EDGE-COLORINGS OF INFINITE COMPLETE GRAPHS

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1. Introduction

This note stems from some thinking in my spare time about edge-colorings of infinite complete graphs. Some of the reults contained in this note are well-known; none is claimed to be original. The note is not necessarily in a final form and may be expanded in the future.

The note deals primarily with colorings of the form $F : [X]^2 \to \omega$, where X is a set and $[X]^2$ denotes the set of all 2-element subsets of X. If x_0, x_1 are distinct elements of X, we will abuse notation and write $F(x_0, x_1)$ instead of $F(\{x_0, x_1\})$. We will focus in particular on colorings which are triangle-free or, more generally, odd-cycle-free in the following sense.

Definition 1.1. Let $F : [X]^2 \to \omega$ be a coloring.

(1) If $k < \omega$, a *mono-chromatic k-cycle* with respect to F is a set of distinct elements $\{x_\ell \mid \ell < k\} \subseteq X$ such that $F(x_0, x_1) = F(x_1, x_2) = \ldots = F(x_{k-2}, x_{k-1}) = F(x_{k-1}, x_0)$.

(2) F is *triangle-free* if there are no mono-chromatic 3-cycles with respect to F. If $k < \omega$, F is *k-cycle-free* if there are no mono-chromatic k-cycles with respect to F.

(3) F is *odd-cycle-free* if for all odd $k \geq 3$, there are no mono-chromatic k-cycles with respect to F. The notion of *even-cycle-free* is defined analogously.

Definition 1.2. ${}^\omega 2$ is the set of all functions $f : \omega \to 2$. If $f \neq g \in {}^\omega 2$, then $\Delta(f, g) = \min(\{n \mid f(n) \neq g(n)\})$. A coloring $F : [{}^\omega 2]^2 \to \omega$ is a δ-coloring if, for all distinct $f, g \in {}^\omega 2$,

$$(F(f, g) = n) \Rightarrow (f(n) \neq g(n)).$$

Remark 1.3. A δ-coloring is easily seen to odd-cycle-free. An example of a δ-coloring is given by $F(f, g) = \Delta(f, g)$.

Definition 1.4. A coloring $F : [X]^2 \to \omega$ is a *maximal triangle-free coloring* if it is triangle-free and, for every $Y \supsetneq X$ and every coloring $G : [Y]^2 \to \omega$ extending F, G is not triangle-free. Maximal odd-cycle-free colorings are defined analogously.

2. A maximal triangle-free coloring

Definition 2.1. If $n < \omega$ and $\sigma \in {}^n 2$, then $C_\sigma = \{f \in {}^\omega 2 \mid f \restriction n = \sigma\}$.

Theorem 2.2. *Let $F : [{}^\omega 2]^2 \to \omega$ be given by $F(f, g) = \Delta(f, g)$. Then F is a maximal triangle-free coloring.*

Proof. Fix some $x \notin {}^\omega 2$, let $X = {}^\omega 2 \cup \{x\}$, and suppose for sake of contradiction that there is a triangle-free coloring $G : [X]^2 \to \omega$ extending F. We will recursively construct a specific element $h \in {}^\omega 2$ and use it to derive a contradiction.

We will define an increasing sequence of natural numbers $\{n_k \mid k < \omega\rangle\}$ and recursively define $h \restriction n_k$. Let $n_0 = 0$. Suppose $k < \omega$ and n_k, $h \restriction n_k$ have been defined. Let m_k be the least $m \geq n_k$ such that, for some $f \in C_{h \restriction n_k}$, $G(x, f) = m$. Let f_k be such an f, and let $n_{k+1} = m_k + 1$. Define $h \restriction n_{k+1}$ by letting $h(i) = f_k(i)$ for all $n \leq i < m$ and $h(m) = 1 - f_k(m)$.

Let $n^* = G(x, h)$.

