically closed and therefore cannot affect Π^1_1-truths by Lemma 3.3. Since $h_1 \restriction \eta, j \in N[\hat{G} * \hat{h}_0 * \hat{h}_2]$, we can define a function $p^* \in N[\hat{G} * \hat{h}_0 * \hat{h}_2]$ such that $\mathrm{dom}(p^*) = j\text{ " }\eta$ by letting $p^*(j(\xi))$ be a $j(\mathbb{R}_\xi)$-name for $D_\xi \cup \{\kappa\}$ for all $\xi < \eta$. In order to verify that $p^* \in j(\mathbb{R}_\eta)$, we must show that for all $\xi < \eta$, $p^* \restriction j(\xi) \Vdash_{j(\mathbb{R}_\xi)} p^*(j(\xi)) \cap j(\dot{S}_\xi) = \varnothing$.

Suppose this is not the case, and let $\xi < \eta$ be the minimal counterexample. It follows that $p^* \restriction j(\xi) \in j(\mathbb{R}_\xi)$ and, for all $p \in h_1 \restriction \xi$ we have $p^* \restriction j(\xi) \leq j(p)$. By assumption,

$$p^* \restriction j(\xi) \not\Vdash_{j(\mathbb{R}_\xi)} p^*(j(\xi)) \cap j(\dot{S}_\xi) = \varnothing$$

and thus we may let $p^{**} \leq_{j(\mathbb{R}_\xi)} p^* \restriction j(\xi)$ be such that

$$p^{**} \Vdash_{j(\mathbb{R}_\xi)} p^*(j(\xi)) \cap j(\dot{S}_\xi) \neq \varnothing.$$

Since $j(\mathbb{R}_\xi)$ is sufficiently strategically closed, we can build a filter $\hat{h}_1 \subseteq j(\mathbb{R}_\xi)$ in $V[G * H]$ which is generic over $N[\hat{G} * \hat{h}_0 * \hat{h}_2]$ with $p^{**} \in \hat{h}_1$ and lift to

$$j : M[G * h_0 * h_2 * (h_1 \restriction \xi)] \to N[\hat{G} * \hat{h}_0 * \hat{h}_2 * \hat{h}_1].$$

It follows that in $N[\hat{G} * \hat{h}_0 * \hat{h}_2 * \hat{h}_1]$ we have $(D_\xi \cup \{\kappa\}) \cap j(S_\xi) \neq \varnothing$, where $S_\xi = (\dot{S}_\xi)_{h_0\restriction\xi}$. Since $j(S_\xi) \cap \kappa = S_\xi$, we know that $D_\xi \cap j(S_\xi) = \varnothing$, so it must be the case that $\kappa \in j(S_\xi)$. However, in $M[G * h_0 * (h_1 \restriction \xi)]$, we have

$$\Vdash_{\mathbb{Q}_{\kappa,2}} \text{“there is a 1-club in } \kappa \text{ disjoint from } \check{S}_\xi\text{”}.$$

Therefore, we can fix such a 1-club E in $M[G * h_0 * h_2 * (h_1 \restriction \xi)]$. Note that E is actually stationary because $M[G * h_0 * h_2 * (h_1 \restriction \xi)]$ is Π^1_1-correct by Theorem 3.7 and Corollary 3.8. But then $\kappa \in j(E)$ since in $N[\hat{G} * \hat{h}_0 * \hat{h}_2 * \hat{h}_1]$, $j(E)$ is 1-club in $j(\kappa)$ and $j(E) \cap \kappa = E$ is stationary in κ. Thus $\kappa \in j(E) \cap j(S_\xi) = \varnothing$, a contradiction.

Thus, $p^* \in j(\mathbb{R}_\eta)$ and we can build a filter $p^* \in \hat{h}_1$ in $V[G * H]$ which is generic over $N[\hat{G} * \hat{h}_0 * \hat{h}_2]$. This implies that the embedding lifts to

$$j : M[G * h_0 * h_2 * (h_1 \restriction \eta)] \to N[\hat{G} * \hat{h}_0 * \hat{h}_2 * \hat{h}_1].$$

As we argued above, $N[G * h_0 * h_2 * (h_1 \restriction \eta)]$ is Π^1_1-correct in $V[G * H]$, and since the forcing

$$\mathbb{P}_{(\kappa,j(\kappa))} * j(\dot{\mathbb{Q}}_{\kappa,0}) * (j(\dot{\mathbb{Q}}_{\kappa,2}) \times j(\dot{\mathbb{R}}_\eta))$$

is $\leq\kappa$-distributive, it follows that $N[\hat{G} * \hat{h}_0 * \hat{h}_2 * \hat{h}_1]$ is Π^1_1-correct in $V[G * H]$. Since $A = \dot{A}_{G * h_0 * (h_1\restriction\eta \times h_2)} \in M[G * h_0 * h_2 * (h_1 \restriction \eta)]$, this shows that κ is Π^1_2-indescribable in $V[G * H]$. $\square$

