EXTREMAL TRIANGLE-FREE AND ODD-CYCLE-FREE COLOURINGS OF UNCOUNTABLE GRAPHS

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Abstract. The optimality of the Erdős-Rado theorem for pairs is witnessed by the colouring $\Delta_\kappa : [2^\kappa]^2 \to \kappa$ recording the least point of disagreement between two functions. This colouring has no monochromatic triangles or, more generally, odd cycles. We investigate a number of questions investigating the extent to which Δ_κ is an *extremal* such triangle-free or odd-cycle-free colouring. We begin by introducing the notion of Δ-regressive and almost Δ-regressive colourings and studying the structures that must appear as monochromatic subgraphs for such colourings. We also consider the question as to whether Δ_κ has the minimal cardinality of any *maximal* triangle-free or odd-cycle-free colouring into κ. We resolve the question positively for odd-cycle-free colourings.

1. Introduction

The starting point of our paper is the classical Erdős-Rado partition relation:

$$(2^\kappa)^+ \to (\kappa^+)^2_\kappa,$$

i.e., for any colouring $c : [(2^\kappa)^+]^2 \to \kappa$, there is a c-monochromatic subset of $(2^\kappa)^+$ of size κ^+ [3]. This result is optimal, in the sense that

$$2^\kappa \not\to (3)^2_\kappa.$$

This negative relation is witnessed by the following natural colouring. For two functions x, y whose domains are sets of ordinals, let

$$\Delta(x, y) = \min\{i \in \operatorname{dom} x \cap \operatorname{dom} y : x(i) \neq y(i)\}$$

if such an i exists (otherwise $\Delta(x, y)$ is undefined). We let Δ_κ denote the restriction of Δ to $[2^\kappa]^2$, where 2^κ denotes the set of all functions from κ to 2. The following is immediate.

Fact 1.1. *For any infinite κ, the colouring $\Delta_\kappa : [2^\kappa]^2 \to \kappa$ has no monochromatic triples or odd cycles.*

We are interested in the extent to which Δ is a *minimal* triangle-free or odd-cycle-free colouring on $[2^\kappa]^2$. One way in which this question can be made precise is via the following definition.

Definition 1.2. We say that $c : [2^\kappa]^2 \to \kappa$ is Δ-*regressive* if

$$c(x, y) < \Delta(x, y)$$

for all x, y such that $\Delta(x, y) > 0$.

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So our primary question about such colourings is: What monochromatic subgraphs must appear in a Δ-regressive colouring on $[2^\kappa]^2$? We will see in Section 2 that, if $\kappa > \omega$, then such a colouring must have monochromatic cycles of every length. This will lead us to introduce the notion of *almost Δ-regressive colourings* and study analogous questions with respect to this larger class of colourings.

In particular, we will prove that

- the existence of an almost Δ-regressive colouring on $[2^\kappa]^2$ with no monochromatic odd cycles is equivalent to the existence of $\mu < \kappa$ such that $2^\mu = 2^\kappa$;
- it is consistent relative to the consistency of a measurable cardinal that, for instance, every almost Δ-regressive colouring on $[2^{\omega_1}]^2$ has a monochromatic set of size $\aleph_1$;
- every almost Δ-regressive κ-Borel colouring on $[2^\kappa]^2$ has a monochromatic set of size κ.

In Section 3, we look at maximality properties of the colouring Δ_κ on $[2^\kappa]^2$.

Definition 1.3. Suppose that $\mathcal{H}$ is a collection of graphs and $c : [X]^2 \to \kappa$ is a colouring.

(1) We say that c is *$\mathcal{H}$-free* if, for every $G \in \mathcal{H}$, there is no subgraph $E \subseteq [X]^2$ isomorphic to G such that $c \restriction E$ is constant.

(2) We say that c is *maximal $\mathcal{H}$-free (into κ)* if it is $\mathcal{H}$-free but, for any $y \notin X$ and any $d : [X \cup \{y\}]^2 \to \kappa$ extending c, d is not $\mathcal{H}$-free.

We are mostly concerned with $\mathcal{H}$ being the single 3-cycle or the collection of all odd cycles. We will see that Δ_κ is a maximal 3-cycle-free colouring (and hence a maximal odd-cycle-free colouring) into κ. Our primary question here is whether there exist *smaller* such maximal colourings, i.e., whether there are maximal 3-cycle-free or odd-cycle-free colourings $c : [X]^2 \to \kappa$ in which $|X| < 2^\kappa$. We answer this question negatively for odd-cycle-free colourings, in the process providing a characterization of odd-cycle-free colourings. The question for 3-cycle-free colourings remains open.

In Section 4, we prove a couple of results about 3-cycle-free colourings $c : [\omega_1]^2 \to \omega$. In particular, we show that, if $\diamondsuit$ holds, then there is a colouring $c : [\omega_1]^2 \to \omega$ so that each colour class $G_n = c^{-1}(\{n\})$ is a Hajnal-Máté graph with uncountable chromatic number, extending previous results of Komjáth ([7], [8]).

We conclude in Section 5 by collecting a number of problems that remain open.

Our notation and terminology is for the most part standard. If X is a set and θ is a cardinal, then $[X]^\theta := \{Y \subseteq X \mid |Y| = \theta\}$. If c is a function on $[X]^2$, then we will frequently abuse notation and write $c(x, y)$ in place of $c(\{x, y\})$. If $c : [X]^2 \to \kappa$ is a colouring and $Y \subseteq X$, then Y is *c-monochromatic* if $c \restriction [Y]^2$ is constant. A c-monochromatic triple (or triangle) is a c-monochromatic subset with exactly three elements. A c-monochromatic *cycle* is a finite injective sequence $\langle x_i \mid i < k\rangle$ of elements of X (with $k \geq 3$) such that c is constant on the set $\{\{x_i, x_{i+1}\} \mid i + 1 < k\} \cup \{\{x_{k-1}, x_0\}\}$. The number k is the *length* of the cycle. Such a cycle is a c-monochromatic *odd* cycle if k is odd.

2. Monochromatic subsets in Δ-regressive colourings

We begin with the simple observation that there are Δ-regressive colourings on $[2^\omega]^2$ without monochromatic triples. The following is easily verified.

Observation 2.1. *Define a colouring* $c : [2^\omega]^2 \to \omega$ *by letting, for all* $\{f, g\} \in [2^\omega]^2$,

$$c(f,g) = \begin{cases} \Delta(f,g) - 1 & \text{if } \Delta(f,g) > 0 \\ f(1) + g(1) \pmod 2 & \text{if } \Delta(f,g) = 0 \end{cases}$$

Then c *is* Δ*-regressive and has no monochromatic triples.*

However, for 2^α with $\alpha > \omega$, the situation is rather different.

Proposition 2.2. *For any* Δ*-regressive* $c : [2^{\omega+1}]^2 \to \omega + 1$, *there is an* $m < \omega$ *such that, for all lengths* $k \geq 3$, c *has monochromatic cycles of colour* m *and length* k.

Proof. Suppose for sake of contradiction that, for all $m < \omega$, there is a natural number $k_m \geq 3$ such that c has no monochromatic cycles of colour m and length k_m.

Claim 2.3. *For all* $m \geq 1$ *and* $\{f, g\} \in [2^{\omega+1}]^2$, *if* $\Delta(f, g) \geq m$, *then* $c(f, g) \geq m - 1$.

Proof. The proof is by induction on m. The claim is trivially true for $m = 1$. Fix $m \geq 1$ and suppose we have established the claim for m. In particular, for all $\{h, h'\} \in [2^{\omega+1}]^2$, if $\Delta(h, h') = m$, then, by the fact that c is Δ-regressive, we have $c(h, h') = m - 1$. To establish the claim for $m + 1$, assume for sake of contradiction that we can find $f, g \in 2^{\omega+1}$ such that $\Delta(f, g) \geq m + 1$ but $c(f, g) < m$. By our inductive hypothesis, it follows that $c(f, g) = m - 1$.

We will reach a contradiction by finding a monochromatic cycle of colour $m - 1$ and length k_{m-1}. If k_{m-1} is even, then simply choose an injective sequence $\langle h_j \mid j < k_{m-1}\rangle$ such that $\Delta(h_j, h_{j+1}) = m$ for all $j < k_{m-1} - 1$ (and hence also $\Delta(h_{k_{m-1}-1}, h_0) = m$). Then $\langle h_j \mid j < k_{m-1}\rangle$ is the desired monochromatic cycle.

If k_{m-1} is odd, then choose an injective sequence $\langle h_j \mid j < k_{m-1} - 2\rangle$ such that

- $\Delta(h_j, h_{j+1}) = m$ for all $j < k_{m-1} - 3$;
- $\Delta(f, h_j) = m$ for all even $j < k_{m-1} - 2$ (and hence $\Delta(g, h_j) = m$ for all even $j < k_{m-1} - 2$ as well).

