ALMOST KUREPA SUSLIN TREES AND DESTRUCTIBILITY OF THE GUESSING MODEL PROPERTY

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Abstract. Building on recent work of Krueger and the second author, we prove the consistency of the Guessing Model Principle at ω_2 together with the existence of an almost Kurepa Suslin tree. In particular, it is consistent that the Guessing Model Principle holds but is destructible by a ccc forcing of size ω_1. We also prove the consistency of the existence of a weak Kurepa tree together with the failure of the Kurepa Hypothesis and a certain guessing model principle that, for example, implies the tree property at ω_2.

Contents

2020 *Mathematics Subject Classification.* 03E05, 03E35, 03E55.

Key words and phrases. Kurepa trees, Suslin trees, guessing models, Mitchell forcing.

Both authors were supported by the Czech Academy of Sciences (RVO 67985840) and the GAČR project 23-04683S.

1. Introduction

Compactness principles play a central role in contemporary combinatorial set theory, particularly around questions concerning large cardinals, forcing, and canonical inner models. A paradigmatic example of a compactness principle at a cardinal κ is the *tree property* at κ, denoted $\mathsf{TP}(\kappa)$, which asserts the non-existence of κ-Aronszajn trees. Among inaccessible cardinals, compactness principles are frequently used to characterize large cardinals. For example, an inaccessible cardinal κ is weakly compact if and only if $\mathsf{TP}(\kappa)$ holds. When they hold at large cardinals, compactness principles are typically quite robust under forcing extensions. For example, if κ is weakly compact and $\mathbb{P}$ is any forcing notion that has the μ-cc for some $\mu < \kappa$, then $\mathsf{TP}(\kappa)$ continues to hold in the forcing extension by $\mathbb{P}$.

However, compactness principles used to characterize large cardinals can also consistently hold at accessible cardinals, such as ω_2, and here questions of their robustness are much less clear. For example, by results of Mitchell and Silver (cf. [19]), $\mathsf{TP}(\omega_2)$ is equiconsistent with the existence of a weakly compact cardinal. Very few definitive answers are known, however, about the (consistent) (in)destructibility of $\mathsf{TP}(\omega_2)$ under ccc forcing extensions. For example, it is unknown whether it is consistent that $\mathsf{TP}(\omega_2)$ holds and is preserved in all ccc forcing extensions, but it is also unknown whether it is consistent that $\mathsf{TP}(\omega_2)$ holds and there exists a ccc forcing poset $\mathbb{P}$ forcing the failure of $\mathsf{TP}(\omega_2)$. For the best partial results on these questions, see [24] and [9].

In this paper, we focus on an important strengthening of the tree property known as the *guessing model property* (denoted $\mathsf{GMP}(\kappa)$). Just as $\mathsf{TP}(\kappa)$ characterizes weakly compact cardinals among inaccessible cardinals, $\mathsf{GMP}(\kappa)$ characterizes supercompact cardinals among inaccessible cardinals. Also, just as $\mathsf{TP}(\kappa)$ at a weakly compact cardinal is robust under forcings with good chain condition, $\mathsf{GMP}(\kappa)$ at a supercompact cardinal κ continues to hold after forcing with any forcing poset that is μ-cc for some $\mu < \kappa$. Here we show that this robustness can consistently fail at ω_2; in particular, letting the simple GMP denote $\mathsf{GMP}(\omega_2)$, we show that it is consistent that GMP holds but there exists a ccc forcing of size ω_1 forcing the failure of GMP.

We achieve this task by showing that GMP is compatible with the existence of a particular combinatorial object of interest in its own right: an almost Kurepa Suslin tree, i.e., a Suslin tree T such that after forcing with T, it becomes a Kurepa tree (see Section 2 for a precise definition). Since GMP implies that there are no Kurepa trees, it cannot hold after forcing with T.

Almost Kurepa Suslin trees were first constructed by Jensen; see [6] for a sketch of the proof that $\diamondsuit^+$ implies their existence. Later, Bilaniuk [3] constructed an almost Kurepa Suslin tree from the assumption that $\diamondsuit$ holds and there exists a Kurepa tree. In both of these constructions, the resulting Suslin tree is almost Kurepa because there are at least ω_2-many automorphisms of the tree which are pairwise almost disjoint. By applying these automorphisms to a generically added cofinal branch, this consequently ensures that the tree will have at least ω_2-many cofinal branches in the generic extension by itself. In the proof of our theorem, the property of being almost Kurepa is also ensured by the existence of ω_2-many pairwise almost disjoint automorphisms of the Suslin tree.

Theorem A. *Suppose that there exists a supercompact cardinal. Then there exists a forcing extension in which*

(1) there exists an almost Kurepa Suslin tree;
(2) GMP *holds.*

In particular, it is consistent that GMP *holds and yet is destructible by a ccc forcing of cardinality* ω_1.

GMP implies the failure of the weak Kurepa hypothesis (i.e., there are no weak Kurepa trees, denoted $\neg$wKH); hence, Theorem A provides a model where there are no weak Kurepa trees and there is an almost Kurepa Suslin tree. However, to achieve $\neg$wKH, the assumption of an inaccessible cardinal is sufficient. The following is an immediate corollary of the proof of Theorem A:

Corollary 1.1. *Suppose that there exists an inaccessible cardinal. Then there exists a forcing extension in which*

(1) there exists an almost Kurepa Suslin tree;
(2) $\neg$wKH *holds.*

In particular, it is consistent that $\neg$wKH *holds and yet is destructible by a ccc forcing of cardinality* ω_1.

Let us mention several results from the literature to put Theorem A in context.

- In [8], Honzik and the authors prove that GMP is always indestructible under adding any number of Cohen reals. This shows that Theorem A is sharp in the sense that a countable forcing poset can never destroy GMP.
- Relative to the consistency of a supercompact cardinal, it is consistent that GMP holds and is indestructible under all ccc forcings of cardinality ω_1. In particular, this holds in the extension by the classical Mitchell forcing $\mathbb{M}(\omega, \kappa)$, where κ is supercompact in the ground model. For an argument of this fact, see [24, Theorem 2]; the theorem proven there is about the tree property rather than GMP, and thus only requires a weakly compact cardinal rather than a supercompact cardinal, but the proof of the stronger fact is essentially the same. This fact is additionally relevant to our main result due to our method of proof: our forcing extension yielding Theorem A utilizes a variation on the classical Mitchell forcing in which the forcing iterands adding Cohen subsets to ω_1 are replaced by iterands adding automorphisms to a distinguished free Suslin tree. This provides an illustration of the versatility of Mitchell-type forcings, and we expect that a number of further consistency results can be established by appropriately varying the iterands in Mitchell-type constructions.
- In [13], Krueger and the second author show that, assuming the consistency of an inaccessible cardinal, it is consistent with CH that the Kurepa Hypothesis fails and yet there exists an almost Kurepa Suslin tree, in the process answering long-open questions of Jin and Shelah [12] and of Moore [20]. Since GMP entails the failure of the weak Kurepa Hypothesis, and hence of the Kurepa Hypothesis, Theorem A and Corollary 1.1 can be seen as a strengthening of this result, at the cost of necessarily forgoing CH, which is incompatible with $\neg$wKH. Our proof of Theorem A is in large part an adaptation of the techniques developed in [13] to the setting of Mitchell-type forcing extensions.
- The notion of a Suslin tree being almost Kurepa is closely connected to the non-saturation of Aronszajn trees. By an observation of Moore, if S

is an almost Kurepa Suslin tree, then $S \otimes S$ is a non-saturated Aronszajn tree; for details, see [13]. Therefore, in the extension from Theorem A, GMP holds and there is a non-saturated Aronszajn tree. In [14], Krueger and the second author establish a related result: under the assumption of the existence of a supercompact cardinal, there is a forcing $\mathbb{P}$ such that in the generic extension by $\mathbb{P}$, GMP holds and there exists a strongly non-saturated Aronszajn tree. Note that in their model, the Aronszajn tree is *strongly* non-saturated[1], but the forcing $\mathbb{P}$ is designed to add an Aronszajn tree which is strongly non-saturated rather than an almost Kurepa Suslin tree, and it is unclear whether an almost Kurepa Suslin tree exists in their model. On the other hand, the product of the almost Kurepa Suslin tree from Theorem A with itself is only non-saturated and not strongly non-saturated.

In the second main theorem of the paper, we separate the weak Kurepa Hypothesis from the Kurepa Hypothesis in the presence of a weakening of GMP (which is still strong enough to imply $\mathsf{TP}(\omega_2)$), refining a previous result of the authors from [16]:

Theorem B. *Suppose that there is a supercompact cardinal κ. Then there is a forcing extension in which*

(1) $2^\omega = \omega_2 = \kappa$;
(2) $\mathsf{GMP}^{\omega_2}(\omega_2)$ *holds;*
(3) there are no Kurepa tree, but there is a weak Kurepa tree.