Now let us argue that κ is κ^+-weakly compact in $V[G * h_0 * h_1]$. Fix $\zeta < \kappa^+$. We must argue that $\mathrm{Tr}^\zeta_1([\kappa])^{V[G*h_0*h_1]} \neq [\varnothing]$. Since κ is Π^1_2-indescribable in $V[G * h_0 * (h_1 \times h_2)]$ by Claim 6.3, and since $\mathbb{Q}_{\kappa,2}$ is κ-strategically closed, it follows that $\mathrm{Tr}^\zeta_1([\kappa])^{V[G*h_0*(h_1\times h_2)]} = [S]$, where $S \in V[G * h_0 * h_1]$ is Π^1_2-indescribable in $V[G * h_0 * (h_1 \times h_2)]$. It follows that S is weakly compact in $V[G * h_0 * h_1]$ by

Proposition 3.5, and clearly $\mathrm{Tr}_1^\zeta([\kappa])^{V[G*h_0*h_1]} = [S]$. Thus, κ is κ^+-weakly compact in $V[G*h_0*h_1]$.

We next argue that $\mathrm{Refl}_1(\kappa)$ holds in $V[G*h_0*h_1]$. Fix a weakly compact set $S \subseteq \kappa$ in $V[G*h_0*h_1]$. Since S intersects every 1-club in κ, our construction of $\mathbb{Q}_{\kappa,1}$ implies that there is $p \in \mathbb{Q}_{\kappa,2}$ such that

$$p \Vdash_{\mathbb{Q}_{\kappa,2}} \text{“there is no 1-club in } \kappa \text{ disjoint from } \check{S}\text{”}.$$

Let $g_2 \subseteq \mathbb{Q}_{\kappa,2}$ be generic over $V[G*h_0*h_1]$ with $p \in g_2$. By the proof of Claim 6.3, κ is Π^1_2-indescribable in $V[G*h_0*h_1*g_2]$. Therefore, in $V[G*h_0*h_1*g_2]$, $\mathrm{Refl}_1(\kappa)$ holds and S is a weakly compact subset of κ, and thus there is some $\alpha < \kappa$ such that $S\cap\alpha$ is a weakly compact subset of α. But $V[G*h_0*h_1*g_2]$ and $V[G*h_0*h_1]$ have the same V_κ, so $S\cap\alpha$ is a weakly compact subset of α in $V[G*h_0*h_1]$. Thus, $\mathrm{Refl}_1(\kappa)$ holds in $V[G*h_0*h_1]$.

Finally, we argue that $\square_1(\kappa)$ holds in $V[G*h_0*h_1]$. The sequence

$$\bigcup h_0 = \vec{C} = \langle C_\alpha : \alpha \in \mathrm{inacc}(\kappa)\rangle$$

is a $\square_1(\kappa)$-sequence in $V[G*h_0]$ by Theorem 5.11 because we can show that $\mathrm{Refl}_0(\kappa)$ holds by essentially the same argument as for $\mathrm{Refl}_1(\kappa)$ above. Suppose that $\vec{C}$ is no longer a $\square_1(\kappa)$-sequence in $V[G*h_0*h_1]$. This implies that there is a condition $p \in h_1$ such that in $V[G*h_0]$,

$$p \Vdash_{\mathbb{Q}_{\kappa,1}} \text{“there is a 1-club } \dot{E} \subseteq \check{\kappa} \text{ that threads } \check{\vec{C}}\text{”}.$$

Let g_1 be generic for $\mathbb{Q}_{\kappa,1}$ over $V[G*h_0*h_1]$ with $p \in g_1$. In $V[G*h_0*(h_1\times g_1)]$, let $E = \dot{E}_{h_1}$ and $E^* = \dot{E}_{g_1}$. By mutual genericity, we may fix $\alpha \in E \setminus E^*$. A proof almost identical to that of Claim 6.3 shows that κ is Π^1_2-indescribable in $V[G*h_0*(h_1\times g_1\times h_2)]$ and hence weakly compact in $V[G*h_0*(h_1\times g_1)]$. Now, in $V[G*h_0*(h_1\times g_1)]$, fix any $j : M \to N$ with critical point κ and $E, E^* \in M$. Since both are 1-clubs, $\kappa \in j(E)\cap j(E^*)$, and so by elementarity there is an inaccessible $\beta \in \kappa\setminus(\alpha+1)$ such that $E\cap\beta$ and $E^*\cap\beta$ are both stationary in β. But then, as they both thread $\vec{C}$, it must be the case that $E\cap\beta = C_\beta = \hat{E}\cap\beta$. This contradicts the fact that $\alpha \in E\setminus E^*$ and finishes the proof of the theorem. $\square$