Then $\langle f, g\rangle^\frown\langle h_j \mid j < k_{m-1} - 2\rangle$ is the desired monochromatic cycle. □

Now fix $f, g \in 2^{\omega+1}$ with $\Delta(f, g) = \omega$. By the claim, $c(f, g) \geq m - 1$ for all $m < \omega$, and hence $c(f, g) = \omega$, contradicting the fact that c is Δ-regressive. □

For this reason, the notion of Δ-regressive seems to be too strong to be of interest for certain questions. We therefore introduce a natural weakening.

Definition 2.4. We say that $c : [2^\kappa]^2 \to \kappa$ is *almost* Δ*-regressive* if there is $\mu < \kappa$ such that

$$c(x, y) < \max\{\Delta(x, y), \mu\}$$

for all $x \neq y$.

Once again, we ask what monochromatic subgraphs must appear in almost Δ-regressive colourings, and in particular whether they must contain monochromatic triangles or odd cycles. Let us first make an easy observation indicating that consistently there are almost Δ-regressive colourings avoiding all monochromatic odd cycles.

Proposition 2.5. *Suppose that* $\mu < \kappa$ *are infinite cardinals and* $2^\mu = 2^\kappa$. *Then there is an almost* Δ*-regressive colouring* $c : [2^\kappa]^2 \to \kappa$ *with no monochromatic odd cycles.*

Proof. We will in fact find such a function c mapping into μ; it will therefore trivially be almost Δ-regressive, as witnessed by μ.

Fix a bijection $F : 2^\kappa \to 2^\mu$, and define $c : [2^\kappa]^2 \to \mu$ by letting $c(x, y) = \Delta(F(x), F(y))$. It is immediate that c is as desired. □

In light of this fact, the most interesting case seems to be when κ is uncountable and $2^\mu < 2^\kappa$ for all $\mu < \kappa$. Note that, under these assumptions, for any function $c : [2^\kappa]^2 \to \mu$ with $\mu < \kappa$, the Erdős-Rado theorem yields monochromatic subsets of cardinality μ^+. Therefore, unlike the situation in Proposition 2.5, an example of an almost Δ-regressive function avoiding small monochromatic sets, if there is one, needs to have range of size κ.

Theorem 2.6. *Suppose that κ is an uncountable cardinal and $2^\mu < 2^\kappa$ for all $\mu < \kappa$. Suppose also that $c : [2^\kappa]^2 \to \kappa$ is* almost Δ-regressive. *Then there are c-monochromatic odd cycles.*

Proof. Suppose this is not the case, and let $\mu < \kappa$ be such that $c(x, y) < \max\{\Delta(x, y), \mu\}$ for all $x \neq y$. For each colour $\xi < \kappa$, the colour class $c^{-1}(\{\xi\})$ is odd-cycle-free and hence, as a graph, bipartite. We can therefore partition 2^κ into disjoint sets $2^\kappa = X^0_\xi \dot\cup X^1_\xi$ so that no pair from X^i_ξ has colour ξ. Define $\pi : 2^\kappa \to 2^\kappa$ by letting, for all $x \in 2^\kappa$ and all $\xi < \kappa$, $\pi(x)(\xi) = i$ if $x \in X^i_\xi$. Thus, if $\{x, y\} \in [2^\kappa]^2$ and $c(x, y) = \xi$, then $\pi(x)(\xi) \neq \pi(y)(\xi)$. Therefore π must be injective, and, if $\Delta(x, y) \geq \mu$, then

$$\Delta(\pi(x), \pi(y)) \leq c(x, y) < \Delta(x, y).$$

So, for any $x \neq y$, there is a minimal $n = n_{xy} < \omega$ so that

$$\xi_{xy} := \Delta(\pi^n(x), \pi^n(y)) < \mu.$$

Indeed, otherwise we could find a strictly decreasing infinite sequence of ordinals.

Apply the Erdős-Rado theorem to the map $g : [2^\kappa]^2 \to \omega \times \mu$ taking a pair $\{x, y\}$ to (n_{xy}, ξ_{xy}) to find three distinct x, y, z in 2^κ and (n, ξ) in $\omega \times \mu$ so that $g``[\{x, y, z\}]^2 = \{(n, \xi)\}$. This means that the function Δ is constant, with value ξ, on the 3-element set $\{\pi^n(x), \pi^n(y), \pi^n(z)\}$, a contradiction to the fact that Δ is triangle-free. □

At this point, we do not know if Theorem 2.6 can be improved to yield the necessary existence of monochromatic triples. The above argument can easily be generalized to prove the following result, whose verification we leave to the reader.

Theorem 2.7. *Suppose that $2 \leq \nu < \kappa$ are cardinals, with κ uncountable, $2^\mu < 2^\kappa$ for all $\mu < \kappa$, and $c : [\nu^\kappa]^2 \to \kappa$ is almost Δ-regressive. Then there is $\xi < \kappa$ such that the graph $(\nu^\kappa, c^{-1}(\{\xi\}))$ has chromatic number greater than ν.*

If we assume that κ is a large cardinal, we can obtain a stronger result. Recall the following large cardinal notion.

Definition 2.8. Let κ be an uncountable cardinal.

(1) A function $f : [\kappa]^2 \to \kappa$ is *regressive* if $f(\alpha, \beta) < \alpha$ for all $0 < \alpha < \beta < \kappa$.
(2) κ is an *almost ineffable cardinal* if, for every regressive function $f : [\kappa]^2 \to \kappa$, there is an f-monochromatic set of cardinality κ.

We note that this is different from the usual definition of almost ineffability but was proven to be equivalent by Baumgartner [1, Theorem 5.2]. To put almost ineffability into the context of possibly more familiar large cardinal notions, it is easily seen that all measurable cardinals are almost ineffable and all almost ineffable cardinals are weakly compact.

Proposition 2.9. *Suppose that κ is an almost ineffable cardinal and $c : [2^\kappa]^2 \to \kappa$ is almost Δ-regressive. Then there are c-monochromatic subsets of size κ.*

Note that, in general, we cannot hope to find monochromatic sets of size κ^+ by the existence of Sierpinski colourings from $[2^\kappa]^2$ to 2 witnessing $2^\kappa \not\to (\kappa^+)^2_2$.

Proof. Let $\mu < \kappa$ witness that c is almost Δ-regressive. Fix an injective sequence $\langle y_\xi \mid \xi < \kappa \rangle$ from 2^κ so that $\Delta(y_\zeta, y_\xi) = \zeta$ for all $\zeta < \xi < \kappa$. Notice that, for all $\mu \le \zeta < \xi < \kappa$, we have $c(y_\zeta, y_\xi) < \zeta$. Now define a function $f : [\kappa]^2 \to \kappa$ by letting $f(\zeta, \xi) = c(y_\zeta, y_\xi)$ if $\mu \le \zeta < \xi < \kappa$ and $f(\zeta, \xi) = 0$ otherwise. Then f is regressive, so, by the almost ineffability of κ, there is an f-monochromatic set $A \subseteq \kappa$ of size κ. But then $\{y_\xi \mid \xi \in A \setminus \mu\}$ is a c-monochromatic set of size κ. □

We can use similar ideas to obtain consistency results for small cardinals κ, assuming the consistency of a measurable cardinal. For example, a special case of the following theorem yields the consistency of the assertion that every almost Δ-regressive colouring of $[2^{\omega_1}]^2$ has uncountable monochromatic subsets.

Theorem 2.10. *Suppose that $\kappa < \lambda$ are regular uncountable cardinals, with λ measurable, and let $\mathbb{P} = \mathrm{Add}(\kappa, \lambda)$ be the poset to add λ-many Cohen subsets to κ. Then, in $V^{\mathbb{P}}$, every almost Δ-regressive colouring $c : [2^\kappa]^2 \to \kappa$ has monochromatic subsets of size κ.*

Proof. We think of conditions in $\mathbb{P}$ as being partial functions from $\lambda \times \kappa$ to 2 of cardinality less than κ, ordered by reverse inclusion. For $p \in \mathbb{P}$, let

$$D_p = \{\alpha < \lambda \mid \text{there is } i < \kappa \text{ such that } (\alpha, i) \in \mathrm{dom}(p)\}.$$

Slightly abusing notation, if $D \subseteq D_p$, then let $p \restriction D$ denote $p \restriction (\mathrm{dom}(p) \cap (D \times \kappa))$. For $\alpha < \lambda$, let $\dot{f}_\alpha$ be the canonical $\mathbb{P}$-name for the α^{th} Cohen subset added. In other words, for all $p \in \mathbb{P}$ and $i < \kappa$, if $(\alpha, i) \in \mathrm{dom}(p)$, then $p \Vdash$ "$\dot{f}_\alpha(i) = p(\alpha, i)$". Fix a $\mathbb{P}$-name $\dot{c}$ for an almost Δ-regressive colouring from $[2^\kappa]^2$ to κ, and fix a condition $p_0 \in \mathbb{P}$ and a cardinal $\mu < \kappa$ such that

$$p_0 \Vdash_{\mathbb{P}} \text{“}\dot{c}(x, y) < \max(\Delta(x, y), \check{\mu}) \text{ for all } x, y \in 2^\kappa\text{”}.$$

Fix also a normal measure U over λ.

