BOUNDED STATIONARY REFLECTION II

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Abstract. Bounded stationary reflection at a cardinal λ is the assertion that every stationary subset of λ reflects but there is a stationary subset of λ that does not reflect at arbitrarily high cofinalities. We produce a variety of models in which bounded stationary reflection holds. These include models in which bounded stationary reflection holds at the successor of every singular cardinal $\mu > \aleph_\omega$ and models in which bounded stationary reflection holds at μ^+ but the approachability property fails at μ.

1. Introduction

The reflection of stationary sets is a topic of fundamental interest in the study of combinatorial set theory, large cardinals, and inner model theory and provides a useful tool for the investigation of the tension between compactness and incompactness phenomena. In this paper, we extend results, inspired by a question of Eisworth, of Cummings and the author [3]. We start by reviewing the relevant definitions and providing an outline of the structure of the paper.

Definition 1.1. Let $\lambda > \omega_1$ be a regular cardinal.

(1) If $S \subseteq \lambda$ is a stationary set and $\alpha < \lambda$ has uncountable cofinality, then S *reflects at* α if $S \cap \alpha$ is stationary in α. S *reflects* if there is $\alpha < \lambda$ with uncountable cofinality such that S reflects at α.

(2) If μ is a singular cardinal and $\lambda = \mu^+$, then $\mathrm{Refl}(\lambda)$ holds if every stationary subset of λ reflects.

(3) If μ is a singular cardinal, $\lambda = \mu^+$, and $S \subseteq \lambda$ is stationary, then S *reflects at arbitrarily high cofinalities* if, for all $\kappa < \mu$, there is $\alpha < \lambda$ such that $\mathrm{cf}(\alpha) \geq \kappa$ and S reflects at α.

(4) If μ is a singular cardinal and $\lambda = \mu^+$, then $\mathrm{bRefl}(\lambda)$ (bounded stationary reflection at λ) holds if $\mathrm{Refl}(\lambda)$ holds but there is a stationary $T \subseteq \lambda$ that does not reflect at arbitrarily high cofinalities.

Eisworth [4] asked whether $\mathrm{bRefl}(\lambda)$ is consistent when λ is the successor of a singular cardinal. $\mathrm{bRefl}(\aleph_{\omega+1})$ is easily seen to be inconsistent, but Cummings and the author showed in [3] that, for other values of λ, $\mathrm{bRefl}(\lambda)$ is consistent modulo large cardinal assumptions. In particular, the following theorem was proven.

Theorem 1.2. *Suppose there is a proper class of supercompact cardinals. Then there is a class forcing extension in which, for every singular cardinal $\mu > \aleph_\omega$ such that μ is not a cardinal fixed point,* $\mathrm{bRefl}(\mu^+)$ *holds.*

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This left open the question of whether it is consistent that $\mathrm{bRefl}(\mu^+)$ holds for every singular cardinal $\mu > \aleph_\omega$. In this paper, we answer this question affirmatively and prove a number of variations on Theorem 1.2.

In Section 2, we briefly discuss the notion of approachability before defining some of the forcing posets to be used throughout the paper and introducing their basic properties. In Section 3, we prove a general lemma about iteratively destroying stationary sets. In Section 4, we prove a dense version of Theorem 1.2 by producing a model in which $\mathrm{Refl}(\aleph_{\omega\cdot 2+1})$ holds and, for every stationary $S \subseteq S^{\aleph_{\omega\cdot 2+1}}_{<\aleph_\omega}$, there is a stationary $T \subseteq S$ that does not reflect at arbitrarily high cofinalities. In Section 5, we prove a global version of Theorem 1.2 by producing a model in which, for every singular cardinal $\mu > \aleph_\omega$, $\mathrm{bRefl}(\mu^+)$ holds.

The proofs of the results in Sections 4 and 5 and in [3] rely heavily on the approachability property holding in the final model. The relationship between approachability and stationary reflection is complicated and interesting, and in the last two sections of this paper we investigate the extent to which we can get bounded stationary reflection together with the failure of approachability. In Section 6, we produce a model with a singular cardinal μ such that AP_μ fails and $\mathrm{bRefl}(\mu^+)$ holds. In this model μ is a limit of cardinals which are supercompact in an outer model. In Section 7, we show that this result can be attained with $\mu = \aleph_{\omega^2\cdot 2}$.

Our notation is for the most part standard. The primary reference for all undefined notions and notations is [7]. If $\kappa < \lambda$ are infinite cardinals, with κ regular, then $S^\lambda_\kappa = \{\alpha < \lambda \mid \mathrm{cf}(\alpha) = \kappa\}$. Expressions such as $S^\lambda_{>\kappa}$ or $S^\lambda_{\geq\kappa}$ are defined in the obvious way. If X is a set of ordinals, then $\mathrm{nacc}(X)$ (the set of non-accumulation points of X) is the set $\{\alpha \in X \mid \sup(X \cap \alpha) < \alpha\}$, X' is the set of limit points of X (i.e. $X \setminus \mathrm{nacc}(X)$), and $\mathrm{otp}(X)$ is the order type of X. If $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\zeta \mid \xi \leq \gamma, \zeta < \gamma\rangle$ is a forcing iteration with supports of size μ for some cardinal μ, we will frequently write $\Vdash_\xi$ instead of $\Vdash_{\mathbb{P}_\xi}$. Conditions of $\mathbb{P}_\gamma$ are thought of as functions p such that $\mathrm{dom}(p) \in [\gamma]^{\leq\mu}$ and, for all $\zeta \in \mathrm{dom}(p)$, $\Vdash_\zeta$ "$p(\zeta) \in \dot{\mathbb{Q}}_\zeta$." For $\zeta < \xi \leq \gamma$, we let $\dot{\mathbb{P}}_{\zeta,\xi}$ be a $\mathbb{P}_\zeta$-name such that $\mathbb{P}_\xi \cong \mathbb{P}_\zeta * \dot{\mathbb{P}}_{\zeta,\xi}$.

2. Approachability and forcing preliminaries

Definition 2.1. Let λ be a regular, uncountable cardinal.

(1) Let $\vec{a} = \langle a_\alpha \mid \alpha < \lambda\rangle$ be a sequence of bounded subsets of λ. If $\gamma < \lambda$, γ is *approachable with respect to* $\vec{a}$ if there is an unbounded $A \subseteq \gamma$ such that $\mathrm{otp}(A) = \mathrm{cf}(\gamma)$ and, for every $\beta < \gamma$, there is $\alpha < \gamma$ such that $A \cap \beta = a_\alpha$.
(2) If $B \subseteq \lambda$, then $B \in I[\lambda]$ if there is a club $C \subseteq \lambda$ and a sequence $\vec{a} = \langle a_\alpha \mid \alpha < \lambda\rangle$ of bounded subsets of λ such that, for every $\gamma \in B \cap C$, $\mathrm{cf}(\gamma) < \gamma$ and γ is approachable with respect to $\vec{a}$.
(3) If μ is a singular cardinal and $\lambda = \mu^+$, then AP_μ is the assertion that $\lambda \in I[\lambda]$.

A wealth of information about approachability, including proofs of the statements in the following remark, can be found in [5, Section 3].

Remark 2.2. Let λ be a regular, uncountable cardinal.

(1) $I[\lambda]$ is a normal, λ-complete ideal extending the non-stationary ideal on λ.
(2) $S^\lambda_\omega \in I[\lambda]$, and, if κ is a regular uncountable cardinal such that $\kappa^+ < \lambda$, then there is a stationary $S \subseteq S^\lambda_\kappa$ such that $S \in I[\lambda]$.

(3) Suppose $\lambda^{<\lambda} = \lambda$, and let $\vec{a} = \langle a_\alpha \mid \alpha < \lambda \rangle$ be a fixed enumeration of all bounded subsets of λ. If $B \subseteq \lambda$, then $B \in I[\lambda]$ iff there is a club $C \subseteq \lambda$ such that every element of $B \cap C$ is approachable with respect to $\vec{a}$.

Definition 2.3. Let θ be a regular cardinal, and let $\lhd$ be a well-ordering of $H(\theta)$. An *internally approachable chain of substructures of* $H(\theta)$ is a $\subseteq$-increasing, continuous sequence $\langle M_i \mid i < \eta \rangle$ such that, for all $i < \eta$:

- $M_i \prec (H(\theta), \in, \lhd)$.
- $\langle M_k \mid k \leq i \rangle \in M_{i+1}$.

The notion of approachability is intimately connected with internally approachable chains. The following result is obtained in the proof of Claim 4.4 in [6].

Lemma 2.4. *Let $\lambda < \theta$ be regular cardinals, let $x \in H(\theta)$, and let $\lhd$ be a well-ordering of $H(\theta)$. Suppose $S \in I[\lambda]$. Then there is a club $C \subseteq \lambda$ such that, for every $\gamma \in C \cap S \cap S^\lambda_{>\omega}$, letting $\kappa = \mathrm{cf}(\gamma)$, there is an internally approachable chain $\langle M_i \mid i < \kappa \rangle$ of substructures of $H(\theta)$ such that:*

(1) *For all $i < \kappa$, $|M_i| < \kappa$.*
(2) *$x \in M_0$.*
(3) *If $M = \bigcup_{i<\kappa} M_i$, then $\gamma = \sup(M \cap \lambda)$.*

Before we introduce specific forcing posets, we recall the notions of directed closure and strategic closure.

Definition 2.5. Let $\mathbb{P}$ be a partial order.

(1) A subset $D \subseteq \mathbb{P}$ is *directed* if, for all $p, q \in D$, there is $r \in D$ such that $r \leq p, q$.
(2) Suppose μ is a cardinal. $\mathbb{P}$ is *μ-directed closed* if, whenever $D \subseteq \mathbb{P}$ is directed and $|D| < \mu$, there is $q \in \mathbb{P}$ such that, for all $p \in D$, $q \leq p$.

Definition 2.6. Let $\mathbb{P}$ be a partial order and let β be an ordinal.

(1) The two-player game $G_\beta(\mathbb{P})$ is defined as follows: Players I and II alternately play entries in $\langle p_\alpha \mid 0 < \alpha < \beta \rangle$, a decreasing sequence of conditions in $\mathbb{P}$. Player I plays at odd stages, and Player II plays at even stages (including all limit stages). If there is a limit stage $\alpha < \beta$ at which $\langle p_\eta \mid 0 < \eta < \alpha \rangle$ has no lower bound, then Player I wins. Otherwise, Player II wins.
(2) $G^*_\beta(\mathbb{P})$ is defined just as $G_\beta(\mathbb{P})$ except that Player I plays at limit stages instead of Player II.
(3) $\mathbb{P}$ is *β-strategically closed* if Player II has a winning strategy for the game $G_\beta(\mathbb{P})$. $\mathbb{P}$ is *strongly β-strategically closed* if Player II has a winning strategy for the game $G^*_\beta(\mathbb{P})$. The notions of *$(< \beta)$-strategically closed* and *strongly $(< \beta)$-strategically closed* are defined in the obvious way.

Remark 2.7. Note that a μ-directed closed forcing notion is strongly $(< \mu)$-strategically closed. Also observe that, if $\mathbb{P}$ is $(< \mu)$-strategically closed, then forcing with $\mathbb{P}$ does not add any new sequences of ordinals of length $< \mu$.

We now define a number of forcing posets to be used throughout the paper and state some of their basic properties. We first introduce a poset used to force AP_μ. Suppose μ is a singular cardinal, $\lambda = \mu^+$, and $\lambda^{<\lambda} = \lambda$. Let $\vec{a} = \langle a_\alpha \mid \alpha < \lambda \rangle$ be an enumeration of the bounded subsets of λ. Let S be the set of ordinals that are approachable with respect to $\vec{a}$. The poset $\mathbb{A}_{\vec{a}}$ consists of closed, bounded subsets of

S and is ordered by end-extension, i.e. $c \leq d$ if $c \cap (\max(d)+1) = d$. The following is due to Shelah. For a proof, see Fact 2.8 of [12]. It is explicitly shown there only that $\mathbb{A}_{\vec{a}}$ is $(<\lambda)$-strategically closed, but the proof easily yields the stronger statement.

Proposition 2.8. *$\mathbb{A}_{\vec{a}}$ is strongly $(<\lambda)$-strategically closed.*

Since $\lambda^{<\lambda} = \lambda$, $|\mathbb{A}_{\vec{a}}| = \lambda$, so $\mathbb{A}_{\vec{a}}$ has the λ^+-c.c., and hence forcing with $\mathbb{A}_{\vec{a}}$ preserves all cardinalities and cofinalities.

Next, we define a poset to add a stationary set that only reflects at points of small cofinality. Let $\kappa < \mu < \lambda$ be infinite, regular cardinals. Conditions in $\mathbb{S}^\lambda_{\kappa,\mu}$ are functions $s \in {}^{\gamma_s+1}2$ such that:

(1) $\gamma_s < \lambda$.
(2) $\{\alpha \leq \gamma_s \mid s(\alpha) = 1\} \subseteq S^\lambda_\kappa$.
(3) $\{\alpha \leq \gamma_s \mid s(\alpha) = 1\} \cap \beta$ is nonstationary in β for all $\beta \in S^\lambda_{\geq\mu}$.

Conditions are ordered by reverse inclusion. We will sometimes abuse notation and identify s with $\{\alpha \leq \gamma_s \mid s(\alpha) = 1\}$ in statements such as "s does not reflect at any $\beta \in S^\lambda_{\geq\mu}$." Note, however, that recovering s from $\{\alpha \leq \gamma_s \mid s(\alpha) = 1\}$ requires the parameter γ_s.

Proofs of the following facts can be found in Section 2 of [3].

Lemma 2.9. *Let $\mathbb{S} = \mathbb{S}^\lambda_{\kappa,\mu}$.*

(1) *$\mathbb{S}$ is μ-directed closed.*
(2) *$\mathbb{S}$ is $(<\lambda)$-strategically closed.*
(3) *Let G be $\mathbb{S}$-generic over V, and let $S = \{\alpha < \lambda \mid$ for some $s \in G$, $s(\alpha) = 1\}$. Then, in $V[G]$, S is a subset of S^λ_κ that does not reflect at any ordinal in $S^\lambda_{\geq\mu}$.*

If $\lambda^{<\lambda} = \lambda$, then $|\mathbb{S}^\lambda_{\kappa,\mu}| = \lambda$, so, in this case, $\mathbb{S}^\lambda_{\kappa,\mu}$ has the λ^+-c.c. and forcing with $\mathbb{S}^\lambda_{\kappa,\mu}$ preserves all cardinalities and cofinalities. Lemma 2.10 below, combined with clause (2) of Remark 2.2, will imply that the generic $S \subseteq S^\lambda_\kappa$ added by $\mathbb{S}^\lambda_{\kappa,\mu}$ is stationary in λ.

For some constructions we will need a variant of $\mathbb{S}^\lambda_{\kappa,\mu}$. Let $\mu < \lambda$ be infinite, regular cardinals, and let $T \subseteq S^\lambda_{<\mu}$ be stationary. $\mathbb{S}_{T,\mu}$ is defined exactly as $\mathbb{S}^\lambda_{\kappa,\mu}$ except that, for $s \in \mathbb{S}_{T,\mu}$, instead of clause (2), we require that, if $s(\alpha) = 1$, then $\alpha \in T$. Note that $\mathbb{S}^\lambda_{\kappa,\mu} = \mathbb{S}_{S^\lambda_\kappa,\mu}$. The purpose of $\mathbb{S}_{T,\mu}$ is to add a subset of T that does not reflect at any ordinals in $S^\lambda_{\geq\mu}$. The same arguments used for $\mathbb{S}^\lambda_{\kappa,\mu}$ show that $\mathbb{S}_{T,\mu}$ is μ-directed closed and $(<\lambda)$-strategically closed and, assuming $\lambda^{<\lambda} = \lambda$, has the λ^+-c.c. If S (with canonical name $\dot{S}$) is the subset of T added by $\mathbb{S}_{T,\mu}$, it is clear that S does not reflect at any ordinals in $S^\lambda_{\geq\mu}$. With an additional assumption, we can ensure as well that S is stationary.

Lemma 2.10. *Suppose there is a stationary $T^* \subseteq T$ such that $T^* \in I[\lambda]$. Then $\Vdash_{\mathbb{S}_{T,\mu}}$ "$\dot{S}$ is stationary."*

Proof. Let $\mathbb{S} = \mathbb{S}_{T,\mu}$. Let $\dot{C}$ be an $\mathbb{S}$-name for a club in λ, and let $s \in \mathbb{S}$. We will find $s^* \leq s$ and $\beta < \lambda$ such that $s^* \Vdash$ "$\beta \in \dot{S} \cap \dot{C}$."

Let θ be a sufficiently large regular cardinal. Suppose first that $T \cap S^\lambda_\omega$ is stationary, and find a countable $M \prec (H(\theta), \in, \lhd)$ such that $\{\mathbb{S}, s, \dot{C}\} \subseteq M$ and $\beta := \sup(M \cap \lambda) \in T$. Let $\langle \beta_n \mid n < \omega \rangle$ be an increasing sequence of ordinals from

M that is cofinal in β. Construct a decreasing sequence $\langle s_n \mid n < \omega \rangle$ of conditions from $\mathbb{S} \cap M$ and an increasing sequence of ordinals $\langle \alpha_n \mid n < \omega \rangle$ from M such that:

- $s_0 \leq s$.
- For all $n < \omega$, $\min(\gamma_{s_n}, \alpha_n) > \beta_n$ and $s_n \Vdash$ "$\alpha_n \in \dot{C}$."

The construction is straightforward. Let $s^* = \{(\beta, 1)\} \cup \bigcup_{n<\omega} s_n$. Then $s^* \leq s_n$ for all $n < \omega$. Therefore, $s^* \Vdash$ "$\{\alpha_n \mid n < \omega\} \subseteq \dot{C}$". In particular, $s^* \Vdash$ "β is a limit point of $\dot{C}$," so, since $\dot{C}$ is forced to be a club and $s^*(\beta) = 1$, $s^* \Vdash$ "$\beta \in \dot{S} \cap \dot{C}$."