Remark 6.4. Observe that κ cannot be Π^1_2-indescribable in $V[G*h_0*h_1]$ because $\square_1(\kappa)$ holds there. Thus, the set S, where $\mathrm{Tr}_1^\zeta([\kappa])^{V[G*h_0*h_1]} = [S]$, cannot be Π^1_2-indescribable in $V[G*h_0*h_1]$, which shows that Proposition 3.5 can fail for Π^1_2-indescribable sets.

7. An application to simultaneous reflection

In this section we will show that the simultaneous reflection principle $\mathrm{Refl}_n(\kappa, 2)$ is incompatible with $\square_n(\kappa)$.

Theorem 7.1. *Suppose that $1 \le n < \omega$, κ is Π^1_n-indescribable and $\square_n(\kappa)$ holds. Then there are two Π^1_n-indescribable subsets $S_0, S_1 \subseteq \kappa$ that do not reflect simultaneously, i.e., there is no $\beta < \kappa$ such that $S_0\cap\beta$ and $S_1\cap\beta$ are both Π^1_n-indescribable subsets of β.*

Proof. Suppose for the sake of contradiction that every pair of Π^1_n-indescribable subsets of κ reflects simultaneously. Already, $\mathrm{Refl}_n(\kappa)$ implies that the set $E = \{\alpha < \kappa : \mathrm{Refl}_{n-1}(\alpha) \text{ holds}\}$ is a Π^1_n-indescribable subset of κ because the set of

Π^1_n-indescribable cardinals below κ is Π^1_n-indescribable and $(n-1)$-reflection holds at each of them (see [Cod19]).

Let $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Tr}_{n-1}(\kappa)\rangle$ be a $\square_n(\kappa)$-sequence. For all $\alpha \in \mathrm{Tr}_{n-1}(\kappa)$, let

$$S^0_\alpha = \{\beta \in \mathrm{Tr}_{n-1}(\kappa) \setminus (\alpha+1) : C_\beta \cap \alpha \in \Pi^1_{n-1}(\alpha)^+\} \text{ and}$$
$$S^1_\alpha = \mathrm{Tr}_{n-1}(\kappa) \setminus ((\alpha+1) \cup S^0_\alpha).$$

Let $A = \{\alpha \in \mathrm{Tr}_{n-1}(\kappa) : S^0_\alpha \in \Pi^1_n(\kappa)^+\}$.

Claim 7.2. *A is Π^1_n-indescribable in κ.*

Proof. Fix an n-club $C \subseteq \kappa$. Since $\mathrm{Tr}_{n-1}(C)$ is an n-club in κ, it follows that $E \cap \mathrm{Tr}_{n-1}(C)$ is Π^1_n-indescribable in κ. For each $\beta \in E \cap \mathrm{Tr}_{n-1}(C)$, $\mathrm{Refl}_{n-1}(\beta)$ holds and $C_\beta \cap C$ is a Π^1_{n-1}-indescribable subset of β. Thus, for $\beta \in E \cap \mathrm{Tr}_{n-1}(C)$, we may let α_β be the least Π^1_{n-1}-indescribable cardinal such that $C_\beta \cap C \cap \alpha_\beta$ is Π^1_{n-1}-indescribable in α_β. Notice that $\alpha_\beta \in C$ for all $\beta \in E \cap \mathrm{Tr}_{n-1}(C)$ because C is an n-club. Since the map $\beta \mapsto \alpha_\beta$ is regressive on $E \cap \mathrm{Tr}_{n-1}(C)$, it follows by the normality of $\Pi^1_n(\kappa)$ that there is a fixed $\alpha \in C$ and a Π^1_n-indescribable set $T \subseteq E \cap \mathrm{Tr}_{n-1}(C)$ such that $\alpha_\beta = \alpha$ for all $\beta \in T$. This implies that $T \subseteq S^0_\alpha$, and thus $\alpha \in A \cap C$. $\square$