Claim 2.3. *There is $0 < k < \omega$ such that $n^* = n_k - 1$.*

Proof. Let $k < \omega$ be least such that $n^* < n_k$, and let $j = k - 1$. We defined m_j to be the least $m \geq n_j$ such that, for some $f \in C_{h \restriction n_j}$, $G(x, f) = m$. Since $g \in C_{h \restriction n_j}$, $n^* \geq n_j$, and $G(x, h) = n^*$, we must have $m_j \leq n^*$. But $n_k - 1 = m_j$, so $n_k - 1 \leq n^* < n_k$ and thus $n^* = n_k$. □

Let $k^* < \omega$ be such that $n^* = n_{k^*} - 1$. Consider the function f_{k^*} used in the definition of h. By construction, we have $f_{k^*} \restriction n^* = h \restriction n^*$ and $h(n^*) = 1 - f_{k^*}(n^*)$. Thus, $G(f_{k^*}, h) = \Delta(h, f_{k^*}) = n^*$. Moreover, we know that $G(x, h) = n^* = G(x, f_{k^*})$. Thus, x, f_{k^*}, and h are distinct elements of X such that $G(x, f_{k^*}) = G(f_{k^*}, h) = G(x, h)$, contradicting the assumption that G is triangle-free. □

3. Maximal odd-cycle-free colorings

Definition 3.1. Let $F : [X]^2 \to \omega$ and $G : [Y]^2 \to \omega$ be colorings. F and G are *isomorphic* if there is a bijection $\pi : F \to G$ such that, for all distinct $x_0, x_1 \in X$, $F(x_0, x_1) = G(\pi(x_0), \pi(x_1))$.

Theorem 3.2. *Let $F : [X]^2 \to \omega$ be an odd-cycle-free coloring. Then there is a δ-coloring $G : [{}^\omega 2]^2 \to \omega$ and a set $Y \subseteq {}^\omega 2$ such that F is isomorphic to $G \restriction [Y]^2$.*

Proof. For each $n < \omega$, let $\Gamma_n = (V_n, E_n)$ be a graph with vertex set X such that, for distinct $x_0, x_1 \in X$, $x_0 E_n x_1$ iff $F(x_0, x_1) = n$. Since F is an odd-cycle-free coloring, Γ_n is a bipartite graph. Thus, we can partition X into disjoint pieces X^n_0, X^n_1 such that, for all $i \in \{0, 1\}$ and distinct $x_0, x_1 \in X^n_i$, $F(x_0, x_1) \neq n$. Define a function $\pi : X \to {}^\omega 2$ by letting $\pi(x)$ be the unique $f \in {}^\omega 2$ such that, for all $n < \omega$, $x \in X^n_{f(n)}$.

Claim 3.3. *π is injective.*

Proof. Suppose for sake of contradiction that $x_0 \neq x_1$ and $\pi(x_0) = \pi(x_1) = f$. Let $n = F(x_0, x_1)$. But then $x_0, x_1 \in X^n_{f(n)}$, which is a contradiction to the definition of X^n_0, X^n_1. □

Let $Y = \pi[X]$, and define $G^* : [Y]^2 \to \omega$ by $G^*(\pi(x_0), \pi(x_1)) = F(x_0, x_1)$. By construction, F is isomorphic to G^*.

Claim 3.4. *If $f \neq g \in Y$ and $G^*(f, g) = n$, then $f(n) \neq g(n)$.*

Proof. Let $f \neq g \in Y$, and let $f = \pi(x_0)$, $g = \pi(x_1)$. If $G^*(f, g) = n$, then $F(x_0, x_1) = n$, so $\neg x_0 E_n x_1$, so $\pi(x_0)(n) \neq \pi(x_1)(n)$, so $f(n) \neq g(n)$. □

Now extend G^* to a full coloring $G : [{}^\omega 2]^2 \to 2$ by letting $G(f, g) = \Delta(f, g)$ for all $\{f, g\} \in [{}^\omega 2]^2 \setminus [Y]^2$. It is clear that G is a δ-coloring and that F is isomorphic to $G \restriction [Y]^2$. □

Corollary 3.5. *If $F : [X]^2 \to \omega$ is a maximal odd-cycle-free coloring, then $|X| = 2^{\aleph_0}$.*

Even-cycle-free colorings behave very differently. Every even-cycle-free coloring with countably many colors has cardinality at most $\aleph_1$. In fact, we have the following strong result.