Suppose next that $T \cap S^\lambda_\omega$ is non-stationary. Since $T^* \in I[\lambda]$, we can apply Lemma 2.4 to find $M \prec H(\theta)$ such that:

- $\beta := \sup(M \cap \lambda) \in T^* \cap S^\lambda_{>\omega}$.
- $|M| = \kappa$, where $\kappa = \mathrm{cf}(\beta)$.
- M is the union of an internally approachable chain $\langle M_i \mid i < \kappa \rangle$ of substructures of $H(\theta)$, where $|M_i| < \kappa$ for all $i < \kappa$.
- $\{\mathbb{S}, s, \dot{C}\} \subseteq M_0$.

We now construct a decreasing sequence of conditions $\langle s_i \mid i < \kappa \rangle$ such that:

- $s_0 \leq s$.
- For all $i < \kappa$, $s_i \in M_{i+1}$.
- For all $i < \kappa$, there is $\alpha_i \geq \sup(M_i \cap \lambda)$ such that $\alpha_i \in M_{i+1}$ and $s_i \Vdash$ "$\alpha_i \in \dot{C}$."

The construction is a straightforward recursion using the μ-closure of $\mathbb{S}$ and maintaining the additional requirement, made possible by the internal approachability of $\langle M_i \mid i < \kappa \rangle$, that, for every $i < \kappa$, $\langle s_j \mid j < i \rangle \in M_{i+1}$. Now let $s^* = \{(\beta, 1)\} \cup \bigcup_{i<\kappa} s_i$. s^* is easily seen to be a member of $\mathbb{S}$, and, as in the previous case, $s^* \Vdash$ "$\beta \in \dot{S} \cap \dot{C}$." □

We now introduce a well-known poset used to destroy the stationarity of subsets of λ, where λ is an uncountable, regular cardinal. Let S be a subset of λ, and let $CU(S)$ consist of closed, bounded $t \subset \lambda$ such that $t \cap S = \emptyset$ and, if $\alpha \in \mathrm{nacc}(t)$, then $\mathrm{cf}(\alpha) = \omega$. (This last condition is not strictly necessary, but it will make certain technical points simpler.) We denote $\max(t)$ by γ_t. If G is $CU(S)$-generic over V, then S is no longer stationary in $V[G]$. In general, forcing with $CU(S)$ can collapse cardinals. However, if S was just added by $\mathbb{S}^\lambda_{\kappa,\mu}$ or $\mathbb{S}_{T,\mu}$, then the forcing is quite nice.

Lemma 2.11. *Let $\mathbb{S} = \mathbb{S}^\lambda_{\kappa,\mu}$ (or $\mathbb{S}_{T,\mu}$), and let $\dot{S}$ be a name for the stationary subset of λ added by $\mathbb{S}$. Then $\mathbb{S} * CU(\dot{S})$ has a λ-closed dense subset.*

Proof. Let $\mathbb{U}$ be the set of $(s, \dot{t}) \in \mathbb{S} * CU(\dot{S})$ such that there is $t \in V$ such that $s \Vdash$ "$\dot{t} = t$" and $\gamma_s = \gamma_t$. We first show that $\mathbb{U}$ is λ-closed. Let $\delta < \lambda$ be a limit ordinal, and suppose $\langle (s_\eta, \dot{t}_\eta) \mid \eta < \delta \rangle$ is a strictly decreasing sequence from $\mathbb{U}$, where, for each $\eta < \delta$, $t_\eta \in V$ witnesses $(s_\eta, \dot{t}_\eta) \in \mathbb{U}$. Let $\gamma^* = \sup(\{\gamma_{s_\eta} \mid \eta < \delta\})$, let $s^* = \{\gamma^*, 0\} \cup \bigcup_{\eta<\delta} s_\eta$, let $t^* = \{\gamma^*\} \cup \bigcup_{\eta<\delta} t_\eta$, and let $\dot{t}^*$ be an $\mathbb{S}$-name forced by s^* to be equal to t^*. Since t^* witnesses that s^* does not reflect at γ^*, it is straightforward to verify that $(s^*, \dot{t}^*) \in \mathbb{U}$ and is a lower bound for $\langle (s_\eta, \dot{t}_\eta) \mid \eta < \delta \rangle$.

We next show that $\mathbb{U}$ is dense in $\mathbb{S} * CU(\dot{S})$. To this end, let $(s, \dot{t}) \in \mathbb{S} * CU(\dot{S})$. We will find $(s^*, \dot{t}^*) \leq (s, \dot{t})$ with $(s^*, \dot{t}^*) \in \mathbb{U}$. Since $\mathbb{S}$ is $(< \lambda)$-strategically closed and hence does not add any new sequences of ordinals of length $< \lambda$, we can find $s_0 \leq s$ and $t \in V$ such that $s_0 \Vdash$ "$\dot{t} = t$." Without loss of generality, $\gamma_{s_0} \geq \gamma_t$. Let

$\gamma^* = \gamma_{s_0} + \omega$. Define $s^* \leq s_0$ with $\gamma_{s^*} = \gamma^*$ by letting $s^*(\alpha) = 0$ for all $\alpha \in (\gamma_{s_0}, \gamma^*]$, let $t^* = t \cup \{\gamma^*\}$, and let $\dot{t}^*$ be an $\mathbb{S}$-name forced by s^* to be equal to t^*. Then $(s^*, \dot{t}^*)$ is as desired. □

We will need the following well-known facts:

Fact 2.12. [9, Lemma 3] *Let κ be a regular cardinal, and let $\kappa < \lambda < \mu$. Suppose that, in $V^{\mathrm{Coll}(\kappa,<\lambda)}$, $\mathbb{P}$ is a separative, strongly κ-strategically closed partial order and $|\mathbb{P}| < \mu$. Let i be the natural complete embedding of* $\mathrm{Coll}(\kappa, < \lambda)$ *into* $\mathrm{Coll}(\kappa, < \mu)$ *(namely, the identity embedding). Then i can be extended to a complete embedding j of* $\mathrm{Coll}(\kappa, < \lambda) * \mathbb{P}$ *into* $\mathrm{Coll}(\kappa, < \mu)$ *so that the quotient forcing* $\mathrm{Coll}(\kappa, < \mu)/j[\mathrm{Coll}(\kappa, < \lambda) * \mathbb{P}]$ *is κ-closed.*

Fact 2.13. [11, Theorem 20] *Let μ and κ be cardinals. Suppose that AP_κ holds, S is a stationary subset of $S^{\kappa^+}_{<\mu}$, and $\mathbb{P}$ is a μ-closed forcing poset. Then S remains stationary in $V^{\mathbb{P}}$.*

3. Destroying stationary sets

At many points in this paper, we will want to use a forcing iteration to destroy the stationarity of many sets. We will also want to ensure that we do not collapse any cardinals in the process. With this in mind, we prove here a general lemma which we will use, either directly or via a modification, throughout the remainder of the paper.

Let λ be a regular cardinal, and assume that $\lambda^{<\lambda} = \lambda$ and $2^\lambda = \lambda^+$. Let $\mathbb{S}$ be a λ^+-c.c. forcing poset, and let $\dot{\mathbb{T}}$ be an $\mathbb{S}$-name for a forcing poset such that $\mathbb{S} * \dot{\mathbb{T}}$ has a dense λ-closed subset (note that, in particular, this implies that $\mathbb{S}$ is λ-distributive and $\Vdash_{\mathbb{S}}$ "$\dot{\mathbb{T}}$ is λ-distributive").

In $V^{\mathbb{S}}$, let $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\zeta \mid \xi \leq \lambda^+, \zeta < \lambda^+ \rangle$ be a forcing iteration with supports of size $< \lambda$ such that, for every $\zeta < \lambda^+$, there is a $\mathbb{P}_\zeta$-name $\dot{T}_\zeta$ for a subset of λ such that $\Vdash_{\mathbb{P}_\zeta * \dot{\mathbb{T}}}$ "$\dot{T}_\zeta$ is non-stationary" and $\Vdash_{\mathbb{P}_\zeta}$ "$\dot{\mathbb{Q}}_\zeta = CU(\dot{T}_\zeta)$." Let $\mathbb{P} = \mathbb{P}_{\lambda^+}$. By an easy Δ-system argument, $\mathbb{P}$ has the λ^+-c.c.

Lemma 3.1. *$\mathbb{S} * \dot{\mathbb{P}} * \dot{\mathbb{T}}$ has a dense λ-closed subset.*

Proof. For each $\zeta < \lambda^+$, let $\dot{C}_\zeta$ be an $\mathbb{S} * \dot{\mathbb{P}}_\zeta * \dot{\mathbb{T}}$-name for a club in λ disjoint from $\dot{T}_\zeta$.

Let $\mathbb{U}_0$ be the dense λ-closed subset of $\mathbb{S} * \dot{\mathbb{T}}$ that exists by assumption. For $\zeta \leq \lambda^+$, let $\mathbb{U}_\zeta$ be the set of $(s, \dot{p}, \dot{t}) \in \mathbb{S} * \dot{\mathbb{P}}_\zeta * \dot{\mathbb{T}}$ such that:

- $(s, \dot{t}) \in \mathbb{U}_0$.
- s decides the value of $\mathrm{dom}(\dot{p})$.
- For every $\xi \in \mathrm{dom}(\dot{p})$, $(s, \dot{p} \restriction \xi, \dot{t}) \Vdash_{\mathbb{S} * \dot{\mathbb{P}}_\xi * \dot{\mathbb{T}}}$ "$\max(\dot{p}(\xi)) \in \dot{C}_\xi$".

We will prove by induction on ζ that $\mathbb{U}_\zeta$ is a dense, λ-closed subset of $\mathbb{S} * \dot{\mathbb{P}}_\zeta * \dot{\mathbb{T}}$ for all $\zeta \leq \lambda^+$.

Thus, fix $\zeta \leq \lambda^+$, and assume we have proven that $\mathbb{U}_\xi$ is a λ-closed dense subset of $\mathbb{S} * \dot{\mathbb{P}}_\xi * \dot{\mathbb{T}}$ for all $\xi < \zeta$. We first show that $\mathbb{U}_\zeta$ is λ-closed. Let $\beta < \lambda$, and let $\langle (s_\alpha, \dot{p}_\alpha, \dot{t}_\alpha) \mid \alpha < \beta \rangle$ be a decreasing sequence of conditions from $\mathbb{U}_\zeta$. Let $X = \bigcup_{\alpha < \beta} \mathrm{dom}(\dot{p}_\alpha)$, and note that $X \in [\zeta]^{<\lambda}$. We will define a lower bound $(s^*, \dot{p}^*, \dot{t}^*) \in \mathbb{U}_\zeta$ such that $\mathrm{dom}(\dot{p}^*) = X$ as follows. First, let $(s^*, \dot{t}^*) \in \mathbb{U}_0$ be a lower bound for $\langle (s_\alpha, \dot{t}_\alpha) \mid \alpha < \beta \rangle$. Next, we define $\dot{p}^*$. For each $\xi \in X$, let α_ξ be the least $\alpha < \beta$ such that $\xi \in \mathrm{dom}(\dot{p}_\alpha)$, let $\dot{\gamma}_\xi$ be be an $\mathbb{S}*\dot{\mathbb{P}}_\xi$-name for $\sup(\{\max(\dot{p}_\alpha(\xi)) \mid \alpha_\xi \leq \alpha < \beta\})$, and let $\dot{p}^*(\xi)$ be an $\mathbb{S} * \dot{\mathbb{P}}_\xi$-name for $\{\dot{\gamma}_\xi\} \cup \bigcup_{\alpha_\xi \leq \alpha < \beta} \dot{p}_\alpha(\xi)$.

We now show by induction on ξ that, for all $\xi \leq \zeta$, $(s^*, \dot{p}^* \restriction \xi, \dot{t}^*) \in \mathbb{U}_\xi$ and, for all $\alpha < \beta$, $(s^*, \dot{p}^* \restriction \xi, \dot{t}^*) \leq (s_\alpha, \dot{p}_\alpha \restriction \xi, \dot{t}_\alpha)$. To this end, fix $\xi \leq \zeta$ and suppose we have established the inductive hypothesis for all $\eta < \xi$. We will establish the inductive hypothesis for ξ. If $\xi = 0$ or ξ is a limit ordinal, there is nothing to show. Thus, suppose $\xi = \xi_0 + 1$. If $\xi_0 \notin X$, there is again nothing to show, so assume $\xi_0 \in X$. It suffices to show that $(s^*, \dot{p}^* \restriction \xi_0, \dot{t}^*) \Vdash_{\mathbb{S}*\dot{\mathbb{P}}_{\xi_0}*\dot{\mathbb{T}}}$ "$\dot{\gamma}_{\xi_0} \in \dot{C}_{\xi_0}$." For all $\alpha_{\xi_0} \leq \alpha < \beta$,

$$(s_\alpha, \dot{p}_\alpha \restriction \xi_0, \dot{t}_\alpha) \Vdash_{\mathbb{S}*\dot{\mathbb{P}}_{\xi_0}*\dot{\mathbb{T}}} \text{“}\max(\dot{p}_\alpha(\xi_0)) \in \dot{C}_{\xi_0}.\text{”}$$

Therefore, by our inductive hypothesis, for all $\alpha_{\xi_0} \leq \alpha < \beta$,

$$(s^*, \dot{p}^* \restriction \xi_0, \dot{t}^*) \Vdash_{\mathbb{S}*\dot{\mathbb{P}}_{\xi_0}*\dot{\mathbb{T}}} \text{“}\max(\dot{p}_\alpha(\xi_0)) \in \dot{C}_{\xi_0}.\text{”}$$

Thus, since $\dot{\gamma}_{\xi_0}$ is a name for $\sup(\{\max(\dot{p}_\alpha(\xi_0)) \mid \alpha_{\xi_0} \leq \alpha < \beta\})$ and $\dot{C}_{\xi_0}$ is forced to be a club, $(s^*, \dot{p}^* \restriction \xi_0, \dot{t}^*) \Vdash_{\mathbb{S}*\dot{\mathbb{P}}_{\xi_0}*\dot{\mathbb{T}}}$ "$\dot{\gamma}_{\xi_0} \in \dot{C}_{\xi_0}$."

We now prove that $\mathbb{U}_\zeta$ is dense in $\mathbb{S} * \dot{\mathbb{P}}_\zeta * \dot{\mathbb{T}}$. Let $(s, \dot{p}, \dot{t}) \in \mathbb{S} * \dot{\mathbb{P}}_\zeta * \dot{\mathbb{T}}$. Since $\mathbb{S}$ is λ-distributive, we may assume that s decides the value of $\mathrm{dom}(\dot{p})$. Suppose first that ζ is a successor ordinal, and let $\zeta = \xi + 1$. If $\xi \notin \mathrm{dom}(\dot{p})$, then we can find $u \leq (s, \dot{p}, \dot{t})$ in $\mathbb{U}_\xi$, and we are done. Thus, suppose $\xi \in \mathrm{dom}(\dot{p})$. Find $(s^*, \dot{p}', \dot{t}^*) \leq (s, \dot{p} \restriction \xi, \dot{t})$ such that:

(1) $(s^*, \dot{p}', \dot{t}^*) \in \mathbb{U}_\xi$.
(2) There is $c \in V$ such that $(s^*, \dot{p}') \Vdash$ "$\dot{p}(\xi) = c$."
(3) There is $\gamma \geq \max(c)$ such that $(s^*, \dot{p}', \dot{t}^*) \Vdash$ "$\gamma \in \dot{C}_\xi$".

This is possible, because $\mathbb{U}_\xi$ is a dense, λ-closed subset of $\mathbb{S} * \dot{\mathbb{P}}_\xi * \dot{\mathbb{T}}$. Now, define $\dot{p}^* \in \dot{\mathbb{P}}_\zeta$ by letting $\dot{p}^* \restriction \xi = \dot{p}'$ and letting $\dot{p}^*(\xi)$ be a name forced by $(s^*, \dot{p}')$ to be equal to $\dot{p}(\xi) \cup \{\gamma\}$. It is easy to see that $(s^*, \dot{p}^*, \dot{t}^*) \leq (s, \dot{p}, \dot{t})$ and $(s^*, \dot{p}^*, \dot{t}^*) \in \mathbb{U}_\zeta$.

Finally, suppose ζ is a limit ordinal. If $\mathrm{cf}(\zeta) \geq \lambda$, then $\mathrm{dom}(\dot{p})$ is bounded below ζ and we are done by the inductive hypothesis. Thus, assume that $\kappa := \mathrm{cf}(\zeta) < \lambda$. Let $\langle \zeta_i \mid i < \kappa \rangle$ be an increasing, continuous sequence of ordinals cofinal in ζ. We will construct a sequence $\langle (s_i, \dot{p}_i, \dot{t}_i) \mid i < \kappa \rangle$ such that:

(1) For every $i < \kappa$, $(s_i, \dot{p}_i, \dot{t}_i) \in \mathbb{U}_{\zeta_i}$.
(2) For every $i < j < \kappa$, $(s_j, \dot{p}_j, \dot{t}_j) \leq (s_i, \dot{p}_i, \dot{t}_i)$.
(3) For every $i < \kappa$, $(s_i, \dot{p}_i {}^\frown \dot{p} \restriction [\zeta_i, \zeta), \dot{t}_i) \leq (s, \dot{p}, \dot{t})$.