Let G be $\mathbb{P}$-generic over V with $p_0 \in G$, and work in $V[G]$. We will recursively construct

- an increasing sequence $\langle \alpha_\xi \mid \xi < \kappa \rangle$ of ordinals below λ;
- sets $X_\xi \in U$ and ordinals $i_\xi < \delta_\xi < \varepsilon_\xi < \kappa$ for each $\xi < \kappa$, with $\langle \delta_\xi \mid \xi < \kappa \rangle$ increasing and continuous;
- conditions q_ξ and $p_{\alpha_\xi \beta}$ in $\mathbb{P}$ for all $\xi < \kappa$ and $\beta \in X_\xi$;
- a $\subseteq$-increasing sequence of functions $\langle g_\xi : \varepsilon_\xi \to 2 \mid \xi < \kappa \rangle$

such that for all $\xi < \kappa$ and $\beta \in X_\xi$,

(1) $\mathrm{dom}(p_{\alpha_\xi \beta}) \subseteq D_{p_{\alpha_\xi \beta}} \times \varepsilon_\xi$;
(2) $\varepsilon_\xi \le \delta_{\xi+1}$;
(3) $p_{\alpha_\xi \beta} \Vdash$ "$\check{\delta}_\xi = \Delta(\dot{f}_{\alpha_\xi}, \dot{f}_\beta)$ and $\dot{f}_\beta \restriction \check{\varepsilon}_\xi = \check{g}_\xi$";
(4) $p_{\alpha_\xi \beta} \le p_0$ and $p_{\alpha_\xi \beta} \Vdash$ "$\dot{c}(\dot{f}_{\alpha_\xi}, \dot{f}_\beta) = \check{i}_\xi$";
(5) $\{D_{p_{\alpha_\xi \beta}} \mid \beta \in X_\xi\}$ forms a Δ-system with root D_ξ, and $p_{\alpha_\xi \beta} \restriction D_\xi = q_\xi$;
(6) for all $\zeta < \xi$, we have
  (a) $X_\xi \cup \{\alpha_\xi\} \subseteq X_\zeta$;
  (b) $p_{\alpha_\xi \beta}$ extends both $p_{\alpha_\zeta \alpha_\xi}$ and $p_{\alpha_\zeta \beta}$;
  (c) $p_{\alpha_\zeta \alpha_\xi}, q_\xi \in G$.

We set $X_{-1} = \lambda$ and $\delta_{-1} = \mu$, and fix $\varepsilon_{-1} < \kappa$ such that $\mathrm{dom}(p_0) \subseteq D_{p_0} \times \varepsilon_{-1}$, and we describe the general step of the recursion. Suppose that $\xi < \kappa$ is fixed and that we have constructed the above objects for all $\zeta < \xi$. Notice that, since $\mathbb{P}$ is κ-closed, the construction thus far all lives in V. Let

$$r_\xi = \bigcup_{\eta < \zeta < \xi} p_{\alpha_\eta \alpha_\zeta} \cup \bigcup_{\zeta < \xi} q_\eta.$$

(If $\xi = 0$, let $r_\xi = p_0$.) By the closure of $\mathbb{P}$, r_ξ is in fact a condition in $\mathbb{P}$ and is in G. Set $\delta_\xi = \sup\{\varepsilon_\zeta \mid \zeta < \xi\}$, and note that $\mathrm{dom}(r_\xi) \subseteq D_{r_\xi} \times \delta_\xi$. Move now to V.

Claim 2.11. *Let E_ξ be the set of $q \leq r_\xi$ for which there exist $\alpha^* < \lambda$, $i^* < \varepsilon^* < \kappa$, $g^* : \varepsilon^* \to 2$, $X^* \in U$, and conditions $p_{\alpha^*\beta}$ for $\beta \in X^*$ such that*

- $i^* < \delta_\xi < \varepsilon^*$;
- $\mathrm{dom}(p_{\alpha^*,\beta}) \subseteq D_{p_{\alpha^*,\beta}} \times \varepsilon^*$ *for all* $\beta \in X^*$;
- $p_{\alpha^*\beta} \Vdash$ "$\check{\delta}_\xi = \Delta(\dot{f}_{\alpha^*}, \dot{f}_\beta)$ *and* $\dot{f}_\beta \restriction \check{\varepsilon}^* = \check{g}^*$" *for all* $\beta \in X^*$;
- $p_{\alpha^*\beta} \Vdash$ "$\dot{c}(\dot{f}_{\alpha^*}, \dot{f}_\beta) = \check{i}^*$" *for all* $\beta \in X^*$;
- $\{D_{p_{\alpha^*\beta}} \mid \beta \in X^*\}$ *forms a* Δ*-system with root* D^*, *and* $p_{\alpha^*\beta} \restriction D^* = q$ *for all* $\beta \in X^*$;
- $X^* \cup \{\alpha^*\} \subseteq \bigcap\{X_\zeta \mid \zeta < \xi\}$;
- q *extends* $p_{\alpha_\zeta \alpha_\xi}$ *for all* $\zeta < \xi$;
- $p_{\alpha^*\beta}$ *extends* $p_{\alpha_\zeta \beta}$ *for all* $\zeta < \xi$ *and* $\beta \in X^*$.

Then E_ξ is dense in $\mathbb{P}$ below r_ξ.

Proof. Let $t \leq r_\xi$ be arbitrary, and let Y be the set of all $\beta \in \bigcap\{X_\zeta \mid \zeta < \xi\}$ such that $\beta > \max\{\alpha_\zeta, \sup(D_t)\}$ and $D_{p_{\alpha_\zeta \beta}} \setminus D_{q_\zeta}$ is disjoint from D_t for all $\zeta < \xi$. Note that, by our recursion hypothesis, we have that, for all $\zeta < \xi$, the sequence $\langle D_{p_{\alpha_\zeta \beta}} \setminus D_{q_\zeta} \mid \beta \in X_\zeta \rangle$ consists of pairwise disjoint sets, and therefore we know that $Y \in U$. Let $\alpha^* = \min(Y)$, and let $t^* = t \cup \bigcup_{\zeta < \xi} p_{\alpha_\zeta \alpha^*}$. By our choice of α^*, t^* is a function and thus a condition in $\mathbb{P}$.

Let $Y^* = Y \setminus (\alpha + 1)$. For all $\beta \in Y^*$, let $t^*_\beta = t^* \cup \bigcup\{p_{\alpha_\zeta \beta} \mid \zeta < \xi\}$. By our choice of Y^*, t^*_β is a function and hence a condition in $\mathbb{P}$. Let $g^- = \bigcup_{\zeta < \xi} g_\zeta$, and notice that $t^*_\beta \Vdash$ "$\dot{f}_{\alpha^*} \restriction \check{\delta}_\xi = \dot{f}_\beta \restriction \check{\delta}_\xi = \check{g}^-$", and t^*_β does not decide the value of $\dot{f}_{\alpha^*}(\eta)$ or $\dot{f}_\beta(\eta)$ for any $\delta_\xi \leq \eta < \kappa$. We can therefore fix a condition $p_{\alpha^*\beta} \leq t^*_\beta$ such that

- $p_{\alpha^*\beta} \Vdash$ "$\Delta(\dot{f}_{\alpha^*} \dot{f}_\beta) = \check{\delta}_\xi$"; and
- $p_{\alpha^*\beta}$ decides the value of $\dot{c}(\dot{f}_{\alpha^*}, \dot{f}_\beta)$ to be equal to some $i^*_\beta < \delta_\xi$.

Let $\epsilon_{\alpha^*\beta} < \kappa$ be such that $\mathrm{dom}(p_{\alpha^*\beta}) \subseteq D_{p_{\alpha^*\beta}} \times \epsilon_{\alpha^*,\beta}$. Without loss of generality, we may assume that $\{\beta\} \times \epsilon_{\alpha^*\beta} \subseteq \mathrm{dom}(p_{\alpha^*\beta})$, so we can define a function $g_{\alpha^*\beta} : \epsilon_{\alpha^*\beta} \to 2$ by letting $g_{\alpha^*\beta}(j) = p_{\alpha^*\beta}(\beta, j)$ for all $j < \epsilon_{\alpha^*\beta}$. Now consider the map h that sends each $\beta \in Y^*$ to the tuple $\left\langle p_{\alpha^*\beta} \restriction (D_{p_{\alpha^*\beta}} \cap \beta),\ i^*_\beta,\ \epsilon_{\alpha^*\beta},\ g_{\alpha^*\beta} \right\rangle$. Then h can be coded as a regressive function, defined on a set in U, so, by the normality of U, we can find a set $X^* \subseteq Y^*$, a condition $q \in \mathbb{P}$, ordinals i^* and ε^* such that $i^* < \delta_\xi < \varepsilon^* < \kappa$, and a function $g^* : \varepsilon^* \to 2$ such that

- $X^* \in U$;
- $h(\beta) = \langle q,\ i^*,\ \varepsilon^*,\ g^* \rangle$ for all $\beta \in X^*$;
- $D_{p_{\alpha^*\beta}} \subseteq \beta'$ for all $\beta < \beta' \in X^*$.