In contrast to Theorem A, the proof of this theorem makes use of the classical Mitchell forcing, together with a natural forcing to add a weak Kurepa tree.

The structure of the remainder of the paper is as follows. In Section 2, we present some relevant background information concerning trees, guessing models, Mitchell forcing, and automorphisms. In Section 3, we prove Theorem A and Corollary 1.1, and in Section 4, we prove Theorem B. In Section 5, we fill some gaps in the literature by surveying known results surrounding the consistent (in)destructibility of failures of the (weak) Kurepa Hypothesis. We end the section by providing a direct proof that the failure of the weak Kurepa Hypothesis is always preserved by σ-centered forcings. In Section 6, we present a few remaining open questions. Finally, Appendix A provides an example, promised in Remark 3.1 below, of a two step iteration $\mathrm{Add}(\omega, 1) * \dot{\mathbb{Q}}$ such that $\dot{\mathbb{Q}}$ is forced to be totally proper but for which forcing with the associated term forcing over V collapses ω_1.

1.1. **Notation and conventions.**

We let On denote the class of all ordinals. Given $\delta \in \mathrm{On}$, let $\Sigma(\delta)$ denote the set of successor ordinals less than δ.

If $\vec{x}$ is a sequence of length β, then by convention, unless explicitly specified otherwise, for all $\alpha < \beta$, the α^{th} element of $\vec{x}$ will be denoted by x_α.

If κ and λ are cardinals, with κ regular and infinite, then $\mathrm{Add}(\kappa, \lambda)$ is the forcing to add λ-many Cohen subsets to κ. Concretely, conditions in $\mathrm{Add}(\kappa, \lambda)$ are partial functions of cardinality less than κ from λ to ${}^{<\kappa}2$. If $p, q \in \mathrm{Add}(\kappa, \lambda)$, then $q \leq p$ iff $\mathrm{dom}(q) \supseteq \mathrm{dom}(p)$ and $q(\gamma) \supseteq p(\gamma)$ for all $\gamma \in \mathrm{dom}(p)$.

[1] For the definition, see [14].

2. Preliminaries

In this section, we review some background information on trees, guessing models, Mitchell forcing, and tree automorphisms.

2.1. Trees and the guessing model property.

Definition 2.1. A *tree* is a partial order $(T, <_T)$ such that, for all $t \in T$, the set $\mathrm{pred}_T(t) = \{s \in T \mid s <_T t\}$ is well-ordered by $<_T$. If T is a tree and $t \in T$, then we let $\mathrm{ht}_T(t)$ denote the order type of $(\mathrm{pred}_T(t), <_T)$. For all $\alpha \in \mathrm{On}$, we let T_α denote $\{t \in T \mid \mathrm{ht}_T(t) = \alpha\}$. Expressions such as $T_{<\alpha}$ or $T_{\leq\alpha}$ are defined in the obvious way. We let $\mathrm{ht}(T) = \min\{\alpha \in \mathrm{On} \mid T_\alpha = \emptyset\}$. $\mathrm{ht}(T)$ is often referred to as the *height* of T.

Given a regular uncountable cardinal κ, a tree T is called a *κ-tree* if $\mathrm{ht}(T) = \kappa$ and $|T_\alpha| < \kappa$ for all $\alpha < \kappa$. A κ-tree T is *normal* if it satisfies the following two conditions:

- for all $\alpha < \beta < \kappa$ and all $t \in T_\alpha$, there exists $s \in T_\beta$ such that $t <_T s$;
- for every limit ordinal $\alpha < \kappa$ and all $s, t \in T_\alpha$, if $\mathrm{pred}_T(s) = \mathrm{pred}_T(t)$, then $s = t$.

If T is a tree and $t \in T$, then we let $\mathrm{succ}_T(t)$ denote the set of immediate successors of t in T, i.e., the set $\{s \in T \mid t <_T s \text{ and } \mathrm{ht}_T(s) = \mathrm{ht}_T(t) + 1\}$. We say that T is *infinitely splitting* if $\mathrm{succ}_T(t)$ is infinite for all $t \in T$.

A *branch* through a tree T is a subset $b \subseteq T$ that is $<_T$-downward closed and linearly ordered by $<_T$. We say that a branch $b \subseteq T$ is a *cofinal* branch if, for all $\alpha < \mathrm{ht}(T)$, we have $b \cap T_\alpha \neq \emptyset$. A subset $A \subseteq T$ is called an *antichain* if elements of A are pairwise incomparable under $<_T$. We say that a subset $E \subset T$ is *dense* if, for all $t \in T$, there is $s \in E$ with $t \leq_T s$, and *open* if, for all $t \in E$ and all $s \in T$ with $t \leq_T s$, we have $s \in E$.

We will mostly be interested in ω_1-trees in this paper, so much of what follows will be in the specific context of ω_1-trees, though it will be clear that essentially everything can be generalized to the setting of κ-trees for other regular uncountable cardinals κ. Given a finite sequence $\langle T(i) \mid i < n\rangle$ of ω_1-trees, let $T(0) \otimes T(1) \otimes \cdots \otimes T(n-1)$ denote the tree T^* defined as follows. The underlying set of T^* is

$$\bigcup_{\alpha<\omega_1} \prod_{i<n} T(i)_\alpha.$$

If $\vec{x}, \vec{y} \in T^*$, then we set $\vec{x} \leq_{T^*} \vec{y}$ if and only if $x_i \leq_{T(i)} y_i$ for all $i < n$. Note that, for all $\alpha < \omega_1$, we have $T^*_\alpha = \prod_{i<n} T(i)_\alpha$.

If T is an ω_1-tree and $x \in T$, then we let T_x denote the tree $\{y \in T \mid x \leq_T y\}$, with the induced order from T. As long as T is normal, it follows that T_x is an ω_1-tree for all $x \in T$. We note that there is a slight abuse of notation here, given our convention that T_α denotes the α^{th} level of the tree T, but in practice there will never be any risk of confusion: we will always use Roman letter such as "x" to denote elements of a tree and Greek letters such as "α" to denote levels of the tree. If $\alpha < \omega_1$ and $\vec{x}$ is a finite sequence from T_α of length $n < \omega$, then we let

$$T_{\vec{x}} := T_{x_0} \otimes T_{x_1} \otimes \cdots \otimes T_{x_{n-1}}.$$

Such a tree $T_{\vec{x}}$ is called a *derived tree of* T. Given $\vec{z} \in T_{\vec{x}}$, we will let $\mathrm{ht}_T(\vec{z})$ denote the unique $\beta < \omega_1$ such that $z_i \in T_\beta$ for all $i < n$. Recall that a *Suslin tree* is an ω_1-tree T with no uncountable branches and no uncountable antichains. It is routine to prove that an (infinitely) splitting ω_1-tree T is Suslin if and only if, for every dense, open subset $E \subseteq T$, there is $\beta < \omega_1$ such that $T_\beta \subseteq E$.

We say that a Suslin tree T is a *free Suslin tree* if, for all $\alpha < \omega_1$ and all finite injective sequences $\vec{x}$ from T_α, the tree $T_{\vec{x}}$ is Suslin. A *Kurepa tree* is an ω_1-tree T with at least ω_2-many cofinal branches. We say that a Suslin tree T is *almost Kurepa* if after forcing with T, T is a Kurepa tree. A *weak Kurepa tree* is a tree T of height and size ω_1 with at least ω_2-many cofinal branches. The *Kurepa Hypothesis* (KH) is the assertion that there exists a Kurepa tree. The *weak Kurepa Hypothesis* (wKH) is the assertion that there exists a weak Kurepa tree.

Given a regular uncountable cardinal κ, we say that the *tree property holds* at κ (denoted $\mathsf{TP}(\kappa)$), if every κ-tree has a cofinal branch. The tree property can be used to characterize weakly compact cardinals: an inaccessible cardinal κ is weakly compact if and only if $\mathsf{TP}(\kappa)$ holds, and, by work of Mitchell and Silver [19], $\mathsf{TP}(\omega_2)$ is equiconsistent with the existence of a weakly compact cardinal. Beginning in the 1970s with work of Jech [10] and Magidor [17], and continuing in the 2000s with work of Weiß [26], two-cardinal analogues of the tree property have been shown to characterize strongly compact and supercompact cardinals in a similar manner.

Let κ be a regular uncountable cardinal, and let X be a set with $|X| \geq \kappa$. We say that a sequence $\langle d_x \mid x \in \mathscr{P}_\kappa X\rangle$ is a (κ, X)-list if $d_x \subseteq x$ for all $x \in \mathscr{P}_\kappa X$.

Definition 2.2. Assume that $D = \langle d_x \mid x \in \mathscr{P}_\kappa X\rangle$ is a (κ, X)-list.