Theorem 3.6. *Suppose $c : [\omega_2]^2 \to \omega$, and let $k < \omega$. Then there are $n^* < \omega$ and sets $X, Y \subseteq \omega_2$ such that $|X| = k$, $|Y| = \omega_2$, and, for every $\alpha \in X$ and $\beta \in Y$, $c(\alpha, \beta) = n^*$. In particular, there are mono-chromatic $2k$-cycles for every $1 < k < \omega$.*

Proof. For $\beta \in [\omega_1, \omega_2)$, let $n_\beta < \omega$ be such that $|\{\alpha < \omega_1 \mid c(\alpha, \beta) = n_\beta\}| = \aleph_1$. Find an unbounded $S \subseteq [\omega_1, \omega_2)$ and an $n^* < \omega$ such that, for all $\beta \in S$, $n_\beta = n^*$. For each $\beta \in S$, find $X_\beta \in [\omega_1]^k$ such that, for all $\alpha \in X_\beta$, $c(\alpha, \beta) = n^*$. Find an unbounded $Y \subseteq S$ and a fixed $X \in [\omega_1]^k$ such that, for all $\beta \in Y$, $X_\beta = X$. Then X, Y, and n^* are as desired.

To show that there is a mono-chromatic $2k$-cycle, take X, Y, and n^* as above. Let $X = \{\alpha_\ell \mid \ell < k\}$. Let $Z \in [Y]^k$, $Z = \{\beta_\ell \mid \ell < k\}$. Then $c(\alpha_0, \beta_0) = c(\beta_0, \alpha_1) = c(\alpha_1, \beta_1) = ... = c(\alpha_{k-1}, \beta_{k-1}) = c(\beta_{k-1}, \alpha_1) = n^*$, giving us a mono-chromatic $2k$-cycle. □

4. Coloring numbers

Definition 4.1. Let $G = (V, E)$ be a graph. The *coloring number of* G, which we will denote $c(G)$ is the least cardinal κ such that there is a well-ordering $\langle v_\alpha \mid \alpha < \eta \rangle$ of V such that, for every $\beta < \eta$, the set $E_\beta := \{\alpha < \beta \mid \{v_\alpha, v_\beta\} \in E\}$ has cardinality less than κ.

Definition 4.2. Let κ and λ be cardinals. Then $K_{\kappa,\lambda}$ is the complete bipartite graph in which the two sides have cardinality κ and λ, respectively.

Theorem 4.3. *Suppose $G = (V, E)$ is a graph, κ is an infinite cardinal, and $c(G) \geq \kappa^+$. Then, for every $n < \omega$, G contains a copy of K_{n,κ^+}.*

Proof. We prove the contrapositive by induction on $|G|$. Thus, suppose $|G| = \lambda$, $n < \omega$, and G contains no copy of K_{n,κ^+}. Suppose we have shown by induction that all subgraphs of G of smaller cardinality have coloring number less than κ^+. We prove that $c(G) < \kappa^+$. If $\lambda \leq \kappa$, then V can be well-ordered in order-type $\leq \kappa$, so $c(G) \leq \kappa < \kappa^+$, and we are done. Thus, suppose $\lambda > \kappa$.

If $X \in [V]^n$, let $f(X) = \{u \in V \mid \text{for every } v \in X,\ \{u, v\} \in E\}$. By our assumption that G contains no copy of K_{n,κ^+}, $|f(X)| \leq \kappa$ for every $X \in [V]^n$. If $Y \subseteq V$, let $F(Y) = \bigcup\{f(X) \mid X \in [Y]^n\}$. Since $|[Y]^n| = |Y|$, we have $|F(Y)| \leq \max(|Y|, \kappa)$. Given $Y \subseteq V$, we define $H(Y)$ as follows. Let $Y_0 = Y$ and, for all $n < \omega$, let $Y_{n+1} = Y_n \cup F(Y_n)$. Let $H(Y) = \bigcup_{n<\omega} Y_n$. Then $|H(Y)| \leq \max(|Y|, \kappa)$, and $H(Y)$ has the property that $F(H(Y)) \subseteq H(Y)$.