The construction is straightforward, by recursion on i. Let $(s_0, \dot{p}_0, \dot{t}_0) \leq (s, \dot{p} \restriction \zeta_0, \dot{t})$ be in $\mathbb{U}_{\zeta_0}$. If $i = k + 1$, let $(s_i, \dot{p}_i, \dot{t}_i) \leq (s_k, \dot{p}_k {}^\frown \dot{p} \restriction [\zeta_k, \zeta_i), \dot{t}_k)$ be in $\mathbb{U}_{\zeta_i}$. If i is a limit ordinal, note that $\langle (s_k, \dot{p}_k, \dot{t}_k) \mid k < i \rangle$ is a decreasing sequence of conditions in $\mathbb{U}_{\zeta_i}$ and thus has a lower bound in $\mathbb{U}_{\zeta_i}$. Let $(s_i, \dot{p}_i, \dot{t}_i)$ be such a lower bound. Finally, at the end of the construction, $\langle (s_i, \dot{p}_i, \dot{t}_i) \mid i < \kappa \rangle$ is a decreasing sequence of conditions in $\mathbb{U}_\zeta$, so, by the λ-closure of $\mathbb{U}_\zeta$, it has a lower bound in $\mathbb{U}_\zeta$. Let $(s^*, \dot{p}^*, \dot{t}^*)$ be such a lower bound. It is easily verified that $(s^*, \dot{p}^*, \dot{t}^*) \leq (s, \dot{p}, \dot{t})$, thus completing the proof. □

4. Dense bounded stationary reflection

In this section, we construct a model in which there are singular cardinals $\delta < \mu$ such that, letting $\lambda = \mu^+$, $\mathrm{Refl}(\lambda)$ holds, and, for every stationary $S \subset S^\lambda_{<\delta}$, there is a stationary $T \subseteq S$ such that T does not reflect at any ordinal in $S^\lambda_{>\delta}$. In our model, we arrange so that $\mu = \aleph_{\omega\cdot 2}$ and $\delta = \aleph_\omega$, though the technique is quite flexible.

Theorem 4.1. *Suppose there is a sequence of supercompact cardinals of order type $\omega \cdot 2$. Then there is a forcing extension in which* $\mathrm{Refl}(\aleph_{\omega\cdot 2+1})$ *holds and, for every stationary $S \subseteq S^{\aleph_{\omega\cdot 2+1}}_{<\aleph_\omega}$, there is a stationary $T \subseteq S$ such that T does not reflect at any ordinals in $S^{\aleph_{\omega\cdot 2+1}}_{>\aleph_\omega}$.*

Proof. Assume GCH. Let $\langle \kappa_i \mid i \leq \omega \cdot 2 + 1\rangle$ be an increasing, continuous sequence of cardinals such that:

- $\kappa_0 = \omega$.
- If i is 0 or a successor ordinal, then κ_{i+1} is supercompact.
- If i is a limit ordinal, then $\kappa_{i+1} = \kappa_i^+$.

For ease of notation, let λ denote $\kappa_{\omega\cdot 2+1}$, $\mu = \kappa_{\omega\cdot 2}$, and $\delta = \kappa_\omega$. The reason for our numbering is that, in the final extension, we will have $\kappa_i = \aleph_i$ for all $i \leq \omega \cdot 2 + 1$.

Let $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j \mid i \leq \omega \cdot 2, j < \omega \cdot 2\rangle$ be a forcing iteration, taken with full supports, in which, if $i < \omega \cdot 2$ and $i \neq \omega$, then $\Vdash_{\mathbb{P}_i}$ "$\dot{\mathbb{Q}}_i = \dot{\mathrm{Coll}}(\kappa_i, <\kappa_{i+1})$" and, if $i = \omega$, $\dot{Q}_i$ is a $\mathbb{P}_i$ name for trivial forcing. Let $\mathbb{P} = \mathbb{P}_{\omega\cdot 2}$. Standard arguments (see, e.g. [3]) show that, if G is $\mathbb{P}$-generic over V, then, for all $i \leq \omega \cdot 2 + 1$, $\kappa_i = (\aleph_i)^{V[G]}$. Let $\dot{\vec{a}}$ be a $\mathbb{P}$-name for an enumeration, of order type λ, of all bounded subsets of λ, and let $\dot{\mathbb{A}}$ be a $\mathbb{P}$-name for $\mathbb{A}_{\dot{\vec{a}}}$, the forcing poset to shoot a club through the set of ordinals below λ that are approachable with respect to $\dot{\vec{a}}$.

Denote $V^{\mathbb{P}*\dot{\mathbb{A}}}$ by V_1. In V_1, we will define a forcing iteration $\langle \mathbb{S}_\xi, \dot{\mathbb{T}}_\zeta \mid \xi \leq \lambda^+, \zeta < \lambda^+\rangle$. The iteration will use supports of size $\leq \mu$. For $\zeta \leq \lambda^+$, let $A_\zeta = \{\eta < \zeta \mid \eta$ is even$\}$. The definition of $\dot{\mathbb{T}}_\zeta$ will depend on whether ζ is even or odd. If $\zeta < \lambda^+$ is even (including limit ordinals), then choose an $\mathbb{S}_\zeta$-name $\dot{T}_\zeta$ for a stationary subset of $S^\lambda_{<\delta}$, and let $\dot{\mathbb{T}}_\zeta$ be an $\mathbb{S}_\zeta$-name for $\mathbb{S}_{\dot{T}_\zeta,\delta^+}$, i.e. the forcing to add a subset of $\dot{T}_\zeta$ that does not reflect at any points in $S^\lambda_{>\delta}$. Let $\dot{S}_\zeta$ be an $\mathbb{S}_{\zeta+1}$-name for this subset of $\dot{T}_\zeta$, and let S_ζ denote its realization in $V_1^{\mathbb{S}_{\zeta+1}}$.

If $\zeta \leq \lambda^+$, then, in $V_1^{\mathbb{S}_\zeta}$, let $\mathbb{U}_\zeta$ be the product $\prod_{\eta\in A_\zeta} CU(S_\eta)$, where the product is taken with supports of size $\leq \mu$. Let $\mathbb{U} = \mathbb{U}_{\lambda^+}$. If $\zeta < \lambda^+$ is odd, we will choose an $\mathbb{S}_\zeta$-name $\dot{T}_\zeta$ for a subset of $S^\lambda_{>\delta}$ such that $\Vdash_{\mathbb{S}_\zeta * \dot{\mathbb{U}}_\zeta}$ "$\dot{T}_\zeta$ is non-stationary," and let $\dot{\mathbb{T}}_\zeta$ be an $\mathbb{S}_\zeta$-name for $CU(\dot{T}_\zeta)$. Also, fix an $\mathbb{S}_\zeta * \dot{\mathbb{U}}_\zeta$-name $\dot{C}_\zeta$ for a club in λ disjoint from $\dot{T}_\zeta$.

Before we discuss the choice of the name $\dot{T}_\zeta$, we describe some of the properties of $\mathbb{S} := \mathbb{S}_{\lambda^+}$. First note that, by a standard Δ-system argument, $\mathbb{S}$ has the λ^+-c.c. Also, $\mathbb{S}$ is easily seen to be δ^+-directed closed. We also claim that it is λ-distributive. To show this, we define another poset. In V_1, for all $\xi \leq \lambda^+$, let $\mathbb{V}_\xi$ be the set of (s, u) such that:

- $s \in \mathbb{S}_\xi$.
- u is a function and $\mathrm{dom}(u) = \mathrm{dom}(s) \cap A_\xi$.
- For all $\zeta \in \mathrm{dom}(u)$, $u(\zeta)$ is a closed, bounded subset of λ such that $s \restriction (\zeta + 1) \Vdash$ "$u(\zeta) \cap \dot{S}_\zeta = \emptyset$" and, if $\alpha \in \mathrm{nacc}(u(\zeta))$, then $\mathrm{cf}(\alpha) = \omega$.

If $(s_0, u_0), (s_1, u_1) \in \mathbb{V}_\xi$, we let $(s_1, u_1) \leq (s_0, u_0)$ iff $s_1 \leq s_0$ in $\mathbb{S}_\xi$ and, for every $\zeta \in \mathrm{dom}(u_0)$, $u_1(\zeta)$ end-extends $u_0(\zeta)$. If $\zeta < \xi \leq \lambda^+$, the map $(s, u) \mapsto (s \restriction \zeta, u \restriction \zeta)$ defines a projection from $\mathbb{V}_\xi$ to $\mathbb{V}_\zeta$.

Lemma 4.2. *For all $\xi \leq \lambda^+$,*

1. *For every $(s, u) \in \mathbb{V}_\xi$, there is $(s^*, u^*) \leq (s, u)$ such that:*
   (a) *For every $\zeta \in \mathrm{dom}(s^*)$, there is $s_\zeta \in V_1$ such that $s^* \restriction \zeta \Vdash$ "$s^*(\zeta) = s_\zeta$."*
   (b) *For every $\zeta \in \mathrm{dom}(s^*) \cap A_\zeta$, $\gamma_{s^*(\zeta)} = \max(u(\zeta))$*
   (c) *For every $\zeta \in \mathrm{dom}(s^*) \setminus A_\zeta$, $(s^* \restriction \zeta, u^* \restriction \zeta) \Vdash_{\mathbb{V}_\zeta}$ "$\max(s^*(\zeta)) \in \dot{C}_\zeta$." (Note that this will make sense if clause (3) below holds at ζ.)*
2. *$\mathbb{S}_\xi$ is λ-distributive.*
3. *$\mathbb{V}_\xi$ is isomorphic to a dense subset of $\mathbb{S}_\xi * \dot{\mathbb{U}}_\xi$.*

Proof. We prove all three statements simultaneously by induction on ξ. First note that, to show (3), it suffices to show that, for every $(s, \dot{u}) \in \mathbb{S}_\xi * \dot{\mathbb{U}}_\xi$, there is $s^* \leq s$ and $u^* \in V_1$ such that $s^* \Vdash$ "$\dot{u} = u^*$". This is easily implied by (2), as $\dot{u}$ can be thought of as a name for a set of pairs of ordinals of size $< \lambda$. Also note that the set of $(s^*, u^*) \in \mathbb{V}_\xi$ as given in the conclusion of (1) is easily seen to be λ-directed closed so, since $(s, u) \mapsto s$ is a projection from $\mathbb{V}_\xi$ to $\mathbb{S}_\xi$, (1) implies (2) for a fixed $\xi \leq \lambda^+$.

Fix $\xi \leq \lambda^+$. Assume we have proven all three statements for all $\zeta < \xi$. We prove (1) for ξ. Assume first that ξ is a successor ordinal, with $\xi = \zeta + 1$ and ζ odd. Let $(s, u) \in \mathbb{V}_\xi$. If $\zeta \notin \mathrm{dom}(s)$, then we are done by (1) for ζ. Thus, assume $\zeta \in \mathrm{dom}(s)$. Since $\mathbb{S}_\zeta$ is λ-distributive, we can find $s' \leq s \restriction \zeta$ and $s_\zeta \in V_1$ such that $s' \Vdash$ "$s(\zeta) = s_\zeta$.". Now find $(\bar{s}, \bar{u}) \leq (s', u)$ in $\mathbb{V}_\zeta$ and $\alpha > \max(s_\zeta)$ such that $\mathrm{cf}(\alpha) = \omega$ and $(\bar{s}, \bar{u}) \Vdash$ "$\alpha \in \dot{C}_\zeta$." Form $(s^*, u^*) \leq (s, u)$ by letting $(s^* \restriction \zeta, u^*) \leq (\bar{s}, \bar{u})$ witness (1) for ζ and by letting $s^*(\zeta)$ be a name forced by $s^* \restriction \zeta$ to be equal to $s_\zeta \cup \{\alpha\}$. It is easily verified that (s^*, u^*) is as desired.

Next, suppose that $\xi = \zeta + 1$ and ζ is even. Let $(s, u) \in \mathbb{V}_\xi$, and again assume that $\zeta \in \mathrm{dom}(s)$. Find $s' \leq s \restriction \zeta$ and $s_\zeta \in V_1$ such that $s' \Vdash$ "$s(\zeta) = s_\zeta$." Find α with $\max(u(\zeta)), \gamma_{s_\zeta} < \alpha < \lambda$ and $\mathrm{cf}(\alpha) = \omega$. Form $(s^*, u^*) \leq (s, u)$ by letting $(s^* \restriction \zeta, u^* \restriction \zeta) \leq (s', u \restriction \zeta)$ witness (1) for ζ, letting $s^*(\zeta)$ be a name forced by $s^* \restriction \zeta$ to be a function in ${}^{\alpha+1}2$ such that $s^*(\zeta) \restriction (\gamma_{s_\zeta} + 1) = s_\zeta$ and $s^*(\zeta)(\beta) = 0$ for all $\beta \in (\gamma_{s_\zeta}, \alpha]$, and letting $u^*(\zeta) = u \cup \{\alpha\}$.

Finally, suppose ξ is a limit ordinal, and let $(s, u) \in \mathbb{V}_\xi$. If $\mathrm{dom}(s)$ is bounded below ξ (in particular, if $\mathrm{cf}(\xi) \geq \lambda$), then we are done by the induction hypothesis. Thus, assume $\mathrm{cf}(\xi) < \lambda$ and $\mathrm{dom}(s)$ is unbounded in ξ. Let $\langle \xi_i \mid i < \mathrm{cf}(\xi) \rangle$ be an increasing, continuous sequence of ordinals cofinal in ξ. Form a sequence of conditions $\langle (s_i, u_i) \mid i < \mathrm{cf}(\xi) \rangle$ such that:

- For all $i < \mathrm{cf}(\xi)$, $(s_i, u_i) \in \mathbb{V}_{\xi_i}$ and $(s_i, u_i) \leq (s \restriction \xi_i, u \restriction \xi_i)$.
- For all $i < k < \mathrm{cf}(\xi)$, $(s_k \restriction \xi_i, u_k \restriction \xi_i) \leq (s_i, u_i)$.
- For all $i < \mathrm{cf}(\xi)$, (s_i, u_i) witnesses (1) for ξ_i.

The construction is a straightforward recursion; we omit the details. At the end of the construction, let $X = \bigcup_{i < \mathrm{cf}(\xi)} \mathrm{dom}(s_i)$. For $\zeta \in X$, let i_ζ be the least i such that $\zeta \in \mathrm{dom}(s_i)$. For $i_\zeta \leq i < \mathrm{cf}(\xi)$, let $\alpha_{\zeta,i}$ be such that, if ζ is even, then $s_i \restriction \zeta \Vdash$ "$\gamma_{s_i(\zeta)} = \alpha_{\zeta,i}$" and, if ζ is odd, then $s_i \restriction \zeta \Vdash$ "$\max(s_i(\zeta)) = \alpha_{\zeta,i}$." Let $\alpha_\zeta = \sup(\{\alpha_{\zeta,i} \mid i_\zeta \leq i < \mathrm{cf}(\xi)\})$. Form (s^*, u^*) as follows. $\mathrm{dom}(s^*) = X$. For

$\zeta \in X$, let $s^*(\zeta)$ be a name forced by $s^* \restriction \zeta$ to be equal to

$$\{(\alpha_\zeta, 0)\} \cup \bigcup_{i_\zeta \leq i < \mathrm{cf}(\xi)} s_i(\zeta)$$

if ζ is even and

$$\{\alpha_\zeta\} \cup \bigcup_{i_\zeta \leq i < \mathrm{cf}(\xi)} s_i(\zeta)$$

if ζ is odd. For $\zeta \in A_\xi \cap X$, let

$$u^*(\zeta) = \{\alpha_\zeta\} \cup \bigcup_{i_\zeta \leq i < \mathrm{cf}(\xi)} u_i(\zeta).$$

(s^*, u^*) is easily seen to be as required by (1). □

Note that the above proof also shows that, in $V^{\mathbb{P} * \dot{\mathbb{A}}}$, $\mathbb{S} * \dot{\mathbb{U}}$ has a dense, λ-directed-closed subset.

Since, in V_1, $\mathbb{S}$ has the λ^+-c.c., we can, by standard bookkeeping arguments, assume that we chose our names $\dot{T}_\zeta$ so that, in $V_1^{\mathbb{S}}$, every stationary subset of $S^\lambda_{<\delta}$ was considered as $\dot{T}_\zeta$ for some even $\zeta < \lambda^+$ and every subset of $S^\lambda_{>\delta}$ was considered as $\dot{T}_\zeta$ for cofinally many odd $\zeta < \lambda^+$. In $V_1^{\mathbb{S}}$, let $A = A_{\lambda^+}$ and $\mathbb{U} = \mathbb{U}_{\lambda^+}$. By the distributivity of $\mathbb{S}$, all conditions of $\mathbb{U}$ are in V_1.

Let G be $\mathbb{P}$-generic over V, let H be $\mathbb{A}$-generic over $V[G]$, and let I be $\mathbb{S}$-generic over $V[G * H]$. $V[G * H * I]$ will be our final model. For $i < \omega \cdot 2$, let G_i be the $\mathbb{P}_i$-generic filter induced by G, and, for $\zeta < \lambda^+$, let I_ζ be the $\mathbb{S}_\zeta$-generic filter induced by I. Note that, in $V[G * H]$ (and hence in all further extensions preserving λ), AP_μ holds. Thus, by Lemma 2.10, for all even $\zeta < \lambda^+$, S_ζ (the set added by $\mathbb{T}_\zeta$) is stationary in $V[G * H * I_{\zeta+1}]$. Because the remainder of the iteration is δ^+-closed and $S \subseteq S^\lambda_{<\delta}$, S_ζ is stationary in $V[G * H * I]$ by Fact 2.13.

We have therefore arranged so that, in $V[G * H * I]$, for every stationary $T \subseteq S^\lambda_{<\delta}$, there is a stationary $S \subseteq T$ that does not reflect at any ordinals in $S^\lambda_{>\delta}$. Also, suppose that, in $V[G * H * I]$, $T \subseteq S^\lambda_{>\delta}$ and $\Vdash_{\mathbb{U}}$ "T is non-stationary." By standard chain condition arguments, there is $\xi < \lambda^+$ such that $T \in V[G * H * I_\xi]$ and, in $V[G * H * I_\xi]$, $\Vdash_{\mathbb{U}_\xi}$ "T is non-stationary." Thus, by our bookkeeping for the choice of $\dot{T}_\zeta$ at odd ζ, we have that T is already non-stationary in $V[G * H * I]$.