(1) The *width* of D is the least cardinal λ such that $|d_x| < \lambda$ for all $x \in \mathscr{P}_\kappa X$.
(2) We say that D is *thin* if there is a closed unbounded set $C \subseteq \mathscr{P}_\kappa X$ such that $|\{d_x \cap y \mid y \subseteq x \in \mathscr{P}_\kappa X\}| < \kappa$ for every $y \in C$.
(3) Let $\mu \leq \kappa$ be an uncountable cardinal. We say that D is *μ-slender* if for all sufficiently large θ there is a club $C \subseteq \mathscr{P}_\kappa H(\theta)$ such that for all $M \in C$ and all $y \in M \cap \mathscr{P}_\mu X$, we have $d_{M\cap X} \cap y \in M$.

Definition 2.3. Assume that $D = \langle d_x \mid x \in \mathscr{P}_\kappa X\rangle$ is a (κ, X)-list and $d \subseteq X$.

(1) We say that d is a *cofinal branch* of D if for all $x \in \mathscr{P}_\kappa X$ there is $z_x \supseteq x$ such that $d \cap x = d_{z_x} \cap x$.
(2) We say that d is an *ineffable branch* of D if the set $\{x \in \mathscr{P}_\kappa X \mid d \cap x = d_x\}$ is stationary.

Definition 2.4. Assume that $\mu \leq \kappa$ is regular. We say that

(1) the (κ, X)-tree property holds, denoted $\mathsf{TP}(\kappa, X)$, if every thin (κ, X)-list has a cofinal branch.
(2) the (κ, X)-ineffable tree property holds, denoted $\mathsf{ITP}(\kappa, X)$, if every thin (κ, X)-list has an ineffable branch.
(3) the (μ, κ, X)-slender tree property holds, denoted $\mathsf{SP}^\mu(\kappa, X)$, if every μ-slender (κ, X)-list has a cofinal branch.
(4) the (μ, κ, X)-ineffable slender tree property holds, denoted $\mathsf{ISP}^\mu(\kappa, X)$, if every μ-slender (κ, X)-list has an ineffable branch.

In this paper, we will be interested only in the ineffable slender tree property; for more about two-cardinal tree properties see [26]. We introduce a couple of conventions. We will use notations such as $\mathsf{ISP}^\mu(\kappa)$ to assert that $\mathsf{ISP}^\mu(\kappa, \lambda)$ holds for all $\lambda \geq \kappa$. We let ISP denote $\mathsf{ISP}^{\omega_1}(\omega_2)$.

By [17], it follows that an inaccessible cardinal κ is supercompact if and only if $\mathsf{ISP}^{\omega_1}(\kappa)$ holds. Weiß proves in [26] that, if one forces with the standard Mitchell forcing $\mathbb{M}(\omega, \kappa)$, where κ is supercompact, then ISP holds in the extension; similarly, Viale and Weiß prove in [25] that the Proper Forcing Axiom (PFA) implies ISP. It is not known whether, in analogy with the tree property, ISP implies the consistency of the existence of a supercompact cardinal; some partial results in this direction can be found in [25].

In [25], Viale and Weiß reformulate ineffable slender tree properties in terms of the existence of combinatorial objects known as *guessing models*, which we now recall.

Definition 2.5. Let $\theta \geq \omega_2$ be a regular cardinal, and let $M \prec H(\theta)$ be an elementary submodel.

(1) Given a set $x \in M$, a subset $d \subseteq x$, and an uncountable cardinal μ, we say that
   (a) d is (μ, M)-*approximated* if, for every $z \in M \cap \mathscr{P}_\mu(M)$, we have $d \cap z \in M$;
   (b) d is M-*guessed* if there is $e \in M$ such that $d \cap M = e \cap M$.
(2) M is a μ-*guessing model for* x if every (μ, M)-approximated subset of x is M-guessed.
(3) M is a μ-*guessing model* if, for every $x \in M$, it is a μ-guessing model for x.

Given uncountable cardinals $\mu \leq \kappa \leq \theta$, with κ and θ regular, we let $\mathsf{GMP}^\mu(\kappa, \theta)$ denote the assertion that the set of $M \in \mathscr{P}_\kappa(H(\theta))$ such that M is a μ-guessing model is stationary in $\mathscr{P}_\kappa(H(\theta))$. Moreover, we let $\mathsf{GMP}^\mu(\kappa)$ denote the assertion that $\mathsf{GMP}^\mu(\kappa, \theta)$ holds for every regular cardinal $\theta \geq \kappa$. We let $\mathsf{GMP}(\kappa)$ denote $\mathsf{GMP}^{\omega_1}(\kappa)$ and let GMP denote $\mathsf{GMP}(\omega_2)$.

Viale and Weiß prove in [25] that the ineffable slender tree property is equivalent to the guessing model property. More precisely, they proved the following:

Fact 2.6. *Suppose that $\mu \leq \kappa$ are regular uncountable cardinals. Then the following are equivalent:*

(1) $\mathsf{ISP}^\mu(\kappa)$;
(2) $\mathsf{GMP}^\mu(\kappa)$.

In particular ISP *is equivalent to* GMP

2.2. **Preservation lemmas.** In this short subsection we recall some useful preservation lemmas regarding product forcings and adding cofinal branches through trees. The following fact is due to Baumgartner (see [2]).

Fact 2.7. *Let κ be a regular cardinal and assume that $\mathbb{P}$ is a κ-Knaster forcing notion. If T is a tree of height κ, then forcing with $\mathbb{P}$ does not add new cofinal branches to T.*

The following lemma appeared in [7].

Fact 2.8. (Easton) *Let $\kappa > \omega$ be a regular cardinal and assume that $\mathbb{P}$ and $\mathbb{Q}$ are forcing notions, where $\mathbb{P}$ is κ-cc and $\mathbb{Q}$ is κ-closed. Then the following hold:*

(i) $\mathbb{P}$ forces that $\mathbb{Q}$ is κ-distributive.
(ii) $\mathbb{Q}$ forces that $\mathbb{P}$ is κ-cc.

The following fact can be found in [5]. The argument is attributed to Magidor.

Fact 2.9. *Let $\kappa > \omega$ be a regular cardinal and assume that $\mathbb{P}$ and $\mathbb{Q}$ are forcing notions such that $\mathbb{P}$ is κ-Knaster and $\mathbb{Q}$ is κ-cc. Then $\mathbb{Q}$ forces that $\mathbb{P}$ is κ-Knaster.*

The following fact is due to Silver (see [1] for more details).

Fact 2.10. *Let κ, λ be regular cardinals with $2^\kappa \geq \lambda$. Assume that $\mathbb{P}$ is a κ^+-closed forcing notion. If T is a λ-tree, then forcing with $\mathbb{P}$ does not add new cofinal branches to T.*

These fact can be generalized as follows [24].

Fact 2.11. *Let $\kappa < \lambda$ be regular cardinals and $2^\kappa \geq \lambda$. Assume that $\mathbb{P}$ and $\mathbb{Q}$ are forcing notions such that $\mathbb{P}$ is κ^+-cc and $\mathbb{Q}$ is κ^+-closed. If T is a λ-tree in $V[\mathbb{P}]$, then forcing with $\mathbb{Q}$ over $V[\mathbb{P}]$ does not add cofinal branches to T.*

In [16], we prove an analogue of the previous lemma for more general trees, and as a corollary we obtain the following fact:

Fact 2.12. *Let ξ be a cardinal and $\mu < \kappa \leq \lambda$ be regular cardinals such that $2^\mu \geq \xi$ and $2^{<\mu} < \kappa$. Let $\mathbb{Q}$ be a μ^+-closed forcing notion and $\mathbb{P}$ be a μ^+-cc forcing notion. If T is a $\mathscr{P}_\kappa\lambda$-tree with width at most ξ in $V[\mathbb{P}]$, then forcing with $\mathbb{Q}$ over $V[\mathbb{P}]$ does not add a cofinal branch through T.*

2.3. **Mitchell forcing.** Let κ be an ordinal with $\mathrm{cf}(\kappa) > \omega$. In practice, κ will typically be (at least) an inaccessible cardinal, but we will sometimes need to consider κ of the form $j(\lambda)$, where $j : V \to M$ is an elementary embedding with critical point λ, in which case κ is inaccessible in M but may not even be a cardinal in V.