Claim 7.3. *There is $\alpha \in A$ such that S^1_α is a Π^1_n-indescribable subset of κ.*

Proof. Suppose not, and let $\alpha_0 < \alpha_1$ be elements of A. Since E is Π^1_n-indescribable in κ and $S^1_{\alpha_0}$ and $S^1_{\alpha_1}$ are both in the Π^1_n-indescribability ideal on κ, we can find $\beta \in E \setminus ((\alpha_1+1) \cup S^1_{\alpha_0} \cup S^1_{\alpha_1})$. It follows that $\beta \in S^0_{\alpha_0} \cap S^0_{\alpha_1}$, so, by the coherence properties of the $\square_n(\kappa)$-sequence, we have $C_\beta \cap \alpha_0 = C_{\alpha_0}$ and $C_\beta \cap \alpha_1 = C_{\alpha_1}$, and hence $C_{\alpha_1} \cap \alpha_0 = C_{\alpha_0}$. But then by Lemma 2.7 and Claim 7.2, we see that $\bigcup_{\alpha \in A} C_\alpha$ is a Π^1_n-indescribable subset of κ. Thus, $\bigcup_{\alpha \in A} C_\alpha$ is a thread through $\vec{C}$, which is a contradiction. $\square$

We can therefore fix $\alpha \in \mathrm{Tr}_{n-1}(\kappa)$ such that both S^0_α and S^1_α are Π^1_n-indescribable subsets of κ. Let $S_0 = S^0_\alpha$ and $S_1 = S^1_\alpha$. We claim that S_0 and S_1 cannot reflect simultaneously. Otherwise, there is γ such that $S_0 \cap \gamma$ and $S_1 \cap \gamma$ are both Π^1_n-indescribable subsets of γ. Consider the n-club C_γ. Since γ is Π^1_n-indescribable, $\mathrm{Tr}_{n-1}(C_\gamma)$ is also an n-club in γ. We can therefore find $\beta_0 < \beta_1$ in $\mathrm{Tr}_{n-1}(C_\gamma)$ such that $\beta_0 \in S_0$ and $\beta_1 \in S_1$. But note that $C_{\beta_0} = C_\gamma \cap \beta_0$ and $C_{\beta_1} = C_\gamma \cap \beta_1$, so $C_{\beta_0} = C_{\beta_1} \cap \beta_0$, contradicting the fact that $C_{\beta_0} \cap \alpha$ is Π^1_{n-1}-indescribable in α whereas $C_{\beta_1} \cap \alpha$ is not Π^1_{n-1}-indescribable in α. $\square$

As a direct consequence of Theorem 1.2 and Theorem 7.1 we obtain the following.

Corollary 7.4. *Suppose κ is Π^1_2-indescribable. Then there is a forcing extension in which $\mathrm{Refl}_1(\kappa)$ and $\neg\mathrm{Refl}_1(\kappa, 2)$ both hold.*

8. Questions

The theorems proved in this article about the principle $\square_1(\kappa)$ do not easily generalize to $\square_n(\kappa)$ because several key technical results about Π^1_1-indescribability which we used crucially in the proofs no longer hold for higher orders of indescribability. For example, given an embedding $j : M \to N$, where N is Π^1_n-correct, we cannot necessarily use a generic G for a poset $\mathbb{P} \in N$ from the ground model to lift j because $N[G]$ may no longer be Π^1_n-correct. An illustration of this is given in

Remark 5.7. Also, while κ-strategically closed forcing cannot make a subset of κ Π^1_1-indescribable if it was not so already in the ground model by Proposition 3.5, a set can become Π^1_2-indescribable after κ-strategically closed forcing by Remark 6.4.

Question 8.1. For $n > 1$, can we force from a strong enough large cardinal that κ is Π^1_n-indescribable and $\square_n(\kappa)$ holds nontrivially?

Question 8.2. Relative to large cardinals, for $n > 1$, is it consistent that $\mathrm{Refl}_n(\kappa)$ and $\square_n(\kappa)$ both hold?

Question 8.3. Relative to large cardinals, is it consistent that $\mathrm{Refl}_1(\kappa, 2)$ holds but $\mathrm{Refl}_1(\kappa, 3)$ fails?

Question 8.4. Can we force any indestructibility of $\mathrm{Refl}_1(\kappa)$?

Question 8.5. For $1 \leq n < \omega$, if κ is Π^1_n-indescribable, does our principle $\square_n(\kappa)$ imply the Brickhill-Welch principle $\square^n(\kappa)$? See Remark 2.3 for a discussion of $\square^n(\kappa)$.

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