We now argue that Refl(λ) holds in $V[G * H * I]$. So, let $T \subseteq \lambda$ be stationary in $V[G * H * I]$. We can assume, by shrinking T if necessary, that there is $i_0 < \omega \cdot 2$ such that $T \subseteq S^\lambda_{\kappa_{i_0}}$. We first consider the case in which $i_0 < \omega$. Let $i^* = i_0 + 2$. In $V[G_{i_0+1}]$, κ_{i^*} remains supercompact. Fix an elementary embedding $j : V[G_{i_0+1}] \to M[G_{i_0+1}]$ witnessing that κ_{i^*} is λ^+-supercompact. In $V[G_{i_0+1}]$, $\mathbb{Q}_{i_0+1} = \mathrm{Coll}(\kappa_{i_0+1}, < \kappa_{i^*})$ and $j(\mathbb{Q}_{i_0+1}) = \mathrm{Coll}(\kappa_{i_0+1}, < j(\kappa_{i^*}))$. Since $\mathbb{P}_{i^*, \omega \cdot 2} * \dot{\mathbb{A}} * \dot{\mathbb{S}}$ is strongly κ_{i_0+1}-strategically closed, we can apply Fact 2.12 to observe that

$$j(\mathbb{Q}_{i_0+1}) \cong \mathbb{P}_{i_0+1, \omega \cdot 2} * \dot{\mathbb{A}} * \dot{\mathbb{S}} * \dot{\mathbb{R}},$$

where $\dot{\mathbb{R}}$ is forced to be κ_{i_0+1}-closed. Thus, letting J be $\mathbb{R}$-generic over $V[G * H * I]$, we can extend j to $j : V[G_{i^*}] \to M[G * H * I * J]$.

We would like to extend j further to have domain $V[G * H * I]$. To do this, we define a master condition $(p^*, \dot{a}^*, \dot{s}^*) \in j(\mathbb{P}_{i^*, \omega \cdot 2} * \dot{\mathbb{A}} * \dot{\mathbb{S}})$ in $M[G * H * I * J]$, i.e. a condition $(p^*, \dot{a}^*, \dot{s}^*)$ such that, for all $(p, \dot{a}, \dot{s}) \in G_{i^*, \omega \cdot 2} * H * I$, $(p^*, \dot{a}^*, \dot{s}^*) \leq j((p, \dot{a}, \dot{s}))$. The definition is straightforward. Let $\eta = \sup(j``\lambda)$. For $i^* \leq i < \omega \cdot 2$, we let $p^*(i)$ be a name for $\bigcup_{p\in G} j(p(i))$. $p^* \in j(\mathbb{P}_{i^*,\omega\cdot 2})$ by the fact that $j(\mathbb{P}_{i^*,\omega\cdot 2})$ is $j(\kappa_{i^*})$-directed closed.

Next, define $\dot{a}^*$ to be a name forced by p^* to be equal to $\{\eta\} \cup \bigcup_{a\in H} j(a)$. The only thing to check here is that η is approachable with respect to $j(\vec{a})$. Note that, in $M[G * H * I * J]$, $\mathrm{cf}(\lambda) = \kappa_{i_0+1}$, and we can find an unbounded $A \subseteq \lambda$ such that $\mathrm{otp}(A) = \kappa_{i_0+1}$ and, for all $\beta < \lambda$, $A \cap \beta \in V[G_{i_0+1}]$. Let $B = j``A$. For all $\beta < \lambda$, $A \cap \beta$ is enumerated in $\vec{a}$ as a_α for some $\alpha < \lambda$. Therefore, $B \cap j(\beta) = j(a_\alpha)$. In particular, every initial segment of B is enumerated in $j(\vec{a})$ with an index smaller than η, so B witnesses that η is approachable with respect to $j(\vec{a})$.

Finally, since $j(\dot{\mathbb{S}})$ is forced to be $j(\delta^+)$-directed-closed, it is straightforward to find a name $\dot{s}^*$ forced by $(p^*, \dot{a}^*)$ to be a lower bound for $\{j(\dot{s}) \mid \dot{s} \in I\}$. $(p^*, \dot{a}^*, \dot{s}^*)$ is then as desired, and, letting $G^+ * H^+ * I^+$ be $j(\mathbb{P}_{i^*,\omega\cdot 2} * \dot{\mathbb{A}} * \dot{\mathbb{S}})$-generic over $V[G * H * I * J]$ with $(p^*, \dot{a}^*, \dot{s}^*) \in G^+ * H^+ * I^+$, we can extend j to

$$j : V[G * H * I] \to M[G * H * I * J * G^+ * H^+ * I^+].$$

Now, by standard arguments (see e.g. Proposition 1.1 in [3]), if T does not reflect in $V[G * H * I]$, then $j``T$ is non-stationary in η in $M[G * H * I * J * G^+ * H^+ * I^+]$, which further implies that T is non-stationary $V[G*H*I*J*G^+*H^+*I^+]$. Since $G^+ * H^+ * I^+$ is generic for $(< j(\kappa_{i^*}))$-strategically-closed forcing, it could not have added any new subsets of λ, so T is already non-stationary in $V[G * H * I * J]$. However, since J is generic for κ_{i^*}-closed forcing, $T \subseteq S^\lambda_{<\kappa_{i^*}}$, and AP_μ holds in $V[G * H * I]$, Fact 2.13 implies that T is non-stationary in $V[G * H * I]$, which is a contradiction. Thus, T reflects in $V[G * H * I]$.

Next, suppose $i_0 > \omega$. Again, let $i^* = i_0 + 2$ and let $j : V[G_{i_0+1}] \to M[G_{i_0+1}]$ witness that κ_{i^*} is λ^+-supercompact. $\mathbb{P}_{i^*,\omega\cdot 2} * \dot{\mathbb{A}} * \dot{\mathbb{S}} * \dot{\mathbb{U}}$ has a dense strongly κ_{i_0+1}-strategically closed subset, so, again applying Fact 2.12,

$$j(\mathbb{Q}_{i_0+1}) \cong \mathbb{P}_{i^*,\omega\cdot 2} * \dot{\mathbb{A}} * \dot{\mathbb{S}} * \dot{\mathbb{U}} * \dot{\mathbb{R}},$$

where $\dot{\mathbb{R}}$ is forced to be κ_{i_0+1}-closed. Note that, by previous arguments, it is not the case that, in $V[G * H * I]$, $\Vdash_{\mathbb{U}}$ "T is non-stationary". Thus, letting J be $\mathbb{U}$-generic over $V[G * H * I]$ such that T remains stationary in $V[G * H * I * J]$, and letting K be $\mathbb{R}$-generic over $V[G*H*I*J]$, we can lift j to $j : V[G_{i^*}] \to M[G*H*I*J*K]$. We can extend j further to

$$j : V[G * H * I] \to M[G * H * I * J * K * G^+ * H^+ * I^+]$$

using a master condition argument as in the previous case, exploiting the fact that $\mathbb{S} * \dot{\mathbb{U}}$ has a dense λ-closed subset in $V^{\mathbb{P}*\dot{\mathbb{A}}}$. And again, exactly as in the previous case, we can argue that T must reflect in $V[G * H * I]$, for otherwise it would be non-stationary in $V[G * H * I * J]$. □

5. Global bounded stationary reflection

In this section, we improve upon Theorem 1.2 by producing, from large cardinal assumptions, a model in which bounded stationary reflection holds at every possible successor of a singular cardinal.

Theorem 5.1. *Suppose there is a proper class of supercompact cardinals. Then there is a class forcing extension in which, for every singular cardinal $\mu > \aleph_\omega$, $\mathrm{Refl}(\mu^+)$ holds and there is a stationary subset of $S^{\mu^+}_\omega$ that does not reflect in $S^{\mu^+}_{>\aleph_\omega}$.*

Proof. Assume GCH. Let $\langle \kappa_i \mid i \in \mathrm{On} \rangle$ be an increasing, continuous sequence of cardinals such that:

- $\kappa_0 = \omega$.
- If i is a limit ordinal or a successor of a limit ordinal, then $\kappa_{i+1} = \kappa_i^+$.
- If i is not a limit ordinal or a successor of a limit ordinal, then κ_{i+1} is supercompact.

We may assume that, if i is a limit ordinal, then κ_i is singular by cutting the universe off at the least regular κ_i with i limit.

We define a class forcing iteration $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_i \mid i \in \mathrm{On} \rangle$, taken with full supports. If $i = 0$, $i = 1$, or i is a successor of a successor ordinal, then let $\dot{\mathbb{Q}}_i$ be such that $\Vdash_i$ "$\dot{\mathbb{Q}}_i = \mathrm{Coll}(\kappa_i, < \kappa_{i+1})$." If $i = \omega$ or i is a successor of a limit ordinal, let $\dot{\mathbb{Q}}_i$ be a $\mathbb{P}_i$-name for trivial forcing.

It remains to define $\dot{\mathbb{Q}}_i$ when $i > \omega$ is a limit ordinal. Fix such an i, and move temporarily to $V^{\mathbb{P}_i}$. Let $\vec{a}_i$ be an enumeration of the bounded subsets of κ_{i+1} in order type κ_{i+1}, and let $\mathbb{A}_i$ be the poset to shoot a club through the set of ordinals below κ_{i+1} that are approachable with respect to $\vec{a}_i$. In $V^{\mathbb{P}_i * \dot{\mathbb{A}}_i}$, let $\mathbb{S}_i$ be $\mathbb{S}^{\kappa_{i+1}}_{\omega, \kappa_{\omega+1}}$. In $V^{\mathbb{P}_i * \dot{\mathbb{A}}_i * \dot{\mathbb{S}}_i}$, we will define an iteration, $\langle \mathbb{T}^i_\xi, \dot{\mathbb{U}}^i_\zeta \mid \xi \leq \kappa^+_{i+1}, \zeta < \kappa^+_{i+1} \rangle$, taken with supports of size κ_i and, letting $\mathbb{T}_i = \mathbb{T}^i_{\kappa^+_{i+1}}$, we will let $\dot{\mathbb{Q}}_i$ be a $\mathbb{P}_i$-name for $\mathbb{A}_i * \dot{\mathbb{S}}_i * \dot{\mathbb{T}}_i$. If $p \in \mathbb{P}_{i+1}$, we will let $p(i)_0$, $p(i)_1$, and $p(i)_2$ denote the $\mathbb{A}_i$, $\dot{\mathbb{S}}_i$, and $\dot{\mathbb{T}}_i$ parts of $p(i)$, respectively. If $\zeta < \kappa^+_{i+1}$, we will let $p(i) \restriction \zeta$ denote $(p(i)_0, p(i)_1, p(i)_2 \restriction \zeta)$. Moreover, for $\zeta < \kappa^+_{i+1}$, and $k < i$ we will let $\mathbb{P}_{k,i+1} \restriction \zeta$ denote $\mathbb{P}_{k,i} * \dot{\mathbb{A}}_i * \dot{\mathbb{S}}_i * \dot{\mathbb{T}}^i_\zeta$.

Suppose $\zeta < \kappa^+_{i+1}$ and we have defined $\mathbb{T}^i_\zeta$. We describe how to define $\dot{\mathbb{U}}^i_\zeta$. For all limit ordinals $\omega < i' \leq i$, let $S_{i'}$ be the stationary subset of $S^{\kappa_{i'+1}}_\omega$ added by $\mathbb{S}_{i'}$. For all $\omega < k < i$, with k a successor ordinal, let X^i_k be the set of limit ordinals in $(k, i]$, and let $\mathbb{C}^\zeta_{k,i}$ be the poset defined in $V^{\mathbb{P}_k}$ as follows. Conditions are pairs (p, c) such that:

- $p \in \mathbb{P}_{k,i+1} \restriction \zeta$.
- c is a function, and $\mathrm{dom}(c) = X^i_k$.
- For all $i' \in \mathrm{dom}(c)$, $c(i')$ is a $\mathbb{P}_{k,i'}$-name for a closed, bounded subset of $\kappa_{i'+1}$ and $p \restriction [k, i')^\frown p(i')_0{}^\frown p(i')_1 \Vdash$ "$c(i') \cap \dot{S}_{i'} = \emptyset$."

$(p', c') \leq (p, c)$ if $p' \leq p$ and, for all $i' \in \mathrm{dom}(c)$, $p' \restriction i' \Vdash$ "$c'(i')$ end-extends $c(i')$."

The map $(p, c) \mapsto p$ is clearly a projection from $\mathbb{C}^\zeta_{k,i}$ to $\mathbb{P}_{k,i+1} \restriction \zeta$. Let $\mathbb{V}^\zeta_{k,i} \in V^{\mathbb{P}_{i+1} \restriction \zeta}$ be the quotient poset, so $\mathbb{C}^\zeta_{k,i} \cong \mathbb{P}_{k,i+1} \restriction \zeta * \dot{\mathbb{V}}^\zeta_{k,i}$. Let $\dot{T}^i_\zeta$ be a $\mathbb{T}^i_\zeta$-name for a subset of $S^{\kappa_{i+1}}_{>\kappa_\omega}$ such that, for every successor ordinal k with $\omega < k < i$, $\Vdash_{\mathbb{V}^\zeta_{k,i}}$ "$\dot{T}^i_\zeta$ is non-stationary," and let $\dot{\mathbb{U}}^i_\zeta$ be a $\mathbb{T}^i_\zeta$-name for $CU(\dot{T}^i_\zeta)$. If k is a successor ordinal with $\omega < k < i$, notice that, if $\zeta < \zeta' \leq \kappa^+_{i+1}$, there is a natural projection from $\mathbb{C}^{\zeta'}_{k,i}$ to $\mathbb{C}^\zeta_{k,i}$. Let $\mathbb{C}_{k,i} = \mathbb{C}^{\kappa^+_{i+1}}_{k,i}$, and let $\mathbb{V}_{k,i}$ be the quotient forcing over $\mathbb{P}_{k,i+1}$ in $V^{\mathbb{P}_{i+1}}$. Notice also that, if $(p, c) \in \mathbb{C}_{k,i}$, then there is $\zeta < \kappa^+_{i+1}$ such that $(p, c) \in \mathbb{C}^\zeta_{k,i}$ and that, if $i' \in (k, i)$ is a limit ordinal, then

$$\{p \restriction (i'+1), c \restriction (i'+1) | \mid (p, c) \in \mathbb{C}^\zeta_{k,i}\} = \mathbb{C}_{k,i'}$$

and $\mathbb{C}^\zeta_{k,i} \cong \mathbb{C}_{k,i'} * \dot{\mathbb{C}}^\zeta_{i'+1,i}$.

Note that, in $V^{\mathbb{P}_i * \dot{\mathbb{A}}_i * \dot{\mathbb{S}}_i}$, $\mathbb{T}_i$ has the κ_{i+1}^+-c.c., so, by standard bookkeeping arguments, we can arrange so that every canonical $\mathbb{T}_i$-name for a subset of $S^{\kappa_{i+1}}_{>\kappa_\omega}$ was considered as $\dot{T}^i_\zeta$ for cofinally many $\zeta < \kappa^+_{i+1}$. Thus, we can arrange that, in $V^{\mathbb{P}_{i+1}}$, if $T \subseteq S^{\kappa_{i+1}}_{>\kappa_\omega}$ is such that, for every successor ordinal $k < i$, $\Vdash_{\mathbb{V}_{k,i}}$ "T is non-stationary," then T is already non-stationary in $V^{\mathbb{P}_{i+1}}$.

Lemma 5.2. *Let $\omega < k < i$, with k a successor ordinal and i a limit ordinal, and let $\zeta \leq \kappa^+_{i+1}$. In $V^{\mathbb{P}_k}$, for every $\ell \in X^i_k$ and every $\xi < \kappa^+_{\ell+1}$, let $\dot{C}^\xi_\ell$ be a $\mathbb{C}^\xi_{k,\ell}$-name for a club in $\kappa_{\ell+1}$ disjoint from $\dot{T}^\ell_\xi$.*

1. *For every $(p,c) \in \mathbb{C}^\zeta_{k,i}$, there is $(p^*, c^*) \leq (p,c)$ such that:*
   - (a) *There is $h \in \prod_{\ell \in X^i_k} \kappa_{\ell+1}$ such that, for all $\ell \in X^i_k$,*

$$p^* \restriction \ell^\frown p^*(\ell)_0 \Vdash \text{“}\gamma_{p^*(\ell)_1} = h(\ell) = \max(c^*(\ell)).\text{”}$$

   - (b) *For every $\ell \in X^i_k$, for every $\xi < \kappa^+_{\ell+1}$ (or $\xi < \zeta$, if $\ell = i$),*

$$(p^* \restriction \ell^\frown p^*(\ell) \restriction \xi, c^* \restriction (\ell+1)) \Vdash \text{“}\xi \notin \mathrm{dom}(p^*(\ell)_2) \textit{ or } \max(p^*(\ell)_2(\xi)) \in \dot{C}^\xi_\ell.\text{”}$$

2. *$\mathbb{P}_{k,i+1} \restriction \zeta$ is κ_k-distributive.*

Proof. First note that, by now-familiar arguments, the set of conditions (p^*, c^*) that satisfy (1)(a) and (1)(b), which we will denote by $\mathbb{C}^{\zeta,*}_{k,i}$ (or just $\mathbb{C}^*_{k,i}$ if $\zeta = \kappa^+_{i+1}$), is easily seen to be strongly κ_k-strategically closed. This will be useful in the inductive proof of (1) and also immediately yields (2) from the corresponding instance of (1).