For this subsection, let $\mathbb{P}$ denote the forcing $\mathrm{Add}(\omega, \kappa)$ for adding κ-many Cohen subsets to ω. For $\gamma < \kappa$, let $\mathbb{P}_\gamma$ denote the suborder $\mathrm{Add}(\omega, \gamma)$. We can now define the version of the Mitchell forcing that we will use, $\mathbb{M} = \mathbb{M}(\omega, \kappa)$, as follows. First, let A be some unbounded subset of κ with $\min(A) > \omega$. Recall that $\mathrm{acc}(A)$, the set of *accumulation points* of A, is defined to be $\{\alpha \in A \mid \sup(A \cap \alpha) = \alpha\}$, and $\mathrm{nacc}(A) := A \setminus \mathrm{acc}(A)$. Also, $\mathrm{acc}^+(A) := \{\alpha < \sup(A) \mid \sup(A \cap \alpha) = \alpha\}$. Though our definition of $\mathbb{M}$ will depend on our choice of A, we suppress mention of A in the notation; if we need to make the choice of A explicit, we will speak of "the Mitchell forcing $\mathbb{M}(\omega, \kappa)$ defined using the set A". Unless otherwise specified, we will always let A be the set of all inaccessible cardinals in the interval (ω, κ), but the definition of $\mathbb{M}$ and the properties listed below work for any choice of A. Conditions in $\mathbb{M}$ are all pairs (p, q) such that

- $p \in \mathbb{P}$;
- q is a function and $\mathrm{dom}(q) \in [\mathrm{nacc}(A)]^{\leq\omega}$;
- for all $\alpha \in \mathrm{dom}(q)$, $p \restriction \alpha \Vdash_{\mathbb{P}_\alpha}$ "$q(\alpha) \in \mathrm{Add}(\omega_1, 1)^{V^{\mathbb{P}_\alpha}}$".

If $(p_0, q_0), (p_1, q_1) \in \mathbb{M}$, then $(p_1, q_1) \leq_{\mathbb{M}} (p_0, q_0)$ if and only if

- $p_1 \leq_{\mathbb{P}} p_0$;
- $\mathrm{dom}(q_1) \supseteq \mathrm{dom}(q_0)$;
- for all $\alpha \in \mathrm{dom}(q_0)$, $p_1 \restriction \alpha \Vdash_{\mathbb{P}_\alpha}$ "$q_1(\alpha) \leq q_0(\alpha)$".

For $\delta < \kappa$, we let $\mathbb{M}_\delta$ denote the suborder of $\mathbb{M}$ consisting of all conditions (p, q) such that the domains of both p and q are contained in δ.

Remark 2.13. The following are some of the key properties of $\mathbb{M}$ (cf. [1], [19] for further details and proofs):

(1) If κ is inaccessible, then $\mathbb{M}$ is κ-Knaster.
(2) $\mathbb{M}$ has the ω_1-covering and ω_1-approximation properties. Together with the previous item, this implies that, if κ is inaccessible, then forcing with $\mathbb{M}$ preserves ω_1 and all cardinals greater than or equal to κ.

(3) If κ is inaccessible, then $\Vdash_{\mathbb{M}}$ "$2^\omega = \kappa = \omega_2$".
(4) There is a projection onto $\mathbb{M}$ from a forcing of the form $\mathrm{Add}(\omega, \kappa) \times \mathbb{Q}$, where $\mathbb{Q}$ is ω_1-closed (here $\mathbb{Q}$ is what is often called the *term forcing* associated with $\mathbb{M}$; it can be thought of as the set of all pairs $(\emptyset, q)$ from $\mathbb{M}$, with the order inherited from $\mathbb{M}$).
(5) For all inaccessible cardinals $\delta \in \mathrm{acc}^+(A)$, there is a projection from $\mathbb{M}$ to $\mathbb{M}_\delta$ and, in $V^{\mathbb{M}_\delta}$, the quotient forcing $\mathbb{M}/\mathbb{M}_\delta$ has the ω_1-approximation property.
(6) For all inaccessible cardinals $\delta \in \mathrm{acc}^+(A)$, let $\delta^\dagger$ denote $\min(A \setminus \delta{+}1)$. Then, in $V^{\mathbb{M}_\delta}$, the quotient forcing $\mathbb{M}/\mathbb{M}_\delta$ is of the form $\mathrm{Add}(\omega, \delta^\dagger - \delta) * \dot{\mathbb{M}}^\delta$, where, in $V^{\mathbb{M} * \mathrm{Add}(\omega, \delta^\dagger - \delta)}$, there is a projection onto $\mathbb{M}^\delta$ from a forcing of the form $\mathrm{Add}(\omega, \kappa - \delta^\dagger) \times \mathbb{Q}^*_\delta$, where $\mathbb{Q}^*_\delta$ is ω_1-closed.

2.4. **Automorphisms of trees.** If T is a tree, then an *automorphism* of T is a bijective map $g : T \to T$ such that g and g^{-1} both preserve the order $<_T$. Note that an automorphism must be level-preserving, i.e., for all $\alpha < \mathrm{ht}(T)$, $g[T_\alpha] = T_\alpha$. We let $\mathrm{Aut}(T)$ denote the set of all automorphisms of T.

Fix for now a normal, infinitely splitting ω_1-tree T. If $\alpha < \beta < \omega_1$ and $x \in T_\beta$, then $x \restriction \alpha$ denotes the unique predecessor of x in T_α. If $X \subseteq T_\beta$, then $X \restriction \alpha$ denotes $\{x \restriction \alpha \mid x \in X\}$. We say that X has *unique drop-downs to* α if, for all distinct $x, y \in X$, we have $x \restriction \alpha \neq y \restriction \alpha$. Notice that, if $\alpha < \omega_1$, then an element of $\mathrm{Aut}(T_{\leq \alpha})$ is uniquely determined by its values on T_α.

Using the normality of T, the following proposition is immediate.

Proposition 2.14. *Suppose that $\beta < \omega_1$ is a limit ordinal and $g \in \mathrm{Aut}(T_{<\beta})$. Then there exists at most one $g' \in \mathrm{Aut}(T_{\leq \beta})$ such that $g' \supseteq g$, and the following are equivalent:*

(1) there exists $g' \in \mathrm{Aut}(T_{\leq \beta})$ such that $g' \supseteq g$;
(2) for all $x \in T_\beta$ and $j \in \{-1, 1\}$, there exists $y \in T_\beta$ such that, for all $\alpha < \beta$, we have $g^j(x \restriction \alpha) = y \restriction \alpha$.

The straightforward proof of the following proposition, showing that we have some control over extensions of automorphisms of initial segments of T, is left to the reader.

Proposition 2.15. *Suppose that $\alpha < \omega_1$ and $g \in \mathrm{Aut}(T_{\leq \alpha})$. Suppose moreover that*

- *$X \subseteq T_{\alpha+1}$ is finite;*
- *for all $x \in X$ and $j \in \{-1, 1\}$, we are given $z^j_x \in T_{\alpha+1}$ in such a way that*
  - *$g^j(x \restriction \alpha) = z^j_x \restriction \alpha$;*
  - *if x_0 and x_1 are distinct elements of X and $j \in \{-1, 1\}$, then $z^j_{x_0} \neq z^j_{x_1}$;*
  - *if $x_0 \in X$, $j \in \{-1, 1\}$, and $z^j_{x_0} = x_1 \in X$, then $z^{1-j}_{x_1} = x_0$.*

Then there exists $h \in \mathrm{Aut}(T_{\leq \alpha+1})$ such that

- *$h \supseteq g$; and*
- *for all $(x, j) \in X \times \{-1, 1\}$, we have $h^j(x) = z^j_x$.*

We now recall some notions and basic results from [13]. Some definitions and lemmas are given descriptive names for ease of later referral; these names also come from [13].

Definition 2.16 (Consistency)**.** Suppose that $\alpha < \alpha' \leq \beta < \omega_1$ and $g \in \mathrm{Aut}(T_{\leq\beta})$. Suppose moreover that $X \subseteq T_{\alpha'}$ has unique drop-downs to α. We say that $X \restriction \alpha$ and X are *g-consistent* if, for all $x, y \in X$, we have $g(x) = y$ iff $g(x \restriction \alpha) = g(y \restriction \alpha)$. Similarly, if $n < \omega$ and $\vec{x}$ is an injective n-tuple from $T_{\alpha'}$, we say that $\vec{x} \restriction \alpha$ and $\vec{x}$ are g-consistent if, for all $i, j < n$, we have $g(x_i) = x_j$ iff $g(x_i \restriction \alpha) = x_j \restriction \alpha$.

Proposition 2.17. *Suppose that $\alpha < \omega_1$, $g \in \mathrm{Aut}(T_{\leq\alpha})$, $X \subseteq T_{\alpha+1}$ is a finite set with unique drop-downs to α, and $Y \subseteq T_{\alpha+1}$ is a finite set disjoint from X. Then there is $g' \in \mathrm{Aut}(T_{\leq\alpha+1})$ such that*

- *$g' \supseteq g$;*
- *$X \restriction \alpha$ and X are g'-consistent;*
- *$g'[Y] \cap (X \cup Y) = \emptyset$.*

Proof. For each $x \in T_\alpha$, let $\langle x_n \mid n < \omega \rangle$ be an injective enumeration of $\mathrm{succ}_T(x)$. Let $X_0 := \{x \in T_\alpha \mid X \cap \mathrm{succ}_T(x)\} \neq \emptyset$. Note that, since X has unique drop-downs to α, for each $x \in X_0$, there is a unique $n < \omega$ such that $x_n \in X$; denote this n by $n_X(x)$. Similarly, let $Y_0 = \{x \in T_\alpha \mid Y \cap \mathrm{succ}_T(x)\} \neq \emptyset$ and, for all $x \in Y_0$, let $A_Y(x) = \{n < \omega \mid x_n \in Y\}$. Then each $A_Y(x)$ is a finite subset of ω and does not contain $n_X(x)$ if the latter is defined. If $x \notin Y_0$, let $A_Y(x) = \emptyset$.