Let $\mu = \mathrm{cf}(\lambda)$, and let $\langle V_i \mid i < \mu \rangle$ be an increasing, continuous sequence of subsets of V such that $V = \bigcup_{i<\mu} V_i$. By recursion, we will define a well-ordering $\langle v_\alpha \mid \alpha < \lambda \rangle$ together with an increasing, continuous sequence of ordinals $\langle \lambda_i \mid i < \mu \rangle$ such that, denoting $\{v_\alpha \mid \alpha < \lambda_i\}$ by U_i, we have:

- for all $\beta < \lambda$, $E_\beta := \{\alpha < \beta \mid \{v_\alpha, v_\beta\} \in E\}$ has cardinality less than κ;
- for all $i < \mu$, $V_i \subseteq U_{i+1}$;
- for all $i < \mu$, $F(U_i) \subseteq U_i$.

Let $U_0 = \emptyset$. $\lambda_0 = 0$. Suppose $j < \mu$ is a limit ordinal and U_i, λ_i, and the relevant well-orderings have been defined for all $i < j$. Let $U_j = \bigcup_{i<j} U_i$ and

$\lambda_j = \sup(\{\lambda_i \mid i < j\})$. The well-ordering $\{v_\alpha \mid \alpha < \lambda_j\}$ of U_j has been defined in the previous steps. Finally, suppose $i < \lambda$ and U_i, λ_i, and $\{v_\alpha \mid \alpha < \lambda_i\}$ has been defined. Let $U_{i+1} = H(U_i \cup V_i)$. Let $W_{i+1} = U_{i+1} \setminus U_i$. Then $|W_{i+1}| < \lambda$ and, since $F(U_i) \subseteq U_i$, every element of W_{i+1} is connected by an edge in E to at most $n-1$ elements of U_i. By our inductive hypothesis, the subgraph of G induced by W_{i+1} has coloring number less than κ^+, so W_{i+1} can be well-ordered as $\{w_\gamma \mid \gamma < \eta\}$ for some $\eta < \lambda$ such that, for all $\delta < \eta$, the set $\{\gamma < \delta \mid \{w_\gamma, w_\delta\} \in E\}$ has cardinality less than κ. Let $\lambda_{i+1} = \lambda_i + \eta$ (ordinal addition), and extend the well-ordering of U_i to a well-ordering of U_{i+1} by letting, for all $\gamma < \eta$, $v_{\lambda_i + \gamma} = w_\gamma$.

At the end of the construction, $\langle v_\alpha \mid \alpha < \lambda \rangle$ is a well-ordering of V that is easily seen to witness $c(G) < \kappa^+$. □

Lemma 4.4. *Suppose $\kappa \leq \lambda$ are infinite cardinals and $G = (V, E)$ is a graph with $|G| = \lambda$ and $c(G) = \kappa$. Then there is a well-ordering of G of order type λ witnessing $c(G) = \kappa$.*

Proof. If $\kappa = \lambda$, then any well-ordering of G of order type λ witnesses $c(G) = \kappa$. Thus, suppose $\kappa < \lambda$. Let $\mu = \mathrm{cf}(\lambda)$, and let $\langle V_i \mid i < \mu \rangle$ be an increasing, continuous sequence of subsets of V such that:

- $\bigcup_{i<\mu} V_i = V$;
- for all $i < \mu$, $|V_i| < \lambda$.

Let $\lhd$ be a well-ordering of V witnessing that $c(G) = \kappa$. For each $v \in V$, let $f(v) = \{u \in V \mid u \lhd v \text{ and } \{u, v\} \in E\}$. By our assumption, $|f(v)| < \kappa$. Given $X \subseteq V$, let $F(X) = \bigcup\{f(v) \mid v \in X\}$. Note that $|F(X)| \leq \max(\{\kappa, |X|\})$.