We proceed by induction on i and, for fixed i, by induction on $\zeta \leq \kappa^+_{i+1}$. Thus, let $k < i$ be given, let $\zeta = 0$, and let $(p,c) \in \mathbb{C}^\zeta_{k,i}$. First, suppose that $i = k + \omega$. In this case, find $p' \leq p \restriction [k,i)^\frown p(i)_0$ and $\alpha_i < \kappa_{i+1}$ such that $\mathrm{cf}(\alpha_i) = \omega$ and $p' \Vdash$ "$\gamma_{p(i)_1}, \max(c(i)) < \alpha_i$.". Form (p^*, c^*) by letting $p^* \restriction i^\frown p^*(i)_0 = p'$, letting $p^*(i)_1$ be a name forced by p' to be equal to the function in ${}^{\alpha_i+1}2$ extending $p(i)_1$ that is constantly zero on $(\gamma_{p(i)_1}, \alpha_i + 1)$, and letting $c^*(i)$ be a name forced by p' to be equal to $c(i) \cup \{\alpha\}$.

Next, suppose $i = i' + \omega$ for some limit ordinal $i' \in (k,i)$. Find $p' \leq p \restriction i^\frown p(i)_0$ and $\alpha_i < \kappa_{i+1}$ such that $\mathrm{cf}(\alpha_i) = \omega$ and $p' \Vdash$ "$\gamma_{p(i)_1}, \max(c(i)) < \alpha_i$.". Define (p^*, c^*) by letting

$$(p^* \restriction [k, i'+1), c^* \restriction [k, i'+1)) \leq (p' \restriction [k, i'+1), c \restriction [k, i'+1))$$

witness (1) for $\mathbb{C}_{k,i'}$, letting

$$p^* \restriction [i'+1, i)^\frown p^*(i)_0 = p' \restriction [i'+1, i)^\frown p'(i)_0,$$

letting $p^*(i)_1$ be a name forced by p' to be equal to the function in ${}^{\alpha_i+1}2$ extending $p(i)_1$ that is constantly zero on $(\gamma_{p(i)_1}, \alpha_i + 1)$, and letting $c^*(i)$ be a name forced by p' to be equal to $c(i) \cup \{\alpha\}$.

Finally, suppose that i is a limit of limit ordinals. We first suppose that $\mathrm{cf}(i) < \kappa_k$. Let $\langle i_\eta \mid \eta < \mathrm{cf}(i) \rangle$ be an increasing, continuous sequence of limit ordinals from (k,i) that is cofinal in i. Find $p' \leq p \restriction i^\frown p(i)_0$ and α_i as in the previous cases. Recursively construct a sequence $\langle (p_\eta, c_\eta) \mid \eta < \mathrm{cf}(i) \rangle$ such that:

- For all $\eta < \mathrm{cf}(i)$, $(p_\eta, c_\eta) \in \mathbb{C}_{k,i_\eta}$ and satisfies (1).
- For all $\eta < \mathrm{cf}(i)$, $(p_\eta, c_\eta) \leq (p' \restriction (i_\eta + 1), c \restriction (i_\eta + 1))$.
- For all $\eta < \eta' < \mathrm{cf}(i)$, $(p_{\eta'} \restriction (i_\eta + 1), c_{\eta'} \restriction (i_\eta + 1)) \leq (p_\eta, c_\eta)$.

- For all $i' \in X^i_k$, if η^* is the least η such that $i' \leq i_\eta$, then, for all $\eta^* \leq \eta < \mathrm{cf}(i)$, $p_\eta \restriction i'$ forces that $\langle p_\delta(i')_0 \mid \eta^* \leq \delta \leq \eta \rangle$ is a partial run of $G^*_{\kappa_k}(\mathbb{A}_{i'})$ with Player II playing according to her winning strategy.

The construction is straightforward by the inductive hypothesis, and it is straightforward to use $\langle (p_\eta, c_\eta) \mid \eta < \mathrm{cf}(i) \rangle$ to, by taking unions (and, where appropriate, closures) along all coordinates and adding α_i to the end of the ith coordinates as in the previous cases, get a (p^*, c^*) as desired.

If $\kappa_k \leq \mathrm{cf}(i)$, then find $k < k' < i$ with k' a limit ordinal and $\mathrm{cf}(i) < \kappa_{k'+1}$. Move temporarily to $V^{\mathbb{P}_{k'+1}}$, and interpret $(p \restriction [k'+1, i+1), c \restriction [k'+1, i+1))$ in $\mathbb{C}^\zeta_{k'+1,i}$ as (p_0, c_0). For every $\ell \in X^i_{k'+1}$ and $\xi < \kappa^+_{\ell+1}$, the quotient forcing of $\mathbb{C}^\xi_{k,\ell}$ over $\mathbb{P}_{k,k'+1} * \mathbb{C}^\xi_{k'+1,\ell}$ has the $\kappa^+_{k'+1}$-c.c., so we can find a $\mathbb{C}^\xi_{k'+1,\ell}$-name $\dot{D}^\xi_\ell$ for a club in $\kappa_{\ell+1}$ that is forced by the quotient forcing of $\mathbb{C}^\xi_{k,\ell}$ over $\mathbb{P}_{k,k'+1}$ to be a subset of $\dot{C}^\xi_\ell$. Use the argument from the previous paragraph to find $(p_1, c_1) \leq (p_0, c_0)$ satisfying (1) for $\mathbb{C}^\zeta_{k'+1,i}$ and the set of $\dot{D}^\xi_\ell$'s as witnessed by $h_1 \in \prod_{\ell \in X^i_{k'+1}} \kappa_{\ell+1}$. Let $(\dot{p}_1, \dot{c}_1)$ and $\dot{h}_1$ be $\mathbb{P}_{k,k'+1}$-names for (p_1, c_1) and h_1, respectively. Since $\mathbb{P}_{k,k'+1}$ satisfies the $\kappa_{k'+2}$-c.c., there is a function $h^*_1 \in \prod_{\ell \in X^i_{k'+1}} \kappa_{\ell+1}$ in $V^{\mathbb{P}_k}$ such that $\Vdash$ "$\dot{h}_1 \leq h^*_1$" and $\mathrm{cf}(h^*_1(\ell)) = \omega$ for all $\ell \in X^i_{k'+1}$. Find $p_2 \leq p \restriction [k, k'+1)$ forcing that $(\dot{p}_1, \dot{c}_1)$ satisfies (1) as witnessed by h^*_1. Now form (p^*, c^*) by letting

$$(p^* \restriction [k, k'+1), c^* \restriction [k, k'+1)) \leq (p_2, c \restriction [k, k'+1))$$

witness (1) for $\mathbb{C}_{k,k'}$ and letting

$$(p^* \restriction [k'+1, i+1), c^* \restriction [k'+1, i+1)) = (\dot{p}_1, \dot{c}_1).$$

We now deal with the case $\zeta > 0$. First, suppose $\zeta = \zeta_0 + 1$. We may assume that $p \restriction i^\frown p(i)_0{}^\frown p(i)_1$ decides whether $\zeta_0 \in \mathrm{dom}(p(i)_2)$. If it decides $\zeta_0 \notin \mathrm{dom}(p(i)_2)$, then we are done by the inductive hypothesis applied to ζ_0. Otherwise, find $(p', c') \leq (p \restriction i^\frown p(i) \restriction \zeta_0, c)$ and $\alpha < \kappa_{i+1}$ such that $\mathrm{cf}(\alpha) = \omega$ and

$$(p', c') \Vdash \text{“}\max(p(i)_2(\zeta_0)) < \alpha \text{ and } \alpha \in \dot{C}^{\zeta_0}_i.\text{”}$$

Form (p^*, c^*) by letting $(p^* \restriction i^\frown p^*(i) \restriction \zeta_0, c^*) \leq (p', c')$ witness (1) for $\mathbb{C}^{\zeta_0}_{k,i}$ and letting $p^*(i)_2(\zeta_0)$ be a name forced by p' to be equal to $p(i)_2(\zeta_0) \cup \{\alpha\}$.

Suppose next that ζ is a limit ordinal. If $\mathrm{cf}(\zeta) \geq \kappa_{i+1}$, then, strengthening p if necessary, we may assume $(p, c) \in \mathbb{C}^{\zeta_0}_{k,i}$ for some $\zeta_0 < \zeta$, and we are done by the induction hypothesis. Thus, suppose $\mu := \mathrm{cf}(\zeta) < \kappa_i$. Also assume that $\mu < \kappa_k$. If $\mu \geq \kappa_k$, the same trick we used in the $\zeta = 0$ case will work. Let $\langle \zeta_\eta \mid \eta < \mu \rangle$ be an increasing, continuous sequence of ordinals, cofinal in ζ, and construct a sequence $\langle (p_\eta, c_\eta) \mid \eta < \mu \rangle$ such that:

- For all $\eta < \mu$, $(p_\eta, c_\eta) \in \mathbb{C}^{\zeta_\eta}_{k,i}$ and satisfies (1).
- For all $\eta < \mu$, $(p_\eta, c_\eta) \leq (p \restriction i^\frown p(i) \restriction \zeta_\eta, c)$.
- For all $\eta < \eta' < \mu$, $(p_{\eta'} \restriction i^\frown p_{\eta'}(i) \restriction \zeta_\eta, c_{\eta'}) \leq (p_\eta, c_\eta)$.
- For all $\ell \in X^i_k$ and all $\eta < \mu$, $p_\eta \restriction \ell$ forces that the sequence $\langle p_\delta(\ell)_0 \mid \delta \leq \eta \rangle$ is a partial run of the game $G^*_{\kappa_k}(\mathbb{A}_\ell)$ with Player II playing according to her winning strategy.

The construction is a straightforward recursion, and, as in the $\zeta = 0$ case, it is easy to see that $\langle (p_\eta, c_\eta) \mid \eta < \mu \rangle$ gives us a $(p^*, c^*) \leq (p, c)$ witnessing (1). □

Note that $\mathbb{C}^*_{k,i}$ has the following closure property. We omit the proof, which is straightforward.

Claim 5.3. *Suppose $\omega < k < i$, with k a successor ordinal and i a limit ordinal. In $V^{\mathbb{P}_k}$, suppose $A \subseteq C^*_{k,i}$ is a directed set of size $< \kappa_k$. For each $\ell \in X^i_k$, let $\gamma_\ell = \sup(\{\gamma_{p(\ell)_1} \mid (p,c) \in A\})$ and suppose that, for all $\ell \in X^i_k$, $\Vdash_{\mathbb{P}_{k,\ell}}$ "γ_ℓ is approachable with respect to $\vec{a}_\ell$." Then A has a lower bound in $C^*_{k,i}$.*

Lemma 5.2 has the following immediate corollary.

Corollary 5.4. *Let $k < i$ be ordinals, with k a successor and i a limit.*

(1) *In $V^{\mathbb{P}_k}$, $\mathbb{C}^*_{k,i}$ is a dense, strongly κ_k-strategically closed subset of $\mathbb{C}_{k,i}$.*
(2) *In $V^{\mathbb{P}_i}$, $\mathbb{P}_{i,i+1}$ is κ_{i+1}-distributive.*

Proof. (1) is immediate. (2) follows from the Lemma 5.2 together with the observation that, if κ is a singular cardinal and a poset $\mathbb{P}$ is κ-distributive, then it is also κ^+-distributive. □

The fact that, for all $i < k$, $\mathbb{P}_{i,k}$ is κ_i-distributive in $V^{\mathbb{P}_i}$ means that $V^{\mathbb{P}} = \bigcup_{i \in \mathrm{On}} V^{\mathbb{P}_i}$ is a model of ZFC. It is also easy to see that, for all $i \in \mathrm{On}$, $\kappa_i = \aleph_i^{V^{\mathbb{P}}}$. If $i > \omega$ is a limit ordinal, then, in $V^{\mathbb{P}_i * \dot{\mathbb{A}}_i * \dot{\mathbb{S}}_i}$, S_i is a stationary subset of $S^{\kappa_{i+1}}_\omega$ that does not reflect at any ordinals in $S^{\kappa_{i+1}}_{>\kappa_\omega}$. Since $\mathbb{T}_i * \dot{\mathbb{P}}_{i+1,k}$ is countably-closed for all $k > i+1$, S_i remains stationary in $V^{\mathbb{P}}$.

It remains to show that, if $i > \omega$ is a limit ordinal, then $\mathrm{Refl}(\kappa_{i+1})$ holds in $V^{\mathbb{P}}$. Thus, fix a limit ordinal $i > \omega$. Since, for every $k > i+1$, $\mathbb{P}_{i+1,k}$ adds no new subsets of κ_{i+1} (recall that $\mathbb{P}_{i+1,i+2}$ is trivial forcing if i is a limit ordinal), it suffices to check that $\mathrm{Refl}(\kappa_{i+1})$ holds in $V^{\mathbb{P}_{i+1}}$.

Let $G = G_{i+1}$ be $\mathbb{P}_{i+1}$-generic over V. For $k < i+1$, let G_k be the $\mathbb{P}_k$-generic filter induced by G. If $\omega < k < i+1$ and k is a limit ordinal, let $G_{k,0}$ be the $\mathbb{P}_k * \dot{\mathbb{A}}_k$-generic induced by G, and let $G_{k,1}$ be the $\mathbb{P}_k * \dot{\mathbb{A}}_k * \dot{\mathbb{S}}_k$-generic induced by G. For $k < k' \leq i+1$, let $G_{k,k'}$ be the $\mathbb{P}_{k,k'}$-generic filter over $V[G_k]$ induced by G. Let $T \in V[G]$ be a stationary subset of κ_{i+1}. By shrinking T if necessary, we may assume that there is $k < i$ such that $T \subseteq S^{\kappa_{i+1}}_{\kappa_k}$.

We first assume that $k < \omega$. Let $k^* = k+2$. In $V[G_{k+1}]$, κ_{k^*} is still supercompact, so fix $j : V[G_{k+1}] \to M[G_{k+1}]$ witnessing that κ_{k^*} is κ_{i+1}-supercompact. In $V[G_{k+1}]$, $j(\mathbb{Q}_{k+1}) = \mathrm{Coll}(\kappa_{k+1}, < j(\kappa_{k^*})$, and $\mathbb{P}_{k^*,i+1}$ is strongly κ_{k+1}-strategically closed, so, by Fact 2.12, $j(\mathbb{Q}_{k+1}) \cong \mathbb{P}_{k+1,i+1} * \dot{\mathbb{R}}$, where $\dot{\mathbb{R}}$ is forced to be κ_{k+1}-closed. Thus, letting H be $\mathbb{R}$-generic over $V[G]$, we can extend j to $j : V[G_{k^*}] \to M[G * H]$.

To lift j further to have domain $V[G]$, we define a condition $p^* \in j(\mathbb{P}_{k^*,i+1})$ such that $p^* \leq j(p)$ for all $p \in G_{k^*,i+1}$. We recursively define $p^*(\alpha)$ for $\alpha \in [k^*, j(i+1))$. Thus, suppose $\alpha \in [k^*, j(i+1))$ and we have defined $p^* \restriction [k^*, \alpha)$. If $\alpha = \omega$ or α is the successor of a limit ordinal, then $p^*(\alpha)$ is a name for the sole condition in the trivial forcing. If α is the successor of a successor ordinal, then the α-th iterand in $j(\mathbb{P}_{k^*,i+1})$ is a Levy collapse that is $j(\kappa_{k^*})$-directed closed, so we can let $p^*(\alpha)$ be a name forced by $p^* \restriction [k^*, \alpha)$ to be a lower bound for $\{j(p)(\alpha) \mid p \in G_{k^*,i+1}\}$.

If α is a limit ordinal, let

$$\gamma_\alpha = \sup(\{j(g)(\alpha) \mid g \in \prod_{\ell \leq i} \kappa_{\ell+1} \cap V\}).$$

Let $p^*(\alpha)_0$ be a name forced by $p^* \restriction [k^*, \alpha)$ to be equal to

$$\{\gamma_\alpha\} \cup \bigcup_{p \in G_{k^*,i+1}} j(p)(\alpha)_0.$$

This will be forced to be a condition provided that γ_α is forced to be approachable with respect to the entry corresponding to α in $j(\langle \vec{a}_\ell \mid \ell \in X^i_k \rangle)$. The argument showing that this is true can be found in Case 1 of the proof of Theorem 3.1 in [3]. The $\mathbb{S}$ and $\mathbb{T}$ parts of the α-th iterand in $j(\mathbb{P}_{k^*,i+1})$ are forced to be $j(\kappa_{\omega+1})$-directed closed, so we can find $(p^*(\alpha)_1, p^*(\alpha)_2)$ that is forced by $p^* \restriction \alpha ^\frown p^*(\alpha)_0$ to be a lower bound for $\{(j(p)(\alpha)_1, j(p)(\alpha)_2) \mid p \in G_{k^*,i+1}\}$.

Let I be $j(\mathbb{P}_{k^*,i+1})$-generic over $V[G * H]$ with $p^* \in I$, and lift j to $j : V[G] \to M[G * H * I]$. By familiar arguments, we can now argue that, if T does not reflect in $V[G]$, then T is not stationary in $V[G * H * I]$, so there is a club C in κ_{i+1} with $C \in V[G * H * I]$ such that $C \cap T = \emptyset$. But I is generic for $j(\kappa^*)$-distributive forcing (recall $j(\kappa^*) > \kappa_{i+1}$) and thus could not have added C, so $C \in V[G * H]$. H is generic for κ_{k+1}-closed forcing and, in $V[G]$, AP_{κ_i} holds and T is a stationary subset of $S^{\kappa_{i+1}}_{<\kappa_{k+1}}$. Thus, by Fact 2.13, T remains stationary in $V[G * H]$. This is a contradiction, so T reflects in $V[G]$.