Now, for each $x \in T_\alpha$, fix a permutation π_x of ω with the following properties:

- if $x, g(x) \in X_0$, then $\pi_x(n_X(x)) = n_X(g(x))$;
- if $x \in Y_0$, then $\pi_x[A_Y(x)]$ is disjoint from $A_Y(g(x))$ and does not contain $n_X(g(x))$ if the latter is defined.

Now define $g' \in \mathrm{Aut}(T_{\leq\alpha+1})$ by setting $g'(x_n) = g(x)_{\pi_x(n)}$ for all $(x, n) \in T_\alpha \times \omega$. It is easily verified that g' is as desired. □

Definition 2.18 (Separation)**.** Suppose that $\alpha \leq \beta < \omega_1$, $\mathcal{G} = \{g_\tau \mid \tau \in I\}$ is an indexed family from $\mathrm{Aut}(T_{\leq\beta})$, $n < \omega$ and $\vec{x}$ is an injective n-tuple from T_α. We say that $\mathcal{G}$ is *separated on $\vec{x}$* if, for all $m < n$, the following statements hold:

(1) for all $\tau \in I$, $g_\tau(x_m) \neq x_m$;
(2) there exists at most one triple (k, j, τ) such that $k < m$, $j \in \{-1, 1\}$, $\tau \in I$, and $g_\tau^j(x_m) = x_k$.

If $X \subseteq T_\alpha$ is finite, then we say that $\mathcal{G}$ is *separated on X* if there is some injective enumeration $\vec{x}$ of X such that $\mathcal{G}$ is separated on $\vec{x}$. We say that $\mathcal{G}$ is *separated* if it is separated on X for every finite $X \subseteq T_\beta$.

The next five technical results are proved in [13]; we direct the reader there for the proofs.

Proposition 2.19 (Persistence)**.** [13, Lemma 5.7] *Suppose that $\alpha < \beta < \omega_1$, $\mathcal{G}$ is an indexed family from $\mathrm{Aut}(T_{\leq\beta})$, and $X \subseteq T_\beta$ is a finite set with unique drop-downs to α. If $\mathcal{G}$ is separated on $X \restriction \alpha$, then $\mathcal{G}$ is separated on X.*

Lemma 2.20 (Key Property)**.** [13, Proposition 5.11] *Suppose that $\alpha < \beta < \omega_1$, $n < \omega$, $\vec{x}$ is an injective n-tuple from T_α, and $\mathcal{G} = \{g_\tau \mid \tau \in I\}$ is a finite indexed set from $\mathrm{Aut}(T_{\leq\beta})$ that is separated on $\vec{x}$. Let $t \subseteq T_\beta$ be finite. Then there is an n-tuple $\vec{y}$ from $T_\beta \setminus t$ such that*

- *for all $i < n$, we have $x_i <_T y_i$;*
- *for all $\tau \in I$, $\vec{x}$ and $\vec{y}$ are g_τ-consistent.*

Lemma 2.21 (1-Key Property)**.** [13, Proposition 5.12] *Suppose that $\alpha < \beta < \omega_1$, $n < \omega$, $\vec{x}$ is an injective n-tuple from T_α, and $\mathcal{G} = \{g_\tau \mid \tau \in I\}$ is a finite indexed set from $\mathrm{Aut}(T_{\leq\beta})$ that is separated on $\vec{x}$. Let $m < n$, and fix $y^* \in T_\beta$ with $x_m <_T y^*$. Then there is an n-tuple $\vec{y}$ from T_β such that*

- *$y_m = y^*$;*
- *for all $i < n$, we have $x_i <_T y_i$;*
- *for all $\tau \in I$, $\vec{x}$ and $\vec{y}$ are g_τ-consistent.*

Lemma 2.22 (Extension)**.** [13, Proposition 5.15] *Suppose that $\alpha < \beta < \omega_1$, $X \subseteq T_\beta$ is a finite set with unique drop-downs to α, $\{f_\tau \mid \tau \in I\}$ is a countable indexed set from $\mathrm{Aut}(T_{\leq\alpha})$, and $A \subseteq I$ is finite. Then there exists a collection $\{g_\tau \mid \tau \in I\}$ from $\mathrm{Aut}(T_{\leq\beta})$ such that*

- *for all $\tau \in I$, we have $f_\tau \subseteq g_\tau$;*
- *for all $\tau \in A$, $X \restriction \alpha$ and X are g_τ-consistent;*
- *if $\{f_\tau \mid \tau \in A\}$ is separated on $X \restriction \alpha$, then $\{g_\tau \mid \tau \in I\}$ is separated.*

We will also need the following slight variation of the preceding lemma; it is a special case of [13, Lemma 5.39], so we refer the reader there for a proof.

Lemma 2.23. *Suppose that $\alpha < \omega_1$, X_0 and X_1 are finite subsets of $T_{\alpha+1}$ with unique drop-downs to α and $X_0 \cap X_1 = \emptyset$ (though possibly $(X_0 \restriction \alpha) \cap (X_1 \restriction \alpha) \neq \emptyset$), and $g \in \mathrm{Aut}(T_{\leq\alpha})$. Then there is $g' \in \mathrm{Aut}(T_{\leq\alpha+1})$ such that $g' \supseteq g$ and, for each $i < 2$, $X_i \restriction \alpha$ and X_i are g'-consistent.*

We now recall the forcing poset designed to add a generic automorphism to a fixed ω_1-tree by initial segments of successor height. This poset will be a building block in the generalized Mitchell forcing used to prove Theorem A in the next section.

Definition 2.24. Given an ω_1-tree T, let $\mathbb{A}(T)$ be the forcing poset whose underlying set is $\bigcup\{\mathrm{Aut}(T_{\leq\alpha}) \mid \alpha < \omega_1\}$, ordered by reverse inclusion. Given $g \in \mathbb{A}(T)$, let $\mathsf{top}(g)$ be the unique $\alpha < \omega_1$ such that $g \in \mathrm{Aut}(T_{\leq\alpha})$. We will sometimes refer to $\mathsf{top}(g)$ as the *top level* of g.

Remark 2.25. If T is a normal, infinitely splitting ω_1-tree and G is an $\mathbb{A}(T)$-generic filter over V, then, by Lemma 2.22 and a standard genericity argument, $g^* := \bigcup G$ is an automorphism of T. Note that, in general, forcing with $\mathbb{A}(T)$ may collapse ω_1, but under certain assumptions on T, for instance, if T is a free Suslin tree, $\mathbb{A}(T)$ will be totally proper and hence will preserve ω_1 (cf. [13, Theorem 5.34]).

Although $\mathbb{A}(T)$ may preserve ω_1, the next proposition shows that, if CH fails, then it always collapses the continuum.

Proposition 2.26. *Suppose that T is an infinitely splitting, normal ω_1-tree. Then*

$$\Vdash_{\mathbb{A}(T)} \text{“}|(2^\omega)^V| = |\omega_1^V|\text{”}.$$

Proof. In V, fix, for each $x \in T$, an enumeration $\langle x_n \mid n < \omega\rangle$ of $\mathrm{succ}_T(x)$. Recall that S_∞ denotes the group of permutations of ω. If h is an automorphism of T, then h induces a map $\rho_h : T \to S_\infty$ as follows: for each $x \in T$ and each $n < \omega$, let $\rho_h(x)(n)$ be the unique $m < \omega$ such that $h(x_n) = h(x)_m$. A standard genericity argument shows that, if G is $\mathbb{A}(T)$-generic over V and $g^* = \bigcup G$, then, in $V[G]$, ρ_{g^*} is a surjection from T to $(S_\infty)^V$. Since $|T| = \omega_1^V$ and $|(S_\infty)^V| = |(2^\omega)^V|$, the desired conclusion follows. □

3. Guessing models with an almost Kurepa Suslin tree

In this section, we present our proof of Theorem A, constructing a model of ZFC in which GMP holds and there exists an almost Kurepa Suslin tree. In order to motivate the construction, let us first recall the construction from [13] of a model in which there exists an almost Kurepa Suslin tree but there does not exist a Kurepa tree. There, the authors begin in a model of ZFC with an inaccessible cardinal κ. They then force with the product $\mathrm{Coll}(\omega_1, {<}\kappa) \times \mathbb{P}$, where $\mathbb{P}$ is a countable support product of κ-many copies of $\mathbb{A}(T)$, where T is a fixed free Suslin tree in V. We note that collapsing an inaccessible cardinal is necessary here: if there are no Kurepa trees in a model W of ZFC, then $(\omega_2)^W$ is inaccessible in L. Note that the model $V[\mathrm{Coll}(\omega_1, {<}\kappa) \times \mathbb{P}]$ satisfies CH; and therefore contains weak Kurepa trees. If we want to arrange a model of GMP, or even of its consequence ¬wKH, then we must simultaneously increase the value of the continuum as we are collapsing the large cardinal κ. This requirement leads us naturally to consider Mitchell forcing. Since we also want our final model to contain an almost Suslin Kurepa tree, we will work with a variation of the classical Mitchell forcing obtained by replacing $\mathrm{Add}(\omega_1, 1)$ from the classical Mitchell forcing with the forcing $\mathbb{A}(T)$ from the end of the previous section.