Given $X \subseteq V$, define $H(X)$ as follows. Let $X_0 = X$. Given X_n, with $n < \omega$, let $X_{n+1} = X_n \cup F(X_n)$. Finally, let $H(X) = \bigcup_{n<\omega} X_n$. Then $|H(X)| \leq \max(\{\kappa, |X|\})$ and $H(X)$ has the property that, if $v \in H(X)$, $u \lhd v$, and $\{u, v\} \in E$, then $u \in H(X)$.

For $i < \mu$, let $U_i = H(V_i)$. Since $|V_i| < \lambda$, we also get $|U_i| < \lambda$. Also note that, if $j < \mu$ is a limit ordinal, then $U_j = \bigcup_{i<j} U_i$. For $i < \mu$, let $W_i = U_{i+1} \setminus U_i$. For each $v \in V$, there is a unique $i < \mu$, which we denote $i(v)$, such that $v \in W_i$. Define a well-ordering $\prec$ on V by letting $u \prec v$ iff one of the following two conditions holds:

1. $i(u) < i(v)$;
2. $i(u) = i(v)$ and $u \lhd v$.

It is easily verified that $\prec$ is a well-order. Every initial segment of $\prec$ is contained in U_i for some $i < \mu$ and therefore has order type less than λ. Thus, the order type of $\prec$ is exactly λ. It remains to show that, for every $v \in V$, the set $\{u \in V \mid u \prec v$ and $\{u, v\} \in E\}$ has cardinality less than κ. This will follow from the following claim.

Claim 4.5. *For all $u, v \in V$, if $u \prec v$ and $\{u, v\} \in E$, then $u \lhd v$.*

Proof. Fix $u, v \in V$ such that $u \prec v$ and $\{u, v\} \in E$. If $i(u) = i(v)$, then $u \lhd v$ by definition of $\prec$. Thus, suppose $i(u) < i(v)$. $v \notin U_{i(u)}$. In particular, $v \notin f(u)$, which means it is not the case that $v \lhd u$ and $\{u, v\} \in E$. Since $\{u, v\} \in E$, this implies that $u \lhd v$. □

Theorem 4.6. *Let $\kappa < \lambda$ be infinite cardinals, with λ regular. Suppose $c : [\lambda^+]^2 \to \kappa$ is a coloring. For all $\eta < \kappa$, let $G_\eta = (\lambda^+, E_\eta)$ be a graph with vertex set λ such that, for $\alpha < \beta < \lambda^+$, $\{\alpha, \beta\} \in E_\eta$ if and only if $c(\alpha, \beta) = \eta$. Then there is $\eta < \kappa$ such that $c(G_\eta) \geq \lambda$.*

Proof. Suppose not. For each $\eta < \kappa$, let $\langle \alpha^\eta_\xi \mid \xi < \lambda^+ \rangle$ be a well-ordering of λ^+ witnessing that $c(G_\eta) < \lambda$. For $\eta < \kappa$ and $\xi < \lambda^+$, let $V^\eta_\xi = \{\alpha^\eta_\zeta \mid \zeta < \xi\}$. Find ξ^* large enough so that, for all $\eta < \kappa$, $\lambda \subseteq V^\eta_{\xi^*}$. Find $\beta^* < \lambda^+$ such that, for all $\eta < \kappa$, $\beta^* \notin V^\eta_{\xi^*}$. Find $\eta^* < \kappa$ and an unbounded $A \subseteq \lambda$ such that, for all $\alpha < \lambda$, $c(\alpha, \beta^*) = \eta^*$. Then, for every $\alpha \in A$, we have that $\{\alpha, \beta^*\} \in E_{\eta^*}$ and α is enumerated before β^* in the well-ordering $\langle \alpha^{\eta^*}_\xi \mid \xi < \lambda^+ \rangle$. This contradicts the fact that $\langle \alpha^{\eta^*}_\xi \mid \xi < \lambda^+ \rangle$ witnesses $c(G_{\eta^*}) < \lambda$. □

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