Next, suppose $k > \omega$. Let $k < i^* < i$, with i^* a successor ordinal large enough so that $\kappa_{i^*} > i$ and it is not the case that $\Vdash_{\mathbb{V}_{i^*+1,i}}$ "T is non-stationary." Let $k^* = i^* + 1$, and fix $j : V[G_{i^*}] \to M[G_{i^*}]$ witnessing that κ_{k^*} is κ_{i+1}-supercompact in $V[G_{i^*}]$. Recall that $\mathbb{C}_{k^*,i}$ has a dense, strongly κ_{i^*}-strategically-closed forcing. Thus, by Fact 2.12,

$$j(\mathbb{Q}_{i^*}) \cong \mathbb{Q}_{i^*} * \dot{\mathbb{C}}_{k^*,i} * \dot{\mathbb{R}} \cong \mathbb{P}_{i^*,i+1} * \dot{\mathbb{V}}_{k^*,i} * \dot{\mathbb{R}},$$

where $\dot{\mathbb{R}}$ is forced to be κ_{i^*}-closed. Let H be $\mathbb{V}_{k^*,i}$-generic over $V[G]$ such that T remains stationary in $V[G * H]$, let I be $\mathbb{R}$-generic over $V[G * H]$, and extend j to $j : V[G_{k^*}] \to M[G * H * I]$.

We again define a condition $p^* \in j(\mathbb{P}_{k^*,i+1})$ such that $p^* \leq j(p)$ for all $p \in G_{k^*,i+1}$. Since $i < \kappa_{k^*}$, the domain of conditions in $j(\mathbb{P}_{k^*,i+1})$ is $[k^*, i+1)$. If $\ell \in [k^*, i+1)$ is a limit ordinal, let $\gamma_\ell = \sup(j``\kappa_{\ell+1})$. Note that

$$\gamma_\ell = \sup(\{\gamma_{p(\ell)_1} \mid (p, c) \in (G * H) \cap \mathbb{C}^*_{k^*,i}\})$$

and that, as in the previous case and proven in [3], γ_ℓ is forced to be approachable with respect to $j(\dot{\vec{a}}_\ell)$. Thus, by Claim 5.3, $\{j(p, c) \mid (p, c) \in (G * H) \cap \mathbb{C}^*_{k^*,i}\}$ has a lower bound. Let (p^*, c^*) be such a lower bound and note that, since $\mathbb{C}^*_{k^*,i}$ is dense in $\mathbb{C}_{k^*,i}$, $p^* \leq j(p)$ for all $p \in G_{k^*,i+1}$.

As in the previous case, let J be $j(\mathbb{P}_{k^*,i+1})$-generic with $p^* \in J$ and lift j to $j : V[G] \to M[G * H * I * J]$. As before, we argue that, if T does not reflect in $V[G]$, then it is non-stationary in $V[G * H * I * J]$. As before, we can pull the non-stationarity back to $V[G * H]$. However, we chose H so that T remains stationary in $V[G * H]$. This is a contradiction, so T reflects in $V[G]$. □

6. Bounded stationary reflection without approachability

In the previous results, in order to obtain a model in which μ is a singular cardinal and bRefl(μ^+) holds, we forced AP_μ. In the final two sections of this paper, we produce models in which bRefl(μ^+) holds and AP_μ fails. We first find such a model in which μ is a limit of large cardinals.

Let $\langle \kappa^0_n \mid n < \omega \rangle$ and $\langle \kappa^1_n \mid n < \omega \rangle$ be increasing sequences of supercompact cardinals such that, letting $\kappa^i_\omega = \sup(\{\kappa^i_n \mid n < \omega\})$ for $i \in \{0,1\}$, we have $\kappa^0_\omega < \kappa^1_0$. For $i \in \{0,1\}$, let $\lambda_i = (\kappa^i_\omega)^+$.

Theorem 6.1. *Assume GCH. There is a cardinal-preserving forcing extension in which:*

1. $\mathrm{Refl}(\lambda_1)$ *holds.*
2. *There is a stationary $S \subseteq S^{\lambda_1}_\omega$ that does not reflect at any ordinals in $S^{\lambda_1}_{\geq \lambda_0}$.*
3. $AP_{\kappa^1_\omega}$ *fails.*

Proof. By first forcing with Laver's preparatory forcing [8], we may assume that, for all $i \in \{0,1\}$ and $n < \omega$, κ^i_n remains supercompact in any forcing extension by a κ^i_n-directed closed forcing poset. Let $\mathbb{S} = \mathbb{S}^{\lambda_1}_{\omega,\lambda_0}$, and let $\dot{S}$ be a name for the stationary subset of $S^{\lambda_1}_\omega$ added by $\mathbb{S}$. In $V^{\mathbb{S}}$, let $\mathbb{T} = CU(S)$.

In $V^{\mathbb{S}}$, define a forcing iteration $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\zeta \mid \xi \leq \lambda_1^+, \zeta < \lambda_1^+ \rangle$, taken with supports of size κ^1_ω, as follows. If $\zeta < \lambda_1^+$ and $\mathbb{P}_\zeta$ has been defined, choose a $\mathbb{P}_\zeta$-name $\dot{T}_\zeta$ for a subset of $S^{\lambda_1}_{\geq\lambda_0}$ such that: $\Vdash_{\mathbb{P}_\zeta * \dot{\mathbb{T}}}$ "$\dot{T}_\zeta$ is not stationary," and let $\dot{\mathbb{Q}}_\zeta$ be a $\mathbb{P}_\zeta$-name forced to be equal to $CU(\dot{T}_\zeta)$. Let $\mathbb{P} = \mathbb{P}_{\lambda_1^+}$. By standard bookkeeping arguments, we can arrange so that, in $V^{\mathbb{S}*\dot{\mathbb{P}}}$, if $X \subseteq S^{\lambda_1}_{\geq\lambda_0}$ is such that $\Vdash_{\mathbb{T}}$ "X is nonstationary," then X is already nonstationary in $V^{\mathbb{S}*\dot{\mathbb{P}}}$.

Let G be $\mathbb{S}$-generic over V, and let H be $\mathbb{P}$-generic over $V[G]$. We claim that $V[G*H]$ is the desired model. By Lemma 3.1, $\mathbb{S} * \dot{\mathbb{P}} * \dot{\mathbb{T}}$ has a λ_1-closed dense subset (in fact, an examination of the proof shows that it actually has a λ_1-directed closed dense subset). Thus, $\mathbb{S}*\dot{\mathbb{P}}$ is λ_1-distributive. Also, $\mathbb{S}*\dot{\mathbb{P}}$ is easily seen to have the λ_1^+-c.c., so forcing with it preserves cardinals. Let S be the set enumerated by $\bigcup G$. In $V[G]$, S is a stationary subset of $S^{\lambda_1}_\omega$ that does not reflect at any ordinal in $S^{\lambda_1}_{\geq\lambda_0}$. Note that $\mathbb{P}$ is λ_0-directed closed, so S remains stationary in $V[G*H]$ and still does not reflect at any ordinals in $S^{\lambda_1}_{\geq\lambda_0}$.

We now verify that $\mathrm{Refl}(\lambda_1)$ holds in $V[G*H]$. To this end, let T be a stationary subset of λ_1. Without loss of generality, by shrinking T if necessary, we can assume that there is $\mu < \kappa^1_\omega$ such that $T \subseteq S^{\lambda_1}_\mu$. First, suppose $\mu < \kappa^0_\omega$. Let $n^* < \omega$ be such that $\mu < \kappa^0_{n^*}$. Since $\mathbb{S} * \dot{\mathbb{P}}$ is λ_0-directed closed in V, $\kappa^0_{n^*}$ remains supercompact in $V[G*H]$. Let $j : V[G*H] \to M$ witness that $\kappa^0_{n^*}$ is λ_1-supercompact. Let $\delta = \sup(j``\lambda_1)$. As in earlier arguments, $j(T)$ reflects at δ in M. By elementarity, T reflects at some ordinal $\alpha < \lambda_1$ in $V[G*H]$.

Next, suppose $\lambda_0 \leq \mu < \kappa^1_\omega$. Let $n^* < \omega$ be such that $\mu < \kappa^1_{n^*}$. As $T \subseteq S^{\lambda_1}_{\geq\lambda_0}$ is stationary in $V[G*H]$, it is not the case that $\Vdash_{\mathbb{T}}$ "T is non-stationary." Thus, let I be $\mathbb{T}$-generic over $V[G*H]$ such that T remains stationary in $V[G*H*I]$. Since $\mathbb{S}*\dot{\mathbb{P}}*\dot{\mathbb{T}}$ has a λ_1-directed closed dense subset, $\kappa^1_{n^*}$ is supercompact in $V[G*H*I]$. Let $j : V[G*H*I] \to M$ witness that $\kappa^1_{n^*}$ is λ_1-supercompact. By the same arguments as in the previous case, T reflects at some ordinal $\alpha < \lambda_1$ in $V[G*H*I]$. Since this statement is obviously downward absolute, it reflects in $V[G*H]$.

It remains to show that $AP_{\kappa^1_\omega}$ fails in $V[G*H]$. However, this follows from the fact that κ^0_0 is supercompact in $V[G*H]$ and the fact that, by a result of Shelah, if $\mathrm{cf}(\mu) < \kappa < \mu$ and κ is supercompact, then AP_μ fails (see [2, Theorem 18.1] for a proof). □

7. Down to smaller cardinals

We would like to bring the results of the previous section down to smaller cardinals. By the following result of Chayut [1], assuming some cardinal arithmetic, $\aleph_{\omega^2+1}$ is the smallest we can hope for.

Theorem 7.1. *Suppose $n < \omega$, $\aleph_{\omega \cdot n}$ is strong limit, $2^{\aleph_{\omega \cdot n}} = \aleph_{\omega \cdot n+1}$, and $\mathrm{Refl}(\aleph_{\omega \cdot n+1})$ holds. Then $AP_{\aleph_{\omega \cdot n}}$ holds.*

We do not succeed in bringing the result from Section 6 down to $\aleph_{\omega^2+1}$, but we can attain it at $\aleph_{\omega^2 \cdot 2+1}$. In this section we adopt the convention, for notational simplicity, that if we are working in a forcing extension $V^{\mathbb{P}}$ of V, then $G(\mathbb{P})$ denotes the $\mathbb{P}$-generic filter over V used to define the extension.

Theorem 7.2. *Suppose there is an increasing sequence of supercompact cardinals of order type $\omega \cdot 2$. Then there is a forcing extension in which $\mathrm{Refl}(\aleph_{\omega^2 \cdot 2+1})$ holds, $AP_{\aleph_{\omega^2 \cdot 2}}$ fails, and there is a stationary subset of $S^{\aleph_{\omega^2 \cdot 2+1}}_{\omega}$ that does not reflect at any ordinals in $S^{\aleph_{\omega^2 \cdot 2+1}}_{\aleph_{\omega^2+1}}$.*

Proof. We follow, to a large extent, Section 3 of [10], and all references to [10] are to Section 3, specifically. Assume GCH. Let $\langle \kappa^0_n \mid n < \omega \rangle$ and $\langle \kappa^1_n \mid n < \omega \rangle$ be two increasing sequences of supercompact cardinals such that, for all $n < \omega$, $\kappa^0_n < \kappa^1_0$. Assume that the supercompactness of each κ^i_n is indestructible under κ^i_n-directed closed forcing. For $i < 2$, let $\kappa^i_\omega = \sup(\{\kappa^i_n \mid n < \omega\})$, and let $\lambda_i = (\kappa^i_\omega)^+$. For notational simplicity, let $\kappa^0_{-1} = \omega_1$ and $\kappa^1_{-1} = \lambda_0$. For $i < 2$ and $n < \omega$, let $\mathbb{C}^i_n = \prod_{m \geq n} \mathrm{Coll}((\kappa^i_{m-1})^{++}, < \kappa^i_m)$, where the product is taken with full support.

Define an equivalence relation on $\mathbb{C}^1_0$ by declaring that $c_0 \equiv c_1$, where $c_i = \langle c_i(n) \mid n < \omega \rangle$, if $c_0(n) = c_1(n)$ for all but finitely many $n < \omega$. Let $\mathbb{C}^*$ be the forcing notion whose conditions are equivalence classes from $\mathbb{C}^1_0$ and such that $[c_1] \leq [c_0]$ if, for all but finitely many $n < \omega$, $c_1(n) \leq c_0(n)$. The following is proven in [10, Lemma 7].

Proposition 7.3. *For all $n < \omega$, there is a projection from $\mathbb{C}^1_n$ onto $\mathbb{C}^*$. Hence, $\mathbb{C}^*$ is λ_1-distributive. Moreover, if G is $\mathbb{C}^*$-generic over V and $n < \omega$, then $\mathbb{C}^1_n / G$ has the λ_1-c.c.*

In $V^{\mathbb{C}^*}$, let $\mathbb{S} = \mathbb{S}^{\lambda_1}_{\omega, \lambda_0}$. Let $\dot{S}$ be an $\mathbb{S}$-name for the stationary subset of $S^{\lambda_1}_\omega$ added by $\mathbb{S}$ and, in $V^{\mathbb{C}^* * \dot{\mathbb{S}}}$, let $\mathbb{T} = CU(\dot{S})$. Also in $V^{\mathbb{C}^* * \dot{\mathbb{S}}}$, let $\mathbb{Q} = \mathbb{Q}_{\lambda^+}$ be an iteration of length λ_1^+ with supports of size κ^1_ω of forcings to destroy certain stationary subsets of $S^{\lambda_1}_{\geq \lambda_0}$. As in Section 3, we can arrange so that, in $V^{\mathbb{C}^*}$, $\mathbb{S} * \dot{\mathbb{Q}} * \dot{\mathbb{T}}$ has a dense λ_1-closed subset and if, in $V^{\mathbb{C}^* * \dot{\mathbb{S}} * \dot{\mathbb{Q}}}$, $T \subseteq S^{\lambda_1}_{\geq \lambda_0}$ and $\Vdash_{\mathbb{T}}$ "T is non-stationary," then T is already non-stationary in $V^{\mathbb{C}^* * \dot{\mathbb{S}} * \dot{\mathbb{Q}}}$. Since $\mathbb{T}$ is weakly homogeneous, we in fact get that, for all $T \subseteq S^{\lambda_1}_{\geq \lambda_0}$ in $V^{\mathbb{C}^* * \dot{\mathbb{S}} * \dot{\mathbb{Q}}}$, if T is stationary, then $\Vdash_{\mathbb{T}}$ "T is stationary."

For $n < \omega$, note that $\mathbb{C}^0_{n+1} \times (\mathbb{C}^1_0 * \dot{\mathbb{S}} * \dot{\mathbb{Q}})$ is $(\kappa^0_n)^{++}$-directed closed. Let $\dot{F}^0_n$ denote a name for a fine, normal ultrafilter on $\mathcal{P}_{\kappa^0_n}(\lambda_1)$ in $V^{\mathbb{C}^0_{n+1} \times (\mathbb{C}^1_0 * \dot{\mathbb{S}} * \dot{\mathbb{Q}})}$. Let U^0_n be its projection to a normal ultrafilter on κ^0_n. Note that $U^0_n \in V$ and, by the homogeneity of the forcing, we may assume that the trivial condition forces U^0_n to be the projection of a fine, normal ultrafilter on $\mathcal{P}_{\kappa^0_n}(\lambda_1)$. Similarly, define a normal ultrafilter U^1_n on κ^1_n such that the trivial condition in $\mathbb{C}^1_{n+1} * \dot{\mathbb{S}} * \dot{\mathbb{Q}} * \dot{\mathbb{T}}$ forces U^1_n to be the projection of a fine, normal ultrafilter on $\mathcal{P}_{\kappa^1_n}(\lambda_1)$.

For $i < 2$ and $n < \omega$, let M^i_n denote the transitive collapse of $\mathrm{Ult}(V, U^i_n)$, and let $j^i_n : V \to M^i_n$ be the associated embedding. Let $\mathbb{T}^0_n$ denote $\mathrm{Coll}((\kappa^0_n)^{+\omega\cdot 2+2}, < j^0_n(\kappa^0_n))$ as defined in M^0_n. $M \models$ "there are $j^0_n(\kappa^0_n)$ maximal antichains of $\mathbb{T}^0_n$." Since $|j^0_n(\kappa^0_n)| = (\kappa^0_n)^+$ and $\mathbb{T}^0_n$ is $(\kappa^0_n)^+$-closed, we can build in V a $\mathbb{T}^0_n$-generic filter over M^0_n. Let G^0_n be such a filter. Similarly, let $\mathbb{T}^1_n$ denote $\mathrm{Coll}((\kappa^1_n)^{+\omega+2}, < j^1_n(\kappa^1_n))$ as defined in M^1_n, and fix G^1_n, a $\mathbb{T}^1_n$-generic filter over M^1_n.

We now define diagonal Prikry forcing notions $\mathbb{P}_0$ and $\mathbb{P}_1$, which are slightly modified versions of the forcing in [10]. Elements of $\mathbb{P}_0$ are of the form $p = \langle \alpha^p_0, \ldots, \alpha^p_{n-1}, \langle A^p_k \mid n \leq k < \omega\rangle, g^p_0, \ldots, g^p_n, f^p_0, \ldots, f^p_{n-1}, \langle F^p_k \mid n \leq k < \omega\rangle, \langle g^p_k \mid n < k < \omega\rangle\rangle$, where

- For all $i < n$, α^p_i is inaccessible and $\kappa^0_{i-1} < \alpha^p_i < \kappa^0_i$.
- For all $n \leq k < \omega$, $A^p_k \in U^0_k$ and, for all $\alpha \in A^p_k$, α is inaccessible.
- For all $i < n$, $g^p_i \in \mathrm{Coll}((\kappa^0_{i-1})^{++}, < \alpha^p_i)$ and $f^p_i \in \mathrm{Coll}((\alpha^p_i)^{+\omega\cdot 2+2}, < \kappa^0_i)$.
- For all $n \leq k < \omega$, $g^p_k \in \mathrm{Coll}((\kappa^0_{k-1})^{++}, < \kappa^0_k)$ is such that, for all $\alpha \in A^p_k$, $g^p_k \in \mathrm{Coll}((\kappa^0_{k+1})^{++}, < \alpha)$.
- For all $n \leq k < \omega$, F^p_k is a function with domain A^p_k such that, for all $\alpha \in A^p_k$, $F^p_k(\alpha) \in \mathrm{Coll}(\alpha^{+\omega\cdot 2+2}, < \kappa^0_k)$ and $j^0_k(F^p_k)(\kappa^0_k) \in G^0_k$.

n is the length of p and is denoted $\ell(p)$. If $q, p \in \mathbb{P}_0$, then $q \leq p$ if:

- $\ell(q) \geq \ell(p)$.
- For all $i < \ell(p)$, $\alpha^q_i = \alpha^p_i$ and $f^q_i \leq f^p_i$.
- For all $i < \omega$, $g^q_i \leq g^p_i$.
- For all $\ell(q) \leq k < \omega$, $A^q_k \subseteq A^p_k$ and, for all $\alpha \in A^q_k$, $F^q_k(\alpha) \leq F^p_k(\alpha)$.
- For all $\ell(p) \leq k < \ell(q)$, $\alpha^q_k \in A^p_k$ and $f^q_k \leq F^p_k(\alpha^q_k)$.