Then $q \leq t$ is as in the statement of the claim, as witnessed by $\alpha^*, i^*, \varepsilon^*, g^*, X^*$, and $\{p_{\alpha^*\beta} \mid \beta \in X^*\}$. □

Now move back to $V[G]$. By the claim and the fact that $r_\xi \in G$, we can find $q_\xi \in E_\xi \cap G$, as witnessed by $\alpha_\xi < \lambda$, $i_\xi < \varepsilon_\xi < \kappa$, $g_\xi : \varepsilon_\xi \to 2$, $X_\xi \in U$, and conditions $p_{\alpha_\xi \beta} \in \mathbb{P}$ for all $\beta \in X_\xi$. It is easily verified that these objects are as desired, thus completing the recursive construction.

Now the map sending δ_ξ to i_ξ for all $\xi < \kappa$ is a regressive function defined on a club of ordinals in κ, so there is a fixed $i < \kappa$ and a stationary $S \subseteq \kappa$ such that $i_\xi = i$ for all $\xi \in S$. It follows that, in $V[G]$, $\{f_{\alpha_\xi} \mid \xi \in S\}$ is a monochromatic subset for c of size κ. □

We end this section with a discussion indicating that *sufficiently nice* almost Δ-regressive functions necessarily have large monochromatic sets, regardless of cardinal arithmetic. We consider 2^κ as a topological space with the $<\kappa$-supported product topology, i.e., basic open sets are of the form $N_s := \{x \in 2^\kappa \mid s \subseteq x\}$ for $s \in 2^{<\kappa} = \bigcup_{\alpha<\kappa} 2^\alpha$, and we consider $[2^\kappa]^2$ as a topological space inheriting the subspace topology from the product space $2^\kappa \times 2^\kappa$. Recall that, in the space 2^κ, the collection of λ-*Borel* sets, where $\lambda \leq \kappa$ is an infinite cardinal, is the smallest collection of sets containing the open sets and closed under complementation and unions and intersections of size λ. Since we will only be working with κ-Borel sets, we will simply use the word "Borel" to mean "κ-Borel". A subset of 2^κ is *meagre* if it is the union of κ-many nowhere dense sets. Then it is readily established that 2^κ satisfies a version of the Baire Category Theorem, i.e., it is not the union of κ-many meagre sets. Also, Borel subsets of 2^κ have the Baire property: if $Y \subseteq 2^\kappa$ is Borel, then there is an open set $U \subseteq 2^\kappa$ such that $Y \triangle U$ is meagre. All of these comments carry over to the space $[2^\kappa]^2$, as well. (See [4], particularly Chapter 4, for these facts and more on the descriptive set theory of 2^κ and κ^κ.)

We also consider κ as a topological space with the discrete topology. With these assumptions, note that $\Delta_\kappa : [2^\kappa]^2 \to \kappa$ is continuous and thus Borel. The following theorem indicates a way in which Δ_κ is provably a minimal Borel coloring with no monochromatic sets of size κ.

Theorem 2.12. *Suppose that κ is an uncountable regular cardinal and that $c : [2^\kappa]^2 \to \kappa$ is almost Δ-regressive and Borel. Then there are c-monochromatic sets of size κ.*

Proof. Let $\mu < \kappa$ witness that c is almost Δ-regressive. By recursion on ξ, we will construct sequences $(x_\xi)_{\xi<\kappa}$, $(\nu_\xi)_{\xi<\kappa}$, $(s_\xi)_{\xi<\kappa}$, $(Y_\xi)_{\xi<\kappa}$, and $(i_\xi)_{\xi<\kappa}$ such that

- for all $\xi < \kappa$, $x_\xi \in 2^\kappa$;
- $(\nu_\xi)_{\xi<\kappa}$ is an increasing, continuous sequence of infinite ordinals below κ;
- $(s_\xi)_{\xi<\kappa}$ is a $\subseteq$-increasing sequence of elements of $2^{<\kappa}$;
- for all $\xi < \kappa$, we have $\nu_\xi < |s_\xi| < |s_\xi|+1 = \nu_{\xi+1}$, $s_\xi \restriction \nu_\xi = x_\xi \restriction \nu_\xi$, and $s_\xi(\nu_\xi) \neq x_\xi(\nu_\xi)$;
- for all $\xi < \kappa$, Y_ξ is a co-meagre subset of N_{s_ξ} and, for all $y \in Y_\xi$, we have $c(x_\xi, y) = i_\xi$;
- for all $\xi < \eta < \kappa$, we have $x_\eta \in Y_\xi$.

Begin by letting x_0 be an arbitrary element of 2^κ and letting $\nu_0 = \mu$. Define $t_0 \in 2^{\mu+1}$ by setting $t_0 \restriction \mu = x_0 \restriction \mu$ and $t_0(\mu) = 1 - x_0(\mu)$. By the Baire Category Theorem applied to N_{t_0}, we can find $i_0 < \kappa$ such that $Y_0^* := \{y \in N_{t_0} \mid c(x_0, y) = i_0\}$ is non-meagre. Since c is Borel, Y_0^* is a Borel subset of 2^κ. Since Borel sets have the Baire property and Y_0^* is a non-meagre Borel set, there is $s_0 \in 2^{<\kappa}$ such that Y_0^* is co-meagre in N_{s_0}. Note that s_0 extends t_0, so $s_0 \restriction \nu_0 = x_0 \restriction \nu_0$ and $s_0(\nu_0) \neq x_0(\nu_0)$. Set $Y_0 = Y_0^* \cap N_{s_0}$.

Suppose next that $\eta < \kappa$ and that $(x_\xi)_{\xi\leq\eta}$, $(\nu_\xi)_{\xi\leq\eta}$, $(s_\xi)_{\xi\leq\eta}$, $(Y_\xi)_{\xi\leq\eta}$, and $(i_\xi)_{\xi\leq\eta}$ have been constructed, satisfying the requirements listed above. For all $\xi \leq \eta$, Y_ξ is co-meagre in N_{s_η}, and hence $\bigcap_{\xi\leq\eta} Y_\xi$ is co-meagre in N_{s_η}. Let $x_{\eta+1}$ be an arbitrary element of $\bigcap_{\xi\leq\eta} Y_\xi$.

(Note that $x_{\eta+1} \in N_{s_\eta}$, since $Y_\eta \subseteq N_{s_\eta}$.) Let $\nu_{\eta+1} = |s_\eta|+1$, and define $t_\eta \in 2^{\nu_{\eta+1}}$ by letting $t_\eta \restriction |s_\eta| = s_\eta$ and $t_\eta(|s_\eta|) = 1 - x_{\eta+1}(|s_\eta|)$. Apply the Baire Category Theorem to N_{t_η} to find $i_{\eta+1} < \kappa$ such that $Y^*_{\eta+1} := \{y \in N_{t_\eta} \mid c(x_{\eta+1}, y) = i_{\eta+1}\}$ is non-meagre. Fix an $s_{\eta+1}$ such that $Y^*_{\eta+1}$ is co-meagre in $N_{s_{\eta+1}}$, and set $Y_{\eta+1} = Y^*_{\eta+1} \cap N_{s_{\eta+1}}$.

Finally, suppose that $\eta < \kappa$ is a limit ordinal and that $(x_\xi)_{\xi<\eta}$, $(\nu_\xi)_{\xi<\eta}$, $(s_\xi)_{\xi<\eta}$, $(Y_\xi)_{\xi<\eta}$, and $(i_\xi)_{\xi<\eta}$ have been constructed. Let $\nu_\eta = \sup\{\nu_\xi \mid \xi < \eta\}$, let $t^*_\eta = \bigcup_{\xi<\eta} s_\eta$, and note that $t^*_\eta \in 2^{\nu_\eta}$. For all $\xi < \eta$, Y_ξ is co-meagre in $N_{t^*_\eta}$, so we can let x_η be an arbitrary element of $N_{t^*_\eta} \cap \bigcap_{\xi<\eta} Y_\xi$. Define $t_\eta \in 2^{\nu_\eta+1}$ by letting $t_\eta \restriction \nu_\eta = t^*_\eta$ and $t_\eta(\nu_\eta) = 1 - x_\eta(\nu_\eta)$. Now proceed exactly as in the previous case to define i_η, s_η, and Y_η. This concludes the construction.