3.1. **A Mitchell variation.** Fix for this section an ω_1-tree T and a cardinal κ. Let $\mathbb{P} = \mathbb{P}_\kappa = \mathrm{Add}(\omega, \kappa)$; for $\gamma < \kappa$, let $\mathbb{P}_\gamma = \mathrm{Add}(\omega, \gamma)$. Define a forcing poset $\mathbb{M} = \mathbb{M}^T_\kappa$ as follows, recalling that $\Sigma(\kappa)$ denotes the set of successor ordinals less than κ:

- Conditions of $\mathbb{M}$ are all pairs (p, q) such that
  - $p \in \mathbb{P}$;
  - q is a function whose domain is a countable subset of $\Sigma(\kappa)$;
  - for all $\alpha \in \mathrm{dom}(q)$, $q(\alpha)$ is a $\mathbb{P}_\alpha$-name for an element of $\mathbb{A}(T)^{V[\mathbb{P}_\alpha]}$.
- Given $(p_0, q_0), (p_1, q_1) \in \mathbb{M}$, we set $(p_1, q_1) \leq_{\mathbb{M}} (p_0, q_0)$ if and only if
  - $p_1 \leq_{\mathbb{P}} p_0$;
  - $\mathrm{dom}(q_1) \supseteq \mathrm{dom}(q_0)$;
  - for all $\alpha \in \mathrm{dom}(q_0)$, $p_1 \restriction \alpha \Vdash_{\mathbb{P}_\alpha}$ "$q_1(\alpha) \leq_{\mathbb{A}(T)} q_0(\alpha)$".

For $\delta < \kappa$, let $\mathbb{M}_\delta$ denote the set of $(p, q) \in \mathbb{M}$ such that $p \in \mathbb{P}_\delta$ and $\mathrm{dom}(q) \subseteq \delta$. Note that $\mathbb{M}_\delta$ is a regular suborder of $\mathbb{M}$. Given $(p, q) \in \mathbb{M}$, we let $(p, q) \restriction \delta$ denote $(p \restriction \delta, q \restriction \delta) \in \mathbb{M}_\delta$. It is easily verified that the map $(p, q) \mapsto (p, q) \restriction \delta$ is a projection from $\mathbb{M}$ to $\mathbb{M}_\delta$.

We let $\mathbb{Q}$ denote the *term forcing* associated with $\mathbb{M}$. More precisely, conditions in $\mathbb{Q}$ are all functions q such that $(1_{\mathbb{P}}, q) \in \mathbb{M}$. If $q, q' \in \mathbb{Q}$, then we let $q' \leq_{\mathbb{Q}} q$ if and only if $(1_{\mathbb{P}}, q') \leq_{\mathbb{M}} (1_{\mathbb{P}}, q)$.

Remark 3.1. As with the classical Mitchell forcing (recall Subsection 2.3), there is a natural projection from $\mathbb{P} \times \mathbb{Q}$ to $\mathbb{M}$. Many arguments involving Mitchell forcing, including those in Section 4 below, proceed via considerations of this product $\mathbb{P} \times \mathbb{Q}$, which is nicely behaved due to the fact that in the classical setting $\mathbb{Q}$ is ω_1-closed. In the setting of this section, though, $\mathbb{Q}$ is certainly not ω_1-closed, and even though we will see that $\mathbb{M}$ is nicely behaved, it is not even clear to us whether forcing with $\mathbb{Q}$ over V preserves ω_1. (For an example of a natural two-step iteration $\mathrm{Add}(\omega, 1) * \dot{\mathbb{Q}}$ such that $\dot{\mathbb{Q}}$ is forced to be ω_1-distributive (and even totally proper) and yet forcing with the associated term forcing over V collapses ω_1, see Appendix A below.) For this reason, the arguments in this section do not proceed via the familiar product analysis of Mitchell-type forcings.

Lemma 3.2. *Let* $\mathbb{Q}' = \{q \in \mathbb{Q} \mid \exists \alpha < \omega_1 \forall \gamma \in \mathrm{dom}(q) \ \Vdash_{\mathbb{P}_\gamma} \text{“}\mathsf{top}(q(\gamma)) = \alpha\text{”}\}$. *Then* $\mathbb{Q}'$ *is dense in* $\mathbb{Q}$.

Proof. Fix $q_0 \in \mathbb{Q}$. For each $\gamma \in \mathrm{dom}(q_0)$, let

$$\alpha_\gamma = \sup\{\beta < \omega_1 \mid \exists p \in \mathbb{P}_\gamma \ [p \Vdash \mathsf{top}(q_0(\gamma)) = \beta]\},$$

and let $\alpha = \sup\{\alpha_\gamma \mid \gamma \in \mathrm{dom}(q_0)\}$. Since each $\mathbb{P}_\gamma$ has the ccc and $\mathrm{dom}(q_0)$ is countable, it follows that $\alpha < \omega_1$ and, for all $\gamma \in \mathrm{dom}(q_0)$, $\Vdash_{\mathbb{P}_\gamma}$ “$\mathsf{top}(q_0(\gamma)) \leq \alpha$”. Let q be a function such that $\mathrm{dom}(q) = \mathrm{dom}(q_0)$ and, for all $\gamma \in \mathrm{dom}(q)$, $q(\gamma)$ is a $\mathbb{P}_\gamma$-name such that

$$\Vdash_{\mathbb{P}_\gamma} \text{“}q(\gamma) \leq q_0(\gamma) \text{ and } \mathsf{top}(q(\gamma)) = \alpha.\text{”}$$

(To see that such a name can be found, repeatedly apply instances of Lemma 2.22.) Then $q \in \mathbb{Q}'$ and $q \leq_{\mathbb{Q}} q_0$. □

Going forward, we will typically assume without comment that the conditions in $\mathbb{Q}$ that we are considering come from $\mathbb{Q}'$, and, if $q \in \mathbb{Q}'$, then we will let $\mathsf{top}(q)$ denote the unique $\alpha < \omega_1$ witnessing that $q \in \mathbb{Q}'$. Similarly, we will typically assume without comment that all conditions $(p, q) \in \mathbb{M}$ that we are considering are such that $q \in \mathbb{Q}'$.

We now introduce some definitions and lemmas to connect this modified Mitchell forcing with some of the concepts and results from the end of the previous section.

Definition 3.3 (Mitchellized consistency)**.** Suppose that $(p, q) \in \mathbb{M}$, $A \subseteq \mathrm{dom}(q)$ is finite, $\beta = \mathsf{top}(q)$, $\alpha < \alpha' \leq \beta$, and $X \subseteq T_{\alpha'}$ has unique drop-downs to α. We say that $X \restriction \alpha$ and X are $((p, q), A)$-*consistent* if, for all $\gamma \in A$,

$$p \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \text{“}X \restriction \alpha \text{ and } X \text{ are } q(\gamma)\text{-consistent”}.$$

We simply say that $X \restriction \alpha$ and X are (q, A)-*consistent* if they are $((1_{\mathbb{P}}, q), A)$-consistent.

As in Definition 2.16, we similarly define the notion of $((p, q), A)$-consistency for $\vec{x} \restriction \alpha$ and $\vec{x}$, where $\vec{x}$ is an injective tuple from $T_{\alpha'}$ with unique drop-downs to α.

Definition 3.4 (Mitchellized separation)**.** Suppose that $(p, q) \in \mathbb{M}$, $A \subseteq \mathrm{dom}(q)$ is finite, $\alpha \leq \beta = \mathsf{top}(q)$, $0 < n < \omega$, and $\vec{x}$ is an injective n-tuple from T_α.