$\mathbb{P}_1$ is defined in the same way, with the following changes:

- For all $-1 \leq i < \omega$, every occurrence of κ^0_i in the definition of $\mathbb{P}_0$ is replaced by κ^1_i in the definition of $\mathbb{P}_1$ and every occurrence of U^0_i is replaced by U^1_i.
- If $p \in \mathbb{P}_1$ and $i < \ell(p)$, then $f^p_i \in \mathrm{Coll}((\alpha^p_i)^{+\omega+2}, < \kappa^1_i)$. If $\ell(p) \leq k < \omega$ and $\alpha \in A^p_k$, then $F^p_k(\alpha) \in \mathrm{Coll}(\alpha^{+\omega+2}, < \kappa^1_j)$ and $j^1_k(F^p_k)(\kappa^1_k) \in G^1_k$.

Following [10], if $p \in \mathbb{P}_i$, we call $\langle \alpha^p_k \mid k < \ell(p)\rangle$ its α-part, $\langle A^p_k \mid \ell(p) \leq k < \omega\rangle$ its A-part, $\langle f^p_k \mid k < \ell(p)\rangle$ its f-part, $\langle g^p_k \mid k \leq \ell(p)\rangle$ its g-part, $\langle F^p_k \mid \ell(p) \leq k < \omega\rangle$ its F-part, and $\langle g^p_k \mid \ell(p) < k < \omega\rangle$ its C-part. The α-part, g-part, and f-part together comprise the lower part of p, denoted $a(p)$. If $k \leq \ell(p)$, let $p \restriction k$ denote $\langle\langle \alpha^p_m \mid m < k\rangle, \langle g^p_m \mid m \leq k\rangle, \langle f^p_m \mid m < k\rangle\rangle$. Note that $p \restriction \ell(p) = a(p)$.

If $q, p \in \mathbb{P}_i$, then we say q is a length-preserving extension of p if $q \leq p$ and $\ell(q) = \ell(p)$. If $k \leq \ell(p)$, then q is a k-length-preserving extension of p if q is a length-preserving extension of p and $q \restriction k = p \restriction k$. We say q is a trivial extension of p if it is an $\ell(p)$-length-preserving extension of p.

$\mathbb{P}_i$ satisfies a form of the Prikry lemma. A proof can be found in [10, Lemma 3].

Lemma 7.4. *Let $p \in \mathbb{P}_i$, let $k \leq \ell(p)$, and let D be a dense open subset of $\mathbb{P}_i$. Then there is a k-length-preserving extension $q \leq p$ such that, if $q^* \leq q$ and $q^* \in D$, then, if $q^{**} \leq q$, $\ell(q^{**}) = \ell(q^*)$, and $q^{**} \restriction k = q^* \restriction k$, then $q^{**} \in D$.*

The Prikry lemma can be applied to see that the only cardinals below λ_i that are collapsed by forcing with $\mathbb{P}_i$ are those explicitly in the scope of the Levy collapses interleaved into the forcing notion. In particular, the following claim immediately follows.

Claim 7.5. *Let $k < \omega$. In $V^{\mathbb{P}_0}$, if $\mathrm{cf}(\alpha) = (\alpha^p_k)^{+\omega\cdot 2+1}$ (equivalently, $\mathrm{cf}^V(\alpha) = (\alpha^p_k)^{+\omega\cdot 2+1}$) for some $p \in G(\mathbb{P}_0)$ and A is an unbounded subset of α, then there is an unbounded $B \subseteq A$ such that $B \in V$. Similarly, in $V^{\mathbb{P}_1}$, if $\mathrm{cf}(\alpha) = (\alpha^p_k)^{+\omega+1}$ for some $p \in G(\mathbb{P}_1)$ and A is an unbounded subset of α, then there is an unbounded $B \subseteq A$ such that $B \in V$.*

Other arguments, found in [10, Lemma 5], imply that all cardinals $\geq \lambda_i$ are preserved as well. Note that, by the Prikry lemma, forcing with $\mathbb{P}_1$ does not add any new bounded subsets of λ_0^{++}, so $\mathbb{P}_0$ has the same basic properties in $V^{\mathbb{P}_1}$ as it has in V.

The following is proven in [10], in Lemma 7 and the discussion following it.

Proposition 7.6. *There is a projection from $\mathbb{P}_1$ onto $\mathbb{C}^*$ such that the quotient forcing has the λ_1-c.c.*

Our final model will be $V^{\mathbb{P}_0\times(\mathbb{P}_1*\dot{\mathbb{S}}*\dot{\mathbb{Q}})}$. Let S be the subset of $S^{\lambda_1}_\omega$ added by $\mathbb{S}$. In $V^{\mathbb{C}^**\dot{\mathbb{S}}}$, S is stationary and does not reflect at any ordinals in $S^{\lambda_1}_{\geq\lambda_0}$. In $V^{\mathbb{C}^**\dot{\mathbb{S}}}$, $\mathbb{Q}$ is λ_0-closed, so S remains stationary in $V^{\mathbb{C}^**\dot{\mathbb{S}}*\dot{\mathbb{Q}}}$. In $V^{\mathbb{C}^*}$, $\mathbb{P}_1/G(\mathbb{C}^*)$ has the λ_1-c.c. and $\mathbb{S}*\dot{\mathbb{Q}}$ is the projection of a forcing poset with a dense λ_1-closed subset (namely $\mathbb{S}*\dot{\mathbb{Q}}*\dot{\mathbb{T}}$), so, by Easton's Lemma, $\mathbb{P}_1/G(\mathbb{C}^*)$ has the λ_1-c.c. in $V^{\mathbb{C}^**\dot{\mathbb{S}}*\dot{\mathbb{Q}}}$, so S is stationary in $V^{\mathbb{P}_1*\dot{\mathbb{S}}*\dot{\mathbb{Q}}}$. Finally, $|\mathbb{P}_0| = \lambda_0$, so S remains stationary in $V^{\mathbb{P}_0\times(\mathbb{P}_1*\dot{\mathbb{S}}*\dot{\mathbb{Q}})}$.

We now verify that every stationary subset of λ_1 reflects in $V^{\mathbb{P}_0\times(\mathbb{P}_1*\dot{\mathbb{S}}*\dot{\mathbb{Q}})}$. Thus, let $T \in V^{\mathbb{P}_0\times(\mathbb{P}_1*\dot{\mathbb{S}}*\dot{\mathbb{Q}})}$ be a stationary subset of λ_1, and let $\dot{T}$ be a name for it. Let (i,n) be the lexicographically least pair such that $T\cap S^{\lambda_1}_{<\kappa^i_n}$ is stationary. By shrinking T if necessary, we may assume that, for some $(p_0,(p_1,\dot{s},\dot{q})) \in G(\mathbb{P}_0 \times (\mathbb{P}_1*\dot{\mathbb{S}}*\dot{\mathbb{Q}}))$, $(p_0,(p_1,\dot{s},\dot{q}))$ forces that $\dot{T}$ is a stationary subset of $S^{\lambda_1}_{<\kappa^i_n}$. Moreover, if $i = 1$, we may assume that $(p_0,(p_1,\dot{s},\dot{q}))$ forces $\dot{T}$ to be a stationary subset of $S^{\lambda_1}_{\geq\lambda_0}$ as well.

For each $\alpha \in T$, let $(p^\alpha_0,(p^\alpha_1,\dot{s}^\alpha,\dot{q}^\alpha)) \in G(\mathbb{P}_0\times(\mathbb{P}_1*\dot{\mathbb{S}}*\dot{\mathbb{Q}}))$, with $(p^\alpha_0,(p^\alpha_1,\dot{s}^\alpha,\dot{q}^\alpha)) \leq (p_0,(p_1,\dot{s},\dot{q}))$, force that $\alpha \in \dot{T}$. Since $|\mathbb{P}_0| = \lambda_0$ and there are only κ^1_ω lower parts in $\mathbb{P}_1$, we may assume there are $p^*_0 \in \mathbb{P}_0$ and a^*, a lower part for $\mathbb{P}_1$, such that, for every $\alpha \in T$, $p^\alpha_0 = p^*_0$ and $a(p^\alpha_1) = a^*$. We may also assume that $p_0 = p^*_0$, $a(p_1) = a^*$, and $(p_0,(p_1,\dot{s},\dot{q}))$ forces that $T^* := \{\alpha \mid$ for some $(p'_0,(p'_1,\dot{s}',\dot{q}')) \in G(\mathbb{P}_0\times(\mathbb{P}_1*\dot{\mathbb{S}}*\dot{\mathbb{Q}}))$ such that $p'_0 = p_0$, p'_1 is a trivial extension of p_1, and $p'_1 \Vdash$ "$(\dot{s}',\dot{q}') \leq (\dot{s},\dot{q})$", $(p'_0,(p'_1,\dot{s}',\dot{q}'))$ forces that $\alpha \in \dot{T}\}$ is stationary. Let $\dot{T}^*$ be a name for T^*. Finally, we may assume that $\ell(p_i) \geq n$. We will find an extension of $(p_0,(p_1,\dot{s},\dot{q}))$ forcing that $\dot{T}$ reflects. There are two cases to consider.

Case 1: i = 0. Let $n^* = \ell(p_0)$. Move to $V^{\mathbb{C}^0_{n^*+1}\times(\mathbb{C}^1_0*\dot{\mathbb{S}}*\dot{\mathbb{Q}})}$, requiring that the C-part of p_0 is in $G(\mathbb{C}^0_{n^*+1})$ and

$$(\langle g^{p_1}_k \mid k < \omega\rangle, \dot{s}, \dot{q}) \in G(\mathbb{C}^1_0 * \dot{\mathbb{S}} * \dot{\mathbb{Q}}).$$

For $n^*+1 \leq m < \omega$, let $G(\mathbb{C}^0_m)$ be the generic filter induced by $G(\mathbb{C}^0_{n^*+1})$. Similarly, for $m < \omega$, let $G(\mathbb{C}^1_m)$ be the generic filter induced by $G(\mathbb{C}^1_0)$. Let $\mathbb{P}^*$ be the set of $(r_0,r_1) \in \mathbb{P}_0\times\mathbb{P}_1$ such that $(r_0,r_1) \leq (p_0,p_1)$, the C-part of r_0 is in $G(\mathbb{C}^0_{\ell(r_0)+1})$, and the C-part of r_1 is in $G(\mathbb{C}^1_{\ell(r_1)+1})$. The proof of Lemma 6 from Section 3 of [10] shows that forcing with $\mathbb{P}^*$ over $V^{\mathbb{C}^0_{n^*+1}\times(\mathbb{C}^1_0*\dot{\mathbb{S}}*\dot{\mathbb{Q}})}$ adds a V-generic filter for $\mathbb{P}_0\times\mathbb{P}_1$. In $V^{\mathbb{C}^0_{n^*+1}\times(\mathbb{C}^1_0*\dot{\mathbb{S}}*\dot{\mathbb{Q}})}$, let $\hat{T}$ be the set of $\alpha < \lambda_1$ such that, for some r_1 such that r_1

is a trivial extension of p_1 and $(p_0, r_1) \in \mathbb{P}^*$, and, for some $(\dot{r}_2, \dot{r}_3)$ such that

$$(p_0, r_1) \Vdash \text{“}(\dot{r}_2, \dot{r}_3) \in G(\dot{\mathbb{S}} * \dot{\mathbb{Q}}) \text{ and } (\dot{r}_2, \dot{r}_3) \leq (\dot{s}, \dot{q})\text{”},$$

we have $(p_0, (r_1, \dot{r}_2, \dot{r}_3)) \Vdash$ “$\alpha \in \dot{T}^*$”

Lemma 7.7. *$\hat{T}$ is stationary.*

Proof. Suppose not. Then, in $V^{\mathbb{C}^0_{n^*+1} \times (\mathbb{C}^1_0 * \dot{\mathbb{S}} * \dot{\mathbb{Q}})}$, there is a club C in λ_1 such that $C \cap \hat{T} = \emptyset$. Force with $\mathbb{P}^*$. In $V^{(\mathbb{C}^0_{n^*+1} \times (\mathbb{C}^1_0 * \dot{\mathbb{S}} * \dot{\mathbb{Q}})) * \mathbb{P}^*}$, $\hat{T} \supseteq T^*$. Thus, $C \cap T^* = \emptyset$. $V^{\mathbb{C}^0_{n^*+1} \times (\mathbb{C}^1_0 * \dot{\mathbb{S}} * \dot{\mathbb{Q}})}$ is a forcing extension of $V^{(\mathbb{C}^* * \dot{\mathbb{S}} * \dot{\mathbb{Q}})}$ by a λ_1-c.c. forcing poset, so there is a club $D \subseteq C$ such that $D \in V^{(\mathbb{C}^* * \dot{\mathbb{S}} * \dot{\mathbb{Q}})}$. But $V^{(\mathbb{C}^* * \dot{\mathbb{S}} * \dot{\mathbb{Q}})} \subseteq V^{\mathbb{P}_0 \times (\mathbb{P}_1 * \dot{\mathbb{S}} * \dot{\mathbb{Q}})}$ and T^* is stationary in $V^{\mathbb{P}_0 \times (\mathbb{P}_1 * \dot{\mathbb{S}} * \dot{\mathbb{Q}})}$. This contradicts the fact that $D \in V^{\mathbb{P}_0 \times (\mathbb{P}_1 * \dot{\mathbb{S}} * \dot{\mathbb{Q}})}$ is club in λ_1 and disjoint from T^*. □

In $V^{\mathbb{C}^0_{n^*+1} \times (\mathbb{C}^1_0 * \dot{\mathbb{S}} * \dot{\mathbb{Q}})}$, $\kappa^0_{n^*}$ remains supercompact, and $\lambda_1 = (\kappa^0_{n^*})^{+\omega \cdot 2+1}$. Fix a fine, normal measure U^* on $\mathcal{P}_{\kappa^0_{n^*}}(\lambda_1)$ such that U^* projects to $U^0_{n^*}$. Let θ be a sufficiently large regular cardinal, and let $\mathfrak{A}$ denote an expansion of $(H(\theta), \in)$ by a well-ordering of $H(\theta)$ and constants for all relevant sets. The following are standard applications of supercompactness.

Lemma 7.8. *Let $E_0 = \{X \in \mathcal{P}_{\kappa^0_{n^*}}(\lambda_1) \mid$ for some $\mathfrak{B} \prec \mathfrak{A}$, we have $X = \mathfrak{B} \cap \lambda_1$, $|X| = |\mathfrak{B}|$, and $X \in \bigcap_{A \in U^* \cap \mathfrak{B}} A\}$. Then $E_0 \in U^*$.*

Lemma 7.9. *Let $E_1 = \{X \in \mathcal{P}_{\kappa^0_{n^*}}(\lambda_1) \mid X \cap \kappa^0_{n^*}$ is inaccessible, $\mathrm{otp}(X) = (X \cap \kappa^0_{n^*})^{+\omega \cdot 2+1}$, and $\hat{T} \cap X$ is stationary in $\sup(X)\}$. Then $E_1 \in U^*$.*

The next lemma follows from the proof of Lemma 13 in [10].

Lemma 7.10. *Let $X \in E_0 \cap E_1$ such that $X \cap \kappa^0_{n^*} \in A^{p_0}_{n^*}$. Let $\mathfrak{B} \prec \mathfrak{A}$ witness that $X \in E_0$. Then there is $(p^*_0, p^*_1) \in \mathbb{P}^*$ such that:*

(1) $(p^*_0, p^*_1) \leq (p_0, p_1)$.
(2) $\ell(p^*_0) = n^* + 1$ *and p^*_1 is a trivial extension of p_1.*
(3) $\alpha^{p^*_0}_{n^*} = X \cap \kappa^0_{n^*}$.
(4) *If $(p_0, p'_1) \in \mathbb{P}^* \cap \mathfrak{B}$ and p'_1 is a trivial extension of p_1, then $(p^*_0, p^*_1) \leq (p_0, q'_1)$.*

Let $X, \mathfrak{B}$, and (p^*_0, p^*_1) be as given in Lemma 7.10. For every $\gamma \in \hat{T} \cap X$, there is p^1_γ, a trivial extension of p_1, and $(\dot{s}'_\gamma, \dot{q}'_\gamma)$ such that (p_0, p^1_γ) forces that $(\dot{s}'_\gamma, \dot{q}'_\gamma) \in G(\dot{\mathbb{S}} * \dot{\mathbb{Q}})$ and $(p_0, (p^1_\gamma, \dot{s}'_\gamma, \dot{q}'_\gamma))$ forces that $\gamma \in \dot{T}^*$. By elementarity of $\mathfrak{B}$, such a p^1_γ exists in $\mathfrak{B}$, and hence, for all $\gamma \in \hat{T} \cap X$, $(p^*_0, (p^*_1, \dot{s}'_\gamma, \dot{q}'_\gamma)) \Vdash$ “$\gamma \in \dot{T}^*$”. Since $\hat{T} \cap X \in V^{\mathbb{C}^0_{n^*+1} \times (\mathbb{C}^1_0 * \dot{\mathbb{S}} * \dot{\mathbb{Q}})}$ and has size less than $\kappa^0_{n^*}$, we have $\{(\dot{s}'_\gamma, \dot{q}'_\gamma) \mid \gamma \in \hat{T} \cap X\} \in V$.