The point of our construction was to arrange so that, for all $\xi < \eta < \kappa$, we have $\Delta(x_\xi, x_\eta) = \nu_\xi$ and $c(x_\xi, x_\eta) = i_\xi$. Since c is almost Δ-regressive, it follows that the mapping $\nu_\xi \mapsto i_\xi$ is regressive, so, since $\{\nu_\xi \mid \xi < \kappa\}$ is a club in κ and hence stationary, we can apply the pressing-down lemma to find an unbounded $I \subseteq \kappa$ and a fixed $i < \kappa$ such that $i_\xi = i$ for all $\xi \in I$. In turn, $\{x_\xi \mid \xi \in I\}$ is a c-monochromatic set of size κ. □

Note that the above argument can be used to generate a monochromatic set that has lexicographic order type $\kappa + 1$. We do not know how far this can be generalized, even for continuous almost Δ-regressive colourings.

3. Maximal odd-cycle and triangle-free colourings

Definition 3.1. Suppose that X is a set and κ is a cardinal. A colouring $c : [X]^2 \to \kappa$ is a *maximal triangle-free colouring into* κ if

(1) c has no monochromatic triangles; and
(2) for any proper superset $Y \supsetneq X$ and any extension of c to $c' : [Y]^2 \to \kappa$, c' does have a monochromatic triangle.

The definition generalizes in the obvious way to *k-cycle free*, *odd-cycle-free*, etc.

The following proposition is immediate.

Proposition 3.2. *Suppose that* $c : [X]^2 \to \kappa$ *is a triangle-free colouring. The following statements are equivalent.*

(1) c *is a maximal triangle-free colouring.*
(2) For every function $d : X \to \kappa$*, there are distinct* $x, y \in X$ *such that* $d(x) = d(y) = c(x, y)$*.*

By the Erdős-Rado theorem, for infinite κ, if $c : [X]^2 \to \kappa$ is triangle-free (or odd-cycle-free, k-cycle-free, etc.), then it must be the case that $|X| \leq 2^\kappa$. Therefore, there must be maximal triangle-free (or odd-cycle-free, etc.) colourings into κ of size at most 2^κ; it turns out we have already seen an example of such a colouring of size exactly 2^κ.

For any infinite cardinal κ, the colouring $\Delta_\kappa : [2^\kappa]^2 \to \kappa$ is an odd-cycle-free colouring. In fact, it provides an example of a *maximal* odd-cycle free colouring and, indeed, a maximal k-cycle free colouring for each odd $k \geq 3$. We provide a proof of this fact for $k = 3$; an easy modification will work for odd $k > 3$, and we leave this to the reader.

Proposition 3.3. *Suppose that* κ *is an infinite cardinal. Then* Δ_κ *is a maximal triangle-free colouring into* κ*.*

Proof. Suppose not. Then, by Proposition 3.2, there is a function $d : 2^\kappa \to \kappa$ such that, for all distinct $x, y \in 2^\kappa$, it is not the case that $d(x) = d(y) = \Delta(x, y)$.

We will now construct an element $z \in 2^\kappa$ such that, for all $\alpha < \kappa$, there is no $x \in 2^\kappa$ such that $x \restriction (\alpha+1) = z \restriction (\alpha+1)$ and $d(x) = \alpha$. This will immediately result in a contradiction, because if $\alpha = d(z)$, then we clearly have $z \restriction (\alpha+1) = z \restriction (\alpha+1)$ and $d(z) = \alpha$.

We will construct z by specifying $z \restriction \alpha$ by recursion on $\alpha < \kappa$. To this end, fix $\alpha < \kappa$ and suppose that we have constructed $z \restriction \alpha$. We claim that there is at most one $i < 2$ for which there exists $x \in 2^\kappa$ such that $x \restriction (\alpha+1) = (z \restriction \alpha)^\frown\langle i\rangle$ and $d(x) = \alpha$. Indeed, otherwise there would be $x_0, x_1 \in 2^\kappa$ such that

- $x_0 \restriction \alpha = x_1 \restriction \alpha = z \restriction \alpha$;
- $x_0(\alpha) = 0$ and $x_1(\alpha) = 1$;
- $d(x_0) = d(x_1) = \alpha$.

But, in this case, we would have $\Delta(x_0, x_1) = \alpha = d(x_0) = d(x_1)$, contradicting our assumptions about d. Therefore, we can choose $i < 2$ such that there is no $x \in 2^\kappa$ with $x \restriction (\alpha+1) = (z \restriction \alpha)^\frown\langle i\rangle$ and $d(x) = \alpha$, and then set $z(\alpha) = i$. This completes the construction of z and thus the proof of the proposition. □

A natural question to ask now is the following: If $c : [X]^2 \to \kappa$ is a maximal triangle-free (or odd-cycle-free, etc.) colouring into κ, must it be the case that $|X| = 2^\kappa$? For the case of odd-cycle-free colourings, we have an affirmative answer. To help us prove this, let us introduce the following notion.

Definition 3.4. Suppose that $X \subseteq 2^\kappa$ and $c : [X]^2 \to \kappa$. We say that c is a δ-colouring if, for all distinct $x, y \in X$, we have $x(c(x, y)) \neq y(c(x, y))$.

Notice the following relevant facts about δ-colourings, which are easily verified:

- All δ-colourings are odd-cycle-free.
- If $c : [X]^2 \to \kappa$ is a δ-colouring, then c can be extended to a δ-colouring $c' : [2^\kappa]^2 \to \kappa$ by letting $c'(x, y) = \Delta(x, y)$ for all $\{x, y\} \in [2^\kappa]^2 \setminus [X]^2$.

Proposition 3.5. *Suppose that X is a set and $c : [X]^2 \to \kappa$ is odd-cycle-free. Then c is isomorphic to a δ-colouring. In other words, there is an injective function $\iota : X \to 2^\kappa$ such that that the function $c' : [\iota``X]^2 \to \kappa$ defined by $c'(\iota(x), \iota(y)) = c(x, y)$ is a δ-colouring.*

Proof. The fact that c is odd-cycle-free is equivalent to the assertion that, for each $\alpha < \kappa$, the graph $G_\alpha = (X, c^{-1}(\{\alpha\}))$ is bipartite. Therefore, for each $\alpha < \kappa$, we can partition X into two sets $X = X_0^\alpha \dot\cup X_1^\alpha$ such that for all $i < 2$ and distinct $x, y \in X_i^\alpha$, we have $c(x, y) \neq \alpha$. For each $x \in X$ and $\alpha < \kappa$, let $i_\alpha(x)$ be the unique $i < 2$ such that $x \in X_i^\alpha$, and define $\iota(x) \in 2^\kappa$ by letting $\iota(x)(\alpha) = i_\alpha(x)$ for all $\alpha < \kappa$.

Claim 3.6. *ι is injective.*

Proof. Fix distinct $x, y \in X$, and let $\alpha = c(x, y)$. Then it must be the case that $i_\alpha(x) \neq i_\alpha(y)$, so $\iota(x) \neq \iota(y)$. □

To finish the proof, it suffices to show that the colouring c' in the statement of the proposition is a δ-colouring. To see this, fix distinct $x, y \in X$ and let $\alpha = c(x, y) = c'(\iota(x), \iota(y))$. Then, by construction, $i_\alpha(x) \neq i_\alpha(y)$, so $\iota(x)(\alpha) \neq \iota(y)(\alpha)$, so c' is in fact a δ-colouring. □

Corollary 3.7. *If $c : [X]^2 \to \kappa$ is a maximal odd-cycle-free colouring into κ, then $|X| = 2^\kappa$.*

Proof. By Erdős-Rado, we know that $|X| \leq 2^\kappa$. For the other inequality, apply Proposition 3.5 to find an injection $\iota : X \to 2^\kappa$ such that the colouring $c' : [\iota``X]^2 \to \kappa$ defined by $c'(\iota(x), \iota(y)) = c(x, y)$ is a $\delta$-colouring. If $|X| < 2^\kappa$, then $\iota``X$ is a proper subset of 2^κ, so c' can be properly extended to a δ-colouring $d' : [2^\kappa]^2 \to \kappa$. But then d' easily induces a proper extension of c to an odd-cycle-free colouring $c' : [X \cup (2^\kappa \setminus \iota``X)]^2 \to \kappa$, contradicting the fact that c is a maximal odd-cycle-free colouring into κ. □

The analogous question about maximal triangle-free colourings remains open. The simplest case of this question asks whether there is always a maximal triangle-free colouring $c : [\omega_1]^2 \to \omega$, or even whether there is consistently such a colouring in a model of ¬CH. One way to ensure that a colouring $c : [\omega_1]^2 \to \omega$ is triangle-free is to require that all of the fibers $c(\cdot, \beta)$ be one-to-one. Such, colourings, however, necessarily fail to be maximal.