- We say that p *decides* q *on* $(A, \vec{x})$ if, for all $\gamma \in A$ and $m < n$, $p \restriction \gamma$ decides the value of $q(\gamma)(x_m)$ and $q(\gamma)^{-1}(x_m)$, say as $g_\gamma(x_m)$ and $g_\gamma^{-1}(x_m)$.
- We say that (p, q) is A-*separated on* $\vec{x}$ if
  - p decides q on $(A, \vec{x})$, with $g_\gamma(x_m)$ and $g_\gamma^{-1}(x_m)$ for $\gamma \in A$ and $m < n$ as in the previous point;
  - for all $\gamma \in A$ and $m < n$, $g_\gamma(x_m) \neq x_m$; and
  - for all $m < n$, there exists at most triple (k, j, γ) such that $k < m$, $j \in \{-1, 1\}$, $\gamma \in A$, and $g_\gamma^j(x_m) = x_k$.
- We say that q is A-*separated on* $\vec{x}$ *below* p if, for all $p' \leq p$, if p' decides q on $(A, \vec{x})$, then (p', q) is A-separated on $\vec{x}$. We will simply say that q is A-*separated on* $\vec{x}$ if it is A-separated on $\vec{x}$ below $1_{\mathbb{P}}$.

If X is a finite subset of T_α, we say that p decides q on (A, X) if for some (equivalently, any) enumeration $\vec{x}$ of X, p decides q on $(A, \vec{x})$, and we say that (p, q) is A-separated on X if there is an enumeration $\vec{x}$ of X such that (p, q) is A-separated on $\vec{x}$. We say that q is A-separated on X below p if, for all $p' \leq p$, if p' decides q on (A, X), then (p', q) is A-separated on X.

Note that, if $(p', q') \leq_{\mathbb{M}} (p, q)$ and (p, q) is A-separated on $\vec{x}$ for some appropriate choice of A and $\vec{x}$, then (p', q') is also A-separated on $\vec{x}$. Similarly, if q is A-separated on $\vec{x}$ below p, then q' is A-separated on $\vec{x}$ below p'. The following proposition is an immediate consequence of the definitions and Proposition 2.19.

Proposition 3.5 (Mitchellized persistence)**.** *Suppose that*

- $(p, q) \in \mathbb{M}$ *with* $\mathsf{top}(q) = \beta$*;*
- $\alpha < \alpha' \leq \beta$*;*
- $A \in [\mathrm{dom}(q)]^{<\omega}$ *and* $X \in [T_{\alpha'}]^{<\omega}$ *has unique dropdowns to* α*.*

If q *is* A*-separated on* $X \restriction \alpha$ *below* p*, then* q *is also* A*-separated on* X *below* p*.* □

3.2. **Technical lemmas.** In this subsection, we present some technical lemmas concerning the construction of conditions in $\mathbb{M}$ having certain desired properties. Since these lemmas are motivated primarily by their eventual deployment in the proof of Theorem A, the reader may wish to skip this subsection on first read and return to it when the relevant lemmas are invoked in later subsections.

Lemma 3.6. *Suppose that*

- $q \in \mathbb{Q}$*, with* $\mathsf{top}(q) = \alpha$*;*
- X *is a finite subset of* $T_{\alpha+1}$ *with unique drop-downs to* α*;*
- $A \in [\mathrm{dom}(q)]^{<\omega}$ *and* $\delta \in \kappa \setminus A$*;*
- q *is* A*-separated on* $X \restriction \alpha$*.*

Then there is $q' \leq_{\mathbb{Q}} q$ *such that*

1. $\mathsf{top}(q') = \alpha + 1$*;*
2. $\delta \in \mathrm{dom}(q')$*;*
3. $X \restriction \alpha$ *and* X *are* (q', A)*-consistent;*
4. q' *is* $(A \cup \{\delta\})$*-separated on* X*.*

Proof. First note that we can assume that $\delta \in \mathrm{dom}(q)$, as otherwise we first extend q to the condition $q^* \leq_{\mathbb{Q}} q$ defined by setting $\mathrm{dom}(q^*) = \mathrm{dom}(q) \cup \{\delta\}$ and $q^* \restriction \mathrm{dom}(q) = q$, and letting $q^*(\delta)$ be an arbitrary $\mathbb{P}_\delta$-name for an element of $\mathrm{Aut}(T_{\leq \alpha})$.

Now let $q' \leq_{\mathbb{Q}} q$ be such that

- $\mathrm{dom}(q') = \mathrm{dom}(q)$;
- for all $\gamma \in A$, $q'(\gamma)$ is a $\mathbb{P}_\gamma$-name for an element of $\mathrm{Aut}(T_{\leq \alpha+1})$ extending $q(\gamma)$ such that
$$\Vdash_{\mathbb{P}_\gamma} \text{“}X \restriction \alpha \text{ and } X \text{ are } q'(\gamma)\text{-consistent”}$$
(this is possible by Lemma 2.22 applied in $V[\mathbb{P}_\gamma]$);
- $q'(\delta)$ is a $\mathbb{P}_\gamma$-name for an element of $\mathrm{Aut}(T_{\leq \alpha+1})$ extending $q(\delta)$ such that, for all $x \in X$, we have
$$\Vdash_{\mathbb{P}_\delta} \text{“}q'(\delta)(x) \notin X.\text{”}$$
(this is possible by Proposition 2.17);
- for $\gamma \in \mathrm{dom}(q) \setminus (A \cup \{\delta\})$, $q'(\gamma)$ is an arbitrary $\mathbb{P}_\gamma$-name for an element of $\mathrm{Aut}(T_{\leq \alpha+1})$ extending $q(\gamma)$.

Then q' is as desired, with requirement (4) following from a combination of Proposition 3.5 and the choice of $q'(\delta)$. □

Lemma 3.7 (Mitchellized Key Property). *Suppose that*

- $(p, q) \in \mathbb{M}$ *with* $\mathsf{top}(q) = \alpha$*;*
- $0 < n < \omega$ *and* $\vec{x}$ *is an injective* n*-tuple from* T_α*;*
- $A \in [\mathrm{dom}(q)]^{<\omega}$*;*
- (p, q) *is* A*-separated on* $\vec{x}$*;*
- $(p', q') \leq (p, q)$ *with* $\mathsf{top}(q') = \beta$*;*
- $t \in [T_\beta]^{<\omega}$.

Then there are $p'' \leq_{\mathbb{P}} p'$ *and an* n*-tuple* $\vec{y}$ *from* $T_\beta \setminus t$ *such that* $\vec{y} \in T_{\vec{x}}$ *and such that* $\vec{x}$ *and* $\vec{y}$ *are* $((p'', q'), A)$*-consistent.*

Proof. If $\beta = \alpha$, then there is nothing to prove, so assume that $\beta > \alpha$. Let G be $\mathbb{P}$-generic over V with $p' \in G$. For all $\gamma \in A$, let g_γ be the evaluation of $q'(\gamma)$ in $V[G]$. Then $\mathcal{G} = \{g_\gamma \mid \gamma \in A\}$ is a finite indexed set of automorphisms of $T_{\leq \beta+1}$ and $\mathcal{G}$ is separated on $\vec{x}$. By Lemma 2.20, we can find an n-tuple $\vec{y}$ from $T_\beta \setminus t$ such that $\vec{y} \in T_{\vec{x}}$ and, for all $\gamma \in A$, $\vec{x}$ and $\vec{y}$ are g_γ-consistent. We can then find $p'' \in G$ such that $p'' \leq_{\mathbb{P}} p'$ and, for all $\gamma \in A$,

$$p'' \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \text{“}\vec{x} \text{ and } \vec{y} \text{ are } q'(\gamma)\text{-consistent”}.$$

Now p'' and $\vec{y}$ are as desired. □

Lemma 3.8 (Mitchellized 1-Key Property). *Suppose that*

- $(p, q) \in \mathbb{M}$ *with* $\mathsf{top}(q) = \alpha$*;*
- $0 < n < \omega$ *and* $\vec{x}$ *is an injective* n*-tuple from* T_α*;*
- $A \in [\mathrm{dom}(q)]^{<\omega}$*;*
- (p, q) *is* A*-separated on* $\vec{x}$*;*
- $(p', q') \leq (p, q)$ *with* $\mathsf{top}(q') = \beta$*;*
- $m < n$*,* $y^* \in T_\beta$*, and* $x_m <_T y^*$.