In $V^{\mathbb{C}^*}$, $\mathbb{S} * \dot{\mathbb{Q}}$ is $\kappa^0_{n^*}$-directed closed. Thus, we can find names $\dot{s}'$ and $\dot{q}'$ such that, for all $\gamma \in \hat{T} \cap X$, $(p^*_0, (p^*_1, \dot{s}', \dot{q}')) \leq (p^*_0, (p^*_1, \dot{s}'_\gamma, \dot{q}'_\gamma))$. Thus, $(p^*_0, (p^*_1, \dot{s}', \dot{q}')) \Vdash$ “$\dot{T} \cap X \supseteq \hat{T} \cap X$”. Since no cardinals between $X \cap \kappa^0_{n^*}$ and $(X \cap \kappa^0_{n^*})^{+\omega \cdot 2+2}$ are collapsed, an application of Claim 7.5 yields that $\hat{T} \cap X$ remains stationary in $\sup(X)$ after forcing over $V^{\mathbb{C}^0_{n^*+1} \times (\mathbb{C}^1_0 * \dot{\mathbb{S}} * \dot{\mathbb{Q}})}$ with $\mathbb{P}^*$ below (p^*_0, p^*_1). Hence, $T \cap X$ is stationary in $\sup(X)$ in $V^{\mathbb{P}_0 \times (\mathbb{P}_1 * \dot{\mathbb{S}} * \dot{\mathbb{Q}})}$ after forcing below $(q_0, (q_1, \dot{s}', \dot{q}'))$, so $(q_0, (q_1, \dot{s}', \dot{q}')) \Vdash$ “$\dot{T}$ reflects.”

Case 2: i = 1. Let $n^* = \ell(p_1)$. Move to $V^{\mathbb{C}^1_{n^*+1} * \dot{\mathbb{S}} * \dot{\mathbb{Q}}}$, requiring that the C-part of p_1 is in $G(\mathbb{C}^1_{n^*+1})$. Let $\mathbb{P}^*$ be the set of $(r_0, r_1) \in \mathbb{P}_0 \times \mathbb{P}_1$ such that $(r_0, r_1) \leq (p_0, p_1)$ and the C-part of r_1 is in $G(\mathbb{C}^1_{\ell(r_1)+1})$. As before, forcing with $\mathbb{P}^*$ over $V^{\mathbb{C}^1_{n^*+1} * \dot{\mathbb{S}} * \dot{\mathbb{Q}}}$ adds a V-generic filter for $\mathbb{P}_0 \times \mathbb{P}_1$. Let $\hat{T}$ be the set of $\alpha < \lambda_1$ such that, for some r_1 such that r_1 is a trivial extension of p_1 and $(p_0, r_1) \in \mathbb{P}^*$, and for some $(\dot{r}_2, \dot{r}_3)$ such that

$$(p_0, r_1) \Vdash \text{“}(\dot{r}_2, \dot{r}_3) \in G(\dot{S} * \dot{Q}) \text{ and } (\dot{r}_2, \dot{r}_3) \leq (\dot{s}, \dot{q})\text{”},$$

we have $(p_0, (r_1, \dot{r}_2, \dot{r}_3)) \Vdash$ “$\alpha \in \dot{T}^*$”. As in Case 1, $\hat{T}$ is stationary in $V^{\mathbb{C}^1_{n^*+1} * \dot{\mathbb{S}} * \dot{\mathbb{Q}}}$.

Lemma 7.11. *$\hat{T}$ is stationary in $V^{\mathbb{C}^1_{n^*+1} * \dot{\mathbb{S}} * \dot{\mathbb{Q}} * \dot{\mathbb{T}}}$.*

Proof. Suppose not, and let $D \in V^{\mathbb{C}^1_{n^*+1} * \dot{\mathbb{S}} * \dot{\mathbb{Q}} * \dot{\mathbb{T}}}$ be club in λ_1 such that $D \cap \hat{T} = \emptyset$. Since $V^{\mathbb{C}^1_{n^*+1} * \dot{\mathbb{S}} * \dot{\mathbb{Q}} * \dot{\mathbb{T}}}$ is a forcing extension of $V^{\mathbb{C}^* * \dot{\mathbb{S}} * \dot{\mathbb{Q}} * \dot{\mathbb{T}}}$ by a λ_1-c.c. forcing, there is a club $D' \subseteq D$, $D' \in V^{\mathbb{C}^* * \dot{\mathbb{S}} * \dot{\mathbb{Q}} * \dot{\mathbb{T}}}$, such that $\Vdash_{\mathbb{C}^1_{n^*+1}/G(\mathbb{C}^*)}$ “$\dot{\hat{T}} \cap D' = \emptyset$”. In $V^{\mathbb{C}^* * \dot{\mathbb{S}} * \dot{\mathbb{Q}}}$, let

$$\hat{\hat{T}} = \{\alpha < \lambda_1 \mid \text{ for some } c \in \mathbb{C}^1_{n^*+1}/G(\mathbb{C}^*), c \Vdash \text{“}\alpha \in \dot{\hat{T}}\text{”}\}$$

. Then, in $V^{\mathbb{C}^* * \dot{\mathbb{S}} * \dot{\mathbb{Q}} * \dot{\mathbb{T}}}$, $\Vdash_{\mathbb{C}^1_{n^*+1}/G(\mathbb{C}^*)}$ “$\dot{\hat{T}} \subseteq \hat{\hat{T}}$”, and $\hat{\hat{T}} \cap D' = \emptyset$. Thus, $\hat{\hat{T}}$ is a subset of $S^{\lambda_1}_{\geq \lambda_0}$ that is non-stationary in $V^{\mathbb{C}^* * \dot{\mathbb{S}} * \dot{\mathbb{Q}} * \dot{\mathbb{T}}}$ and is thus already non-stationary in $V^{\mathbb{C}^* * \dot{\mathbb{S}} * \dot{\mathbb{Q}}}$. But $V^{\mathbb{C}^* * \dot{\mathbb{S}} * \dot{\mathbb{Q}}} \subseteq V^{\mathbb{P}_0 \times (\mathbb{P}_1 * \dot{\mathbb{S}} * \dot{\mathbb{Q}})}$ and, in $V^{\mathbb{P}_0 \times (\mathbb{P}_1 * \dot{\mathbb{S}} * \dot{\mathbb{Q}})}$, $\hat{\hat{T}} \supseteq T$, contradicting the fact that T is stationary in $V^{\mathbb{P}_0 \times (\mathbb{P}_1 * \dot{\mathbb{S}} * \dot{\mathbb{Q}})}$. □

The rest of the proof is much as in Case 1. We provide some details for completeness. In $V^{\mathbb{C}^1_{n^*+1} * \dot{\mathbb{S}} * \dot{\mathbb{Q}} * \dot{\mathbb{T}}}$, $\kappa^1_{n^*}$ is supercompact and $\lambda_1 = (\kappa^1_{n^*})^{+\omega+1}$. Fix a fine, normal measure U^* on $\mathcal{P}_{\kappa^1_{n^*}}(\lambda_1)$ such that U^* projects to $U^1_{n^*}$. Let θ be a sufficiently large, regular cardinal, and let $\mathfrak{A}$ be an expansion of $(H(\theta), \in)$ by a well-ordering and constants for all relevant sets. The next lemmas are as before.

Lemma 7.12. *Let $E_0 = \{X \in \mathcal{P}_{\kappa^1_{n^*}}(\lambda_1) \mid$ for some $\mathfrak{B} \prec \mathfrak{A}$, we have $X = \mathfrak{B} \cap \lambda_1$, $|X| = |\mathfrak{B}|$, and $X \in \bigcap_{A \in U^* \cap \mathfrak{B}} A\}$. Then $E_0 \in U^*$.*

Lemma 7.13. *Let $E_1 = \{X \in \mathcal{P}_{\kappa^1_{n^*}}(\lambda_1) \mid X \cap \kappa^0_{n^*}$ is inaccessible, $\mathrm{otp}(X) = (X \cap \kappa^0_{n^*})^{+\omega+1}$, and $\hat{T} \cap X$ is stationary in $\sup(X)\}$. Then $E_1 \in U^*$.*

Lemma 7.14. *Let $X \in E_0 \cap E_1$ such that $X \cap \kappa^1_{n^*} \in A^{p_1}_{n^*}$. Let $\mathfrak{B} \prec \mathfrak{A}$ witness that $X \in E_0$. Then there is $(p_0, p^*_1) \in \mathbb{P}^*$ such that:*

1. *$(p_0, p^*_1) \leq (p_0, p_1)$.*
2. *$\ell(p^*_1) = n^* + 1$.*
3. *$\alpha^{p^*_1}_{n^*} = X \cap \kappa^0_{n^*}$.*
4. *If $(p_0, p'_1) \in \mathbb{P}^* \cap \mathfrak{B}$ and p'_1 is a trivial extension of p_1, then $(p_0, p^*_1) \leq (p_0, p'_1)$.*

Let $X, \mathfrak{B}$, and (p_0, p^*_1) be as given in Lemma 7.14. As in Case 1, we get that, for every $\gamma \in \hat{T} \cap X$, there is $(\dot{s}_\gamma, \dot{q}_\gamma)$ such that (p_0, p^*_1) forces that $(\dot{s}_\gamma, \dot{q}_\gamma) \in G(\dot{\mathbb{S}} * \dot{\mathbb{Q}})$ and $(p_0, (p^*_1, \dot{s}_\gamma, \dot{q}_\gamma))$ forces that $\gamma \in \dot{T}^*$. Moreover, we may assume that, for every such γ, there is $\dot{t}_\gamma$ such that (p_0, p^*_1) forces $(\dot{s}_\gamma, \dot{q}_\gamma, \dot{t}_\gamma)$ is in the dense λ_1-directed closed subset of $\dot{\mathbb{S}} * \dot{\mathbb{Q}} * \dot{\mathbb{T}}$ and in $G(\dot{\mathbb{S}} * \dot{\mathbb{Q}} * \dot{\mathbb{T}})$. We can thus find names $\dot{s}'$ and $\dot{q}'$ such that, for all $\gamma \in \hat{T} \cap X$, $(p_0, (p^*_1, \dot{s}', \dot{q}')) \leq (p_0, (p^*_1, \dot{s}_\gamma, \dot{q}_\gamma))$. Since, when forcing with

$\mathbb{P}^*$ below (p_0, q_1), no cardinals between $X \cap \kappa^1_{n^*}$ and $(X \cap \kappa^1_{n^*})^{+\omega+1}$ are collapsed, another application of Claim 7.5 yields that $(p_0, (q_1, \dot{s}', \dot{q}'))$ forces that $\dot{T}$ reflects at $\sup(X)$.

It remains to show that $AP_{\kappa^1_\omega}$ fails in $V^{\mathbb{P}_0 \times (\mathbb{P}_1 * \dot{\mathbb{S}} * \dot{\mathbb{Q}})}$. We will use an equivalent alternative formulation of approachability, due to Shelah.

Definition 7.15. Suppose κ is a singular cardinal of countable cofinality, and let $d : [\kappa^+]^2 \to \omega$.

(1) d is *normal* if, for all $\beta < \kappa^+$ and all $n < \omega$, $|\{\alpha < \beta \mid d(\alpha, \beta) \leq n\}| < \kappa$.
(2) d is *subadditive* if, for all $\alpha < \beta < \gamma < \kappa^+$, $d(\alpha, \gamma) \leq \max(d(\alpha, \beta), d(\beta, \gamma))$.
(3) $S_0(d)$ is the set of $\gamma < \kappa^+$ such that, for some unbounded sets $A, B \subseteq \gamma$, for every $\beta \in B$, there is $n_\beta < \omega$ such that, for all $\alpha \in A \cap \beta$, $d(\alpha, \beta) \leq n_\beta$.

Lemma 7.16. *(Shelah) Suppose κ is a singular, strong limit cardinal of countable cofinality.*

(1) *There is a normal, subadditive function $d : [\kappa^+]^2 \to \omega$.*
(2) *If d, $d' : [\kappa^+]^2 \to \omega$ are two normal, subadditive functions, then $S_0(d) \triangle S_0(d')$ is non-stationary.*
(3) *AP_κ is equivalent to the existence of a normal, subadditive $d : [\kappa^+]^2 \to \omega$ such that $S_0(d)$ contains a club.*

In V, fix a normal, subadditive $d : [\lambda_1] \to \omega$. Note that d remains normal and subadditive in $V^{\mathbb{P}_0 \times (\mathbb{P}_1 * \dot{\mathbb{S}} * \dot{\mathbb{Q}})}$. Let $(p_0, (p_1, \dot{s}, \dot{q})) \in \mathbb{P}_0 \times (\mathbb{P}_1 * \dot{\mathbb{S}} * \dot{\mathbb{Q}})$. Move to $V^{\mathbb{C}^1_{\ell(p_1)+1} * \dot{\mathbb{S}} * \dot{\mathbb{Q}} * \dot{\mathbb{T}}}$, requiring that, letting c be the C-part of p_1, $(c, \dot{s}, \dot{q}) \in G(\mathbb{C}^1_{\ell(p_1)+1} * \dot{\mathbb{S}} * \dot{\mathbb{Q}})$. $\kappa^1_{\ell(p_1)}$ remains supercompact in $V^{\mathbb{C}^1_{\ell(p_1)+1} * \dot{\mathbb{S}} * \dot{\mathbb{Q}} * \dot{\mathbb{T}}}$, and a standard application of supercompactness yields that, if $A = \{\alpha < \kappa^1_{\ell(p_1)} \mid S^{\lambda_1}_{\alpha^{+\omega+1}} \setminus S_0(d) \text{ is stationary}\}$, then $A \in U^1_{\ell(p)}$. Note that, since this is true in $V^{\mathbb{C}^1_{\ell(p_1)+1} * \dot{\mathbb{S}} * \dot{\mathbb{Q}} * \dot{\mathbb{T}}}$, it must be true in V as well. Moreover, by previous arguments, for any club D in λ_1 in $V^{\mathbb{P}_0 \times (\mathbb{P}_1 * \dot{\mathbb{S}} * \dot{\mathbb{Q}})}$, there must be a club $C \subseteq D$ in $V^{\mathbb{C}^1_{\ell(p_1)+1} * \dot{\mathbb{S}} * \dot{\mathbb{Q}} * \dot{\mathbb{T}}}$.

Putting this together, working in V, there are $U^1_{\ell(p_1)}$-many $\alpha < \lambda_1$ such that $(S^{\lambda_1}_{\alpha^{+\omega+1}} \setminus S_0(d))^V$ is stationary in $V^{\mathbb{P}_0 \times (\mathbb{P}_1 * \dot{\mathbb{S}} * \dot{\mathbb{Q}})}$. Find $\alpha \in A \cap A^{p_1}_{\ell(p_1)}$, and find an extension $q_1 \leq p_1$ such that $\ell(q_1) = \ell(p_1) + 1$ and $\alpha^{q_1}_{\ell(p_1)} = \alpha$. It suffices to show that, forcing below $(p_0, (q_1, \dot{s}, \dot{q}))$,

$$(S^{\lambda_1}_{\alpha^{+\omega+1}} \setminus S_0(d))^{V^{\mathbb{P}_0 \times (\mathbb{P}_1 * \dot{\mathbb{S}} * \dot{\mathbb{Q}})}} = (S^{\lambda_1}_{\alpha^{+\omega+1}} \setminus S_0(d))^V.$$

To this end, let $\beta \in (S^{\lambda_1}_{\alpha^{+\omega+1}} \cap S_0(d))^{V^{\mathbb{P}_0 \times (\mathbb{P}_1 * \dot{\mathbb{S}} * \dot{\mathbb{Q}})}}$. Let A, B be unbounded in β witnessing $\beta \in S_0(d)$. Since all cardinals in the interval $(\alpha, \alpha^{+\omega+2})$ are preserved by the forcing, Claim 7.5 yields unbounded $A' \subseteq A$ and $B' \subseteq B$ such that $A', B' \in V$. But then A', B' witness that $\beta \in S_0(d)$ in V. Thus,

$$(S^{\lambda_1}_{\alpha^{+\omega+1}} \setminus S_0(d))^{V^{\mathbb{P}_0 \times (\mathbb{P}_1 * \dot{\mathbb{S}} * \dot{\mathbb{Q}})}} = (S^{\lambda_1}_{\alpha^{+\omega+1}} \setminus S_0(d))^V,$$

so $AP_{\kappa^1_\omega}$ fails in $V^{\mathbb{P}_0 \times (\mathbb{P}_1 * \dot{\mathbb{S}} * \dot{\mathbb{Q}})}$. □

8. Questions

Many questions remain about the possible patterns of stationary reflection at the successor of a singular cardinal. We ask only a few of them here.

Question 8.1. *Is it consistent that* $\mathrm{bRefl}(\aleph_{\omega^2+1})$ *and* $\neg AP_{\aleph_{\omega^2}}$ *hold simultaneously?*

Question 8.2. *Is it consistent that* $\mathrm{Refl}(\aleph_{\omega^2+1})$ *holds and, for every stationary* $S \subseteq \aleph_{\omega^2+1}$*, there is a stationary* $T \subseteq S$ *that does not reflect at arbitrarily high cofinalities?*

Question 8.3. *Is it consistent that* $\mathrm{Refl}(\aleph_{\omega\cdot 2+1})$ *holds and there is a stationary* $S \subseteq S^{\aleph_{\omega\cdot 2+1}}_{\omega}$ *that does not reflect at any ordinal in* $S^{\aleph_{\omega\cdot 2+1}}_{<\aleph_\omega}$*?*

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