Proposition 3.8. *Suppose that $c : [\omega_1]^2 \to \omega$ has the property that, for all $\beta < \omega_1$, the map $c(\cdot, \beta) : \beta \to \omega$ is one-to-one. Then c is not a maximal triangle-free colouring.*

Proof. Suppose that c is triangle-free. To show that c is not maximal, it suffices to construct a function $d : \omega_1 \to \omega$ such that, for all $\alpha < \beta < \omega_1$, it is not the case that $d(\alpha) = d(\beta) = c(\alpha, \beta)$. To begin, fix a colour $i < \omega$ such that $i \neq c(0, 1)$, and let $d(0) = d(1) = i$. Next, for each $1 < \alpha < \omega_1$, there must be $\epsilon_\alpha < 2$ such that $c(\epsilon_\alpha, \alpha) \neq i$. Let $d(\alpha) = c(\epsilon_\alpha, \alpha)$.

To verify that d is as desired, fix $\alpha < \beta < \omega_1$. If $\alpha = 0$ and $\beta = 1$, then $d(\alpha) = d(\beta) = i$ and $c(\alpha, \beta) \neq i$, so the requirement is satisfied. If $\alpha < 2$ and $\beta \geq 2$, then $d(\alpha) = i$ and $d(\beta) \neq i$, so again the requirement is satisfied. Finally, if $2 \leq \alpha$, then $d(\beta) = c(\epsilon_\beta, \beta) \neq c(\alpha, \beta)$, since $c(\cdot, \beta)$ is injective, so the requirement is satisfied once again. □

At this point, it is unclear whether Proposition 3.8 can be strengthened to apply to maps with finite-to-one fibers. We do, however, have the following result.

Theorem 3.9. *Suppose that $c : [\omega_1]^2 \to \omega$ has the property that, for all $\beta < \omega_1$, the map $c(\cdot, \beta) : \beta \to \omega$ is finite-to-one. Then there is a ccc forcing notion $\mathbb{P}$ such that*

$$\Vdash_{\mathbb{P}} \text{“}\check{c} \textit{ is not a maximal triangle-free colouring”}.$$

Proof. Suppose that c is triangle-free. Our forcing notion $\mathbb{P}$ will consist of finite attempts to extend the colouring c. More precisely, conditions of $\mathbb{P}$ are pairs $p = (s^p, f^p)$ such that

- $s^p \in [\omega_1]^{<\omega}$;
- $f^p : s^p \to \omega$;
- for all $\alpha < \beta$ in s^p, it is not the case that $f^p(\alpha) = f^p(\beta) = c(\alpha, \beta)$.

If $p, q \in \mathbb{P}$, then $q \leq_{\mathbb{P}} p$ if $s^q \supseteq s^p$ and $f^q \supseteq f^p$.

Claim 3.10. *$\mathbb{P}$ has the ccc.*

Proof. Suppose for sake of contradiction that $\mathcal{A} = \{p_\eta \mid \eta < \omega_1\}$ is an antichain in $\mathbb{P}$. For $\eta < \omega_1$, let $s^\eta = s^{p_\eta}$ and $f^\eta = f^{p_\eta}$. By thinning out $\mathcal{A}$ if necessary, we can assume that the sets $\{s^\eta \mid \eta < \omega_1\}$ form a head-tail-tail Δ-system with root r. More precisely, for all $\eta < \xi < \omega_1$, we have

- $s^\eta \cap s^\xi = r$; and
- $r < s^\eta \setminus r < s^\xi \setminus r$.

By thinning out further, we can also assume that there is a single function $g : r \to \omega$ such that $f^\eta \restriction r = f$ for all $\eta < \omega_1$.

It follows that, for all $\eta < \xi < \omega_1$, $f^\eta \cup f^\xi$ is a function. Let $q_{\eta\xi} = (s^\eta \cup s^\xi, f^\eta \cup f^\xi)$. If $q_{\eta\xi}$ were a condition in $\mathbb{P}$, then it would be a common lower bound to p_η and p_ξ, contradicting the assumption that $\mathcal{A}$ is an antichain. Therefore, by the definition of $\mathbb{P}$, there must be $\alpha_{\eta\xi} \in s^\eta \setminus r$ and $\beta_{\eta\xi} \in s^\xi \setminus r$ such that $f^\eta(\alpha_{\eta\xi}) = f^\xi(\beta_{\eta\xi}) = c(\alpha_{\eta\xi}, \beta_{\eta\xi})$.

Now, if $\omega \leq \xi < \omega_1$, there must be a fixed $\beta_\xi \in s^\xi \setminus r$ such that the set $X = \{\eta < \xi \mid \beta_{\eta\xi} = \beta_\xi\}$ is infinite. But then, for all $\eta \in X$, we have $c(\alpha_{\eta\xi}, \beta_\xi) = f^\xi(\beta_\xi)$ and, for all $\eta < \eta'$ in X, we have $\alpha_{\eta\xi} < \alpha_{\eta'\xi} < \beta$. Therefore, $c(\cdot, \beta)$ is not finite-to-one, with the failure witnessed by the colour $f^\xi(\beta_\xi)$ and the infinite set $\{\alpha_{\eta\xi} \mid \eta \in X\}$. This is a contradiction. □

For each $\alpha < \omega_1$, let $D_\alpha = \{p \in \mathbb{P} \mid \alpha \in s^p\}$. It is easily verified that D_α is dense in $\mathbb{P}$ for all $\alpha < \omega_1$. Thus, if G is $\mathbb{P}$-generic over V, then $f_G = \bigcup\{f^p \mid p \in G\}$ is a function from ω_1 to ω such that, for all $\alpha < \beta < \omega_1$, it is not the case that $f_G(\alpha) = f_G(\beta) = c(\alpha, \beta)$. By Proposition 3.2, it follows that c is not a maximal triangle-free colouring in $V[G]$. □

Corollary 3.11. *If MA_{ω_1} holds, then there are no maximal triangle-free colourings $c : [\omega_1]^2 \to \omega$ with finite-to-one fibers.*

Proof. Assume that MA_{ω_1} holds, and fix a triangle-free colouring $c : [\omega_1]^2 \to \omega$ with finite-to-one fibers. Apply MA_{ω_1} to the poset $\mathbb{P}$ and the dense sets $\{D_\alpha \mid \alpha < \omega_1\}$ isolated in the proof of Theorem 3.9 to obtain a function $f : \omega_1 \to \omega$ witnessing that c is not maximal. □

Though we do not know of the consistency of a maximal triangle-free colouring $c : [\omega_1]^2 \to \omega$ in a model of $\neg$CH, we can arrange the consistency of the existence of a a maximal triangle-free colouring of some proper subset of $[\omega_1]^2$. Here, we say that an ω-colouring F whose domain is a subset of $[\omega_1]^2$ is maximal if, for every $d : \omega_1 \to \omega$, there are $\alpha < \beta < \omega_1$ such that $\{\alpha, \beta\} \in \mathrm{dom}(F)$ and $d(\alpha) = d(\beta) = F(\alpha, \beta)$.

Theorem 3.12. *There is a ccc poset $\mathbb{P}$ of size $\aleph_1$ such that*

$\Vdash_{\mathbb{P}}$"*There is a maximal monochromatic triangle-free ω-colouring F with* $\mathrm{dom}\, F \subset [\omega_1]^2$".

In particular, the continuum can be arbitrary large. However, we lack techniques to define a maximal F on all of $[\omega_1]^2$.

Proof. For each $\delta \in \lim \omega_1$, pick $\varepsilon_\delta < \delta$ so that the set $\{\delta \mid \varepsilon_\delta = \varepsilon\}$ is stationary for all ε.

Let $\mathbb{P}$ consist of all $p = (s^p, (g^p_k)_{k \in n^p})$ so that

(1) $s^p \in [\omega_1]^{<\omega}$, $n^p \in \omega$, $g^p_k \subset [s^p]^2$;
(2) g^p_k is triangle-free;
(3) $g^p_k \cap g^p_\ell = \emptyset$ for all $k < \ell < n^p$; and
(4) for $\delta' < \delta$, if $\delta'\delta \in g^p_k$, then $\varepsilon_\delta \leq \delta'$.

If $H \subset \mathbb{P}$ is a generic filter, we let $F(\delta'\delta) = k$ for some $\delta' < \delta$ if there is $p \in H$ so that $k < n^p$ and $\delta'\delta \in g^p_k$.[1]

Let us show that F is maximal, which will also imply that ω_1 is not collapsed.

Claim 3.13. *For any partition $\omega_1 = \bigcup_{k \in \omega} X_k$ there is some $k < \omega$ and $\delta' < \delta \in X_k$ so that $F(\delta'\delta) = k$.*

Proof. Working back in V, fix $\mathbb{P}$-names $(\dot{X}_k)_{k \in \omega}$ for $(X_k)_{k \in \omega}$, and fix a condition p such that $p \Vdash$"$\omega_1 = \bigcup_{k \in \omega} \dot{X}_k$". Take a continuous, increasing sequence of elementary submodels $(M_\alpha)_{\alpha < \omega_1}$ of some sufficiently large $H(\theta)$ so that $p, (\dot{X}_k)_{k \in \omega} \in M_0$.