Then there are $p'' \leq_{\mathbb{P}} p'$ *and an* n*-tuple* $\vec{y}$ *from* T_β *such that* $y_m = y^*$ *and* $\vec{y} \in T_{\vec{x}}$*, and such that* $\vec{x}$ *and* $\vec{y}$ *are* $((p'', q'), A)$*-consistent.*

Proof. This is proven in exactly the same way as Lemma 3.7, using Lemma 2.21 in place of Lemma 2.20. □

Lemma 3.9. *Suppose that*

- $\delta < \kappa$ *and* $\alpha < \omega_1$*;*
- $(p_0, q_0), (p_1, q_1) \in \mathbb{M}$ *are such that*
  - $\mathsf{top}(q_0) = \mathsf{top}(q_1) = \alpha$*;*
  - $\mathrm{dom}(q_0) = \mathrm{dom}(q_1)$*;*
  - $(p_0, q_0) \restriction \delta = (p_1, q_1) \restriction \delta$*;*
- $X \in [T_{\alpha+1}]^{<\omega}$ *and* $y \in T_{\alpha+1}$ *are such that* $X \cup \{y\}$ *has unique drop-downs to* α*;*
- $A \in [\mathrm{dom}(q_0)]^{<\omega}$*, and* (p_0, q_0) *and* (p_1, q_1) *are both* A*-separated on* $X \restriction \alpha$.

Then there exist $(\hat{p}_0, \hat{q}_0), (\hat{p}_1, \hat{q}_1) \in \mathbb{M}$ *and* $Y \in [T_{\alpha+1}]^{<\omega}$ *such that*

- $X \cup \{y\} \subseteq Y$*;*
- *for each* $i < 2$*, we have*
  - $(\hat{p}_i, \hat{q}_i) \leq (p_i, q_i)$*;*
  - $\mathsf{top}(\hat{q}_i) = \alpha + 1$*;*
  - $\mathrm{dom}(\hat{q}_i) = \mathrm{dom}(q_i)$*;*
  - $(\hat{p}_i, \hat{q}_i)$ *is* A*-separated on* Y*;*
  - $X \restriction \alpha$ *and* X *are* $((\hat{p}_i, \hat{q}_i), A)$*-consistent;*

– *for all* $\gamma \in A$, $x \in X \cup \{y\}$, *and* $j \in \{-1, 1\}$,

$$\hat{p}_i \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \text{“}\hat{q}_i(\gamma)^j(x) \in Y\text{”};$$

- $(\hat{p}_0, \hat{q}_0) \restriction \delta = (\hat{p}_1, \hat{q}_1) \restriction \delta$.

Proof. By extending p_0 if necessary (and correspondingly extending $p_1 \restriction \delta$ to maintain the fact that $p_0 \restriction \delta = p_1 \restriction \delta$), we can assume that p_0 decides q_0 on $(A, \{y \restriction \alpha\})$. Similarly, we can assume that p_1 decides q_1 on $(A, \{y \restriction \alpha\})$. For each $i < 2$, $\gamma \in A$, $x \in X \cup \{y\}$, and $j \in \{-1, 1\}$, let $z^j_{i,\gamma,x} \in T_\alpha$ be such that

$$p_i \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \text{“}q_i(\gamma)^j(x \restriction \alpha) = z^j_{i,\gamma,x}\text{”}.$$

Note that, if $\gamma \in A \cap \delta$, $x \in X \cup \{y\}$, and $j \in \{-1, 1\}$, then $z^j_{0,\gamma,x} = z^j_{1,\gamma,x}$. Let W_0 be the set of all $(i, \gamma, x, j) \in 2 \times A \times X \times \{-1, 1\}$ such that $z^j_{i,\gamma,x} \in X \restriction \alpha$, and let $W_1 = (2 \times A \times (X \cup \{y\}) \times \{-1, 1\}) \setminus W_0$. For each $(i, \gamma, x, j) \in W_1$, choose $\hat{z}^j_{i,\gamma,x} \in T_{\alpha+1}$ in such a way that

- $\hat{z}^j_{i,\gamma,x} \restriction \alpha = z^j_{i,\gamma,x}$;
- $\hat{z}^j_{i,\gamma,x} \notin X \cup \{y\}$;
- if $(0, \gamma, x, j) \in W_1$ and $\gamma < \delta$, then $\hat{z}^j_{0,\gamma,x} = \hat{z}^j_{1,\gamma,x}$ (note that the hypothesis implies that $(1, \gamma, x, j) \in W_1$ as well);
- if (i, γ, x, j) and (i', γ', x', j') are distinct elements of W_1 such that $(\gamma, x, j) \neq (\gamma', x', j')$ or $\gamma \geq \delta$, (i.e., the pair does not fall into the scope of the previous bullet point), then $\hat{z}^j_{i,\gamma,x} \neq \hat{z}^{j'}_{i',\gamma',x'}$.

Let $Y = X \cup \{y\} \cup \{\hat{z}^j_{i,\gamma,x} \mid (i, \gamma, x, j) \in W_1\}$. For each $\gamma \in A$ and $i < 2$, let $\hat{q}_i(\gamma)$ be a $\mathbb{P}_\gamma$-name for an element of $\mathrm{Aut}(T_{\leq \alpha+1})$ that is forced by $p_i \restriction \gamma$ to have the following properties, with the additional requirement that $\hat{q}_0(\gamma) = \hat{q}_1(\gamma)$ if $\gamma \in A \cap \delta$:

- $\hat{q}_i(\gamma) \supseteq q_i(\gamma)$;
- $X \restriction \alpha$ and X are $\hat{q}_i(\gamma)$-consistent;
- for all $(x, j) \in (X \cup \{y\}) \times \{-1, 1\}$ such that $(i, \gamma, x, j) \in W_1$, we have
  - $\hat{q}_i(\gamma)^j(x) = \hat{z}^j_{i,\gamma,x}$;
  - $\hat{q}_i(\gamma)^j(\hat{z}^j_{i,\gamma,x}) \notin Y$;
- for all $(i', \gamma', x', j') \in W_1$ such that $(i', \gamma') \neq (i, \gamma)$ and it is not the case that $\gamma' = \gamma \in A \cap \delta$ (i.e., $\hat{z}^{j'}_{i',\gamma',x'}$ does not fall within the scope of the previous bullet point), and for all $j \in \{-1, 1\}$, we have $\hat{q}_i(\gamma)^j(\hat{z}^{j'}_{i',\gamma',x'}) \notin Y$.

It is possible to find such a $\hat{q}_i(\gamma)$ by Proposition 2.15. For all $i < 2$ and $\gamma \in \mathrm{dom}(q_i) \setminus A$, let $\hat{q}_i(\gamma)$ be an arbitrary $\mathbb{P}_\gamma$-name for an element of $\mathrm{Aut}(T_{\leq \alpha+1})$ extending $q_i(\gamma)$, again subject to the requirement that $\hat{q}_0(\gamma) = \hat{q}_1(\gamma)$ for all $\gamma \in \mathrm{dom}(q_i) \cap \delta$. Finally, for each $i < 2$, find $\hat{p}_i \leq_{\mathbb{P}} p_i$ such that

- $\hat{p}_i$ decides $\hat{q}_i$ on (A, Y);
- $\hat{p}_0 \restriction \delta = \hat{p}_1 \restriction \delta$.

Then $(\hat{p}_0, \hat{q}_0)$,$(\hat{p}_1, \hat{q}_1)$, and Y are as desired. For instance, to see that $(\hat{p}_0, \hat{q}_0)$ is A-separated on Y, first let $\vec{x}$ be an injective enumeration of X such that $\vec{x} \restriction \alpha$ witnesses that (p_0, q_0) is A-separated on $X \restriction \alpha$, let $\vec{z}$ be an arbitrary injective enumeration of $Y \setminus (X \cup \{y\})$, and let $\vec{y} = \vec{x} ^\frown \langle y \rangle ^\frown \vec{z}$. The construction and choice of $\vec{x}$ then easily yield the conclusion that $(\hat{p}_0, \hat{q}_0)$ is A-separated on $\vec{y}$. □

3.3. **Dense subsets of derived trees.** This subsection contains two crucial lemmas proving that certain subsets of derived trees of T are dense. The first plays a key role in the eventual proof that $\mathbb{M}$ and its quotients have the ω_1-approximation property, which will later lead to the fact that, if κ is supercompact, then GMP holds in $V[\mathbb{M}]$. The second plays a similar role in the proof that, if T is a free Suslin tree, then it remains a Suslin tree in $V[\mathbb{M}]$.

Lemma 3.10. *Suppose that*

- $(p, q) \in \mathbb{M}$ *and* $\alpha = \mathsf{top}(q)$*;*
- $\delta < \kappa$*;*
- A *is a finite subset of* $\mathrm{dom}(q)$*,* $\vec{x}$ *is a finite sequence from* T_α*, and* (p, q) *is* A*-separated on* $\vec{x}$*;*
- λ *is a cardinal and* $\dot{b}$ *is an* $\mathbb{M}$*-name for a subset of* λ *such that* $\Vdash_{\mathbb{M}} \dot{b} \notin V[\mathbb{M}_\delta]$*.*

Let D *be the set of all* $\vec{z} \in T_{\vec{x}}$ *for which there exist* $(p_0, q_0), (p_1, q_1) \in \mathbb{M}$ *and* $\varepsilon < \lambda$ *such that*