Let $\varepsilon = \omega_1 \cap M_0$ and find $\alpha < \omega_1$ so that $\varepsilon = \varepsilon_\delta$ where $\delta = M_\alpha \cap \omega_1$. Now, find some $q \leq p$ and $k \in \omega$ so that $\delta \in s^q$ and $q \Vdash$"$\delta \in \dot{X}_k$". By extending q further, we can assume that $k < n^q$.

[1]So, in a condition p, g^p_k approximates the colour class $F^{-1}(k)$.

We can write s^q as the union of the three sets $s_0 < s_1 < s_2$ where $s_0 = s^q \cap M_0$, $s_1 = s^q \cap M_\alpha \setminus M_0$, and $s_2 = s^q \setminus M_\alpha$.

Subclaim 3.13.1. *There is a $q' \in \mathbb{P}$ with $s^{q'} = s_0 \cup s'_1 \cup s'_2$ with $\delta' = \min s'_2$ and*
(i) q and q' are isomorphic,
(ii) $s_0 < s'_1 \subset M_0$, $s_1 < s'_2 \subset M_\alpha$,
(iii) $q' \Vdash$ "$\delta' \in \dot{X}_k$".

The proof is a double reflection argument using elementarity.

Now, we can define a condition r that extends both q and q' such that $r \Vdash$ "$\dot{F}(\delta'\delta) = k$". Indeed, we let $s^r = s^q \cup s^{q'}$, $n^r = n^q = n^{q'}$ and $g^r_\ell = g^q_\ell \cup g^{q'}_\ell$ for $k \neq \ell < n^q$ and let $g^r_k = g^q_k \cup g^{q'}_k \cup \{\delta'\delta\}$.

The only way that r can fail to be a condition in $\mathbb{P}$ is if g^r_k is not triangle-free. However, any triangle in g^r_k must contain the new edge $\delta'\delta$ and their common neighbour must lie in s_0. However $s_0 \subset \varepsilon_\delta$ while any neighbour of δ is at least ε_δ. □

Proving the ccc of $\mathbb{P}$ is very similar but we don't even need to add any edge when amalgamating isomorphic conditions. □

Note that in the above proof, we proved that each colour class of F has uncountable chromatic number.

4. Further remarks on triangle-free colourings

In this section, we prove some further results about triangle-free colourings on $[\omega_1]^2$ motivated by the question from the previous section about whether there necessarily exists a maximal triangle-free colouring $c : [\omega_1]^2 \to \omega$. This can be seen as a specific instance of a more general question: Do there exist triangle-free colourings $c : [\omega_1]^2 \to \omega$ in which all colour classes $c^{-1}(\{k\})$ are "large"? This leads naturally to the consideration of square bracket partition relations, whose definition we now recall.

Definition 4.1. For cardinals κ, λ, μ, and θ, the partition relation $\kappa \to [\lambda]^\theta_\mu$ is the assertion that, for every $f : [\kappa]^\theta \to \mu$, there is $X \in [\kappa]^\lambda$ such that $f``[X]^\theta \neq \mu$, i.e., f omits at least one colour on $[X]^\theta$. If κ is a regular cardinal, then the relation $\kappa \to [Stat]^\theta_\mu$ is obtained by replacing the requirement $X \in [\kappa]^\lambda$ above with the requirement that X is stationary in κ.

Much work has been done analyzing colourings on $[\omega_1]^2$ witnessing the failure of square bracket partition relations, with the most notable result being Todorčević's proof of $\omega_1 \not\to [\omega_1]^2_{\omega_1}$ in [11]. Recall from the previous section that, if $c : [\omega_1]^2 \to \omega$ is triangle-free, then it is a *maximal* triangle-free colouring into ω if, for every function $d : \omega_1 \to \omega$, there are $\alpha < \beta < \omega_1$ such that $d(\alpha) = d(\beta) = c(\alpha, \beta)$. This latter condition is easily seen to be satisfied if c witnesses $\omega_1 \not\to [\omega_1]^2_\omega$. However, the following easy observation shows that such colourings can never be triangle-free.

Observation 4.2. *Suppose that $c : [\omega_1]^2 \to \omega$ witnesses $\omega_1 \not\to [Stat]^2_\omega$. Then c has infinite monochromatic sets in all colours.*

Proof. This follows from the Dushnik-Miller relation: $\omega_1 \to (Stat, \omega)^2_2$, i.e., for every function $f : [\omega_1]^2 \to 2$, there is either a stationary $X \subseteq \omega_1$ such that $f``[X]^2 = \{0\}$ or there is an infinite $Y \subseteq \omega_1$ such that $f``[Y]^2 = \{1\}$. If we could not find an infinite monochromatic set of colour k then there would be a stationary set that omits colour k, which contradicts the assumption on c. □

On the other hand, the following holds, where $\mathfrak{s}$ denotes the *splitting number*.[2]

Theorem 4.3 (D. Raghavan)**.** *Suppose that $\mathfrak{s} = \aleph_1$. Then there is a triangle-free $c : [\omega_1]^2 \to \omega$ so that for any uncountable $X \subset \omega_1$, $c``[X]^2$ is co-finite.*

We wonder if the conclusion of this theorem holds in ZFC.

One way in which a subset of $[\omega_1]^2$ can be considered "large" is by having uncountable chromatic number as a graph. It is easily seen that there are always triangle-free colourings $c : [\omega_1]^2 \to \omega$ for which all colour classes have uncountable chromatic number.

Proposition 4.4. *There is a colouring $c : [\omega_1]^2 \to \omega$ so that each colour class $G_n = c^{-1}(n)$ is triangle-free of uncountable chromatic number.*

Proof. Let $\omega_1 = \bigcup\{S_n : n < \omega\}$ be a partition into uncountable sets, and, for each $n < \omega$, let H_n be a triangle-free graph of uncountable chromatic number with vertex set S_n (for an example of such a graph, see [2]). Now, define c so that $c(\alpha, \beta) = n$ if $\alpha\beta \in H_n$ and so that for any $\beta \in S_n$,

$$c(\cdot, \beta) \restriction \{\alpha < \beta : \alpha\beta \notin \bigcup_{n<\omega} H_n\}$$

is injective and maps into $\omega \setminus \{n\}$. It is easy to see that c satisfies our requirements. □

Next, we prove that using some additional assumptions, we can make each colour class G_n quite thin. If $G \subseteq [\omega_1]^2$ is a graph and $\alpha < \omega_1$, then we let $G(\alpha) = \{\beta < \alpha \mid \{\beta, \alpha\} \in G\}$. Recall that a graph $G \subseteq [\omega_1]^2$ is a *Hajnal-Máté graph* if, for every $\alpha < \omega_1$, $G(\alpha)$ is either a finite set or an ω-sequence converging to α.

The existence of Hajnal-Máté graphs with uncountable chromatic number turns out to be independent of ZFC. In [6], Hajnal and Máté prove that $\diamondsuit^+$ implies the existence of Hajnal-Máté graphs with uncountable chromatic number, while Martin's Axiom, $\mathrm{MA}_{\aleph_1}$, implies that every Hajnal-Máté graph has countable chromatic number. Komjáth, in [7], improves upon the first result by proving that, if $\diamondsuit$ holds, then there are triangle-free Hajnal-Máté graphs with uncountable chromatic number. We improve this result further with the following theorem.

Theorem 4.5. *Suppose that $\diamondsuit$ holds. Then there is a partition $[\omega_1]^2 = \bigcup_{n<\omega} G_n$ such that each G_n is a triangle-free Hajnal-Máté graph with uncountable chromatic number.*

Proof. Since $\diamondsuit$ holds, we can find pairwise disjoint stationary sets $\{S^n_\delta \mid n < \omega,\ \delta < \omega_1\}$ such that $\diamondsuit(S^n_\delta)$ holds for each $n < \omega$ and $\delta < \omega_1$. For each n and δ, we can assume that S^n_δ consists solely of limit ordinals greater than δ, and we can fix a sequence $\langle f^n_{\delta,\alpha} : \alpha \to \omega \mid \alpha < \omega_1\rangle$ such that, for every $f : \omega_1 \to \omega$, there are stationarily many $\alpha \in S^n_\delta$ for which $f^n_{\delta,\alpha} = f \restriction \alpha$.

We are going to define a function $g : [\omega_1]^2 \to \omega$ such that, for all $n < \omega$, $G_n = g^{-1}\{n\}$ will be as desired. We will define $g \restriction [\alpha]^2$ by recursion on $\alpha < \omega_1$. For each $\alpha < \omega_1$, let $g_\alpha : \alpha \to \omega$ denote the function defined by letting $g_\alpha(\beta) = g(\beta, \alpha)$ for all $\beta < \alpha$.

Suppose that $\alpha < \omega_1$ and that we have defined $g \restriction [\alpha]^2$. We now describe how to define $g \restriction [\alpha + 1]^2$, which amounts to defining $g_\alpha : \alpha \to \omega$. If there are no $n < \omega$ and $\delta < \omega_1$ such that $\alpha \in S^n_\delta$, then simply let g_α be an arbitrary injective function. Note that this introduces no triangles to any G_n and ensures that $|G_n(\alpha)| = 1$ for every $n < \omega$.

[2]Unpublished result from personal communication.

Otherwise, let $n < \omega$ and $\delta < \omega_1$ be such that $\alpha \in S^n_\delta$. We first specify the set of $\beta < \alpha$ for which $g_\alpha(\beta) = n$. Begin by fixing an increasing ω-sequence $\langle \alpha_k \mid k < \omega \rangle$ converging to α with $\delta = \alpha_0$. By recursion on $k < \omega$, we will construct a set $A_\alpha \subseteq \omega$ and a sequence $\langle \beta^\alpha_k \mid k \in A_\alpha \rangle$ such that

- for all $k \in A_\alpha$, we have $\max\{\alpha_k, \max\{\beta^\alpha_j \mid j \in A_\alpha \cap k\}\} < \beta^\alpha_k < \alpha$;
- for all $k \in A_\alpha$, we have $f^n_{\delta,\alpha}(\beta^\alpha_k) = k$;
- for all $j < k$ in A_α, we have $g(\beta^\alpha_j, \beta^\alpha_k) \neq n$.

The construction is straightforward. If $k < \omega$ and we have specified $A_\alpha \cap k$ and $\{\beta^\alpha_j \mid j \in A_\alpha \cap k\}$, then ask whether there is β such that $\max\{\alpha_k, \max\{\beta^\alpha_j \mid j \in A_\alpha \cap k\}\} < \beta < \alpha$, $f^n_{\delta,\alpha}(\beta) = k$, and, for all $j \in A_\alpha \cap k$, $g(\beta^\alpha_j, \beta) \neq n$. If there is, then put k into A_α and let β^α_k be the least such β. Otherwise, leave k out of A_α and leave β^α_k undefined.

Now define $g_\alpha : \alpha \to \omega$ by first requiring that $G_n(\alpha) = \{\beta^\alpha_k \mid k \in A_\alpha\}$. Note that this set is either finite or an ω-sequence converging to α. Now define g_α on $\alpha \setminus G_n(\alpha)$ to be an injective function into $\omega \setminus \{n\}$. Also note that our construction adds no new triangles to any G_n.

This finishes the construction of g. It is clear that each G_n is a triangle-free Hajnal-Máté graph. It remains to show that each G_n is uncountably chromatic. Suppose for sake of contradiction that $n < \omega$ and $f : \omega_1 \to \omega$ is a proper colouring for G_n. For each $\delta < \omega_1$, we introduce the following notation.

- T_δ is the stationary set of $\alpha \in S^n_\delta$ for which $f^n_{\delta,\alpha} = f \restriction \alpha$.
- For all $k < \omega$, $T_{\delta,k}$ is the set of $\alpha \in T_\delta$ for which $f(\alpha) = k$.

It is easy to see that there must be $k < \omega$ such that, for unboundedly many $\delta < \omega_1$, $T_{\delta,k}$ is stationary in ω_1. Fix such a k. Let $T = \bigcup_{\delta<\omega_1} T_{\delta,k}$, and let $E = \{\delta < \omega_1 \mid T_{\delta,k} \text{ is unbounded in } \omega_1\}$. By our choice of k, T is stationary and E is unbounded in ω_1.

Using the normality of the club filter, we can find $\alpha \in T \cap \lim(E) \cap (\Delta_{\delta \in E} \lim(T_{\delta,k}))$. Let $\delta^* < \omega_1$ be such that $\alpha \in S^n_{\delta^*}$. Since $f(\alpha) = k$, it must be the case that, in our construction of g_α, we left β^α_k undefined, because otherwise we would have $f(\beta^\alpha_k) = f^n_{\delta^*,\alpha}(\beta^\alpha_k) = k = f(\alpha)$ and $\{\beta^\alpha_k, \alpha\} \in G_n$, contradicting the fact that f is a good colouring for G_n. But now we can find $\delta \in E \cap \alpha$ such that $\beta^\alpha_j < \delta$ for all $j \in A_\alpha \cap k$. By our choice of α, we can find $\beta \in T_{\delta,k} \cap \alpha$ with $\beta > \alpha_k$. By our construction of g_β, it follows that, for every $\gamma < \delta$, $g_\beta(\gamma) \neq n$. It also follows that $f^n_{\delta^*,\alpha}(\beta) = f(\beta) = k$. But then it is easily seen that β gives a positive answer to the question asked at stage k of the construction of A_α, in which case β^α_k is in fact defined. This contradiction completes the proof. □

Note that the graphs G_n defined in the proof above actually satisfy the following strengthening of triangle-freeness: for all $3 \leq \ell < \omega$, there are no cycles $\langle \alpha_0, \alpha_1, \ldots, \alpha_{\ell-1} \rangle$ for which $\alpha_0 < \alpha_1 < \ldots < \alpha_{\ell-1}$.

5. Open problems

In this final section, we collect some remaining open problems. We start with the most important questions stemming directly from our investigations. First, on regressive and almost-regressive colourings we ask the following.

Problem 5.1. *Suppose that κ is a regular uncountable cardinal, $2^\mu < 2^\kappa$ for every $\mu < \kappa$, and c is an almost Δ-regressive colouring on $[2^\kappa]^2$.*

(1) Does c necessarily have monochromatic triples?

(2) Does c necessarily have infinite monochromatic subsets?
(3) Does c necessarily have monochromatic subsets of size κ?

Problem 5.2. *Suppose that κ is a regular uncountable cardinal and c is a Δ-regressive colouring on $[2^\kappa]^2$. Must there exist a c-monochromatic set of size 4? What about an infinite c-monochromatic set? What about uncountable c-monochromatic sets?*

Regarding maximal triangle-free colourings, the next questions are the most natural.

Problem 5.3. *Is there a maximal triangle-free colouring $c : [\omega_1]^2 \to \omega$?*

Problem 5.4. *Is there, consistently, a maximal triangle-free colouring of $[\omega_1]^2$ which embeds no uncountable δ-colourings?*

Problem 5.5. *Assume $MA_{\aleph_1}$ (or even PFA). Suppose that $X \subset \mathbb{R}$ has size $\aleph_1$ and $c : [X]^2 \to \omega$ is a continuous/Borel. Can c be maximal triangle-free?*

It would also be natural to look at colouring triples and in general, $[2^\kappa]^n$ for some finite $n < \omega$ and look for critical examples. We mention the following results.

Theorem 5.6 (Todorcevic, [12])**.** *There is a colouring $c : [2^{\omega_1}]^3 \to 10$ such that all colours appear on any uncountable $X \subseteq 2^{\omega_1}$. More generally, for every $r \geq 3$, there is a colouring $c : [2^{\omega_1}]^r \to r!\,(r-1)! - 2$ such that all colours appear on any uncountable $X \subseteq 2^{\omega_1}$.*

Proposition 5.7. *There is a colouring $c : [2^{\omega_1}]^3 \to \omega_1$ such that all colours appear on any dense-in-itself $X \subseteq 2^{\omega_1}$.*

Proof. Use a colouring witnessing $\omega_1 \not\to [\omega_1]^2_{\omega_1}$ on the two Δ-values determined by any triple in $[2^{\omega_1}]^3$. $\square$

Next, we mention a prominent open problem concerning uncountable triangle-free graphs that is tangentially related to our work.

Problem 5.8 (Erdős)**.** *Is there, in ZFC, a graph G of uncountable chromatic number so that any triangle-free subgraph of G has countable chromatic number?*

Komjáth and Shelah [9] proved that, consistently, the answer is yes. They also proved that if $\chi(G) \geq \mathfrak{c}^+$ then either G contains a K_4 or a triangle free subgraph of uncountable chromatic number. We wonder if similarly to this and [10] where the finite case is dealt with, one can show:

Problem 5.9. *Suppose that κ is strongly inaccessible, and let G have chromatic number κ. Is there, for any $\lambda < \kappa$, a triangle-free subgraph of G of chromatic number at least λ?*

Finally, we end with two questions about colourings $c : [\omega_1]^2 \to \omega$ with large colour classes.

Problem 5.10. *Is there, in ZFC, a triangle-free $c : [\omega_1]^2 \to \omega$ so that for any uncountable $X \subset \omega_1$, $c \restriction [X]^2$ assumes all but finitely many colours.*

Problem 5.11. *Is there a triangle-free $c : [\omega_1]^2 \to \omega$ so that for any partition $\omega_1 = \bigcup_{i<\omega} X_i$, there is some $i < \omega$ so that $c \restriction [X_i]^2$ assumes all values.*

This last question seems closely related to the simultaneous chromatic number problems studied by Hajnal and Komjáth [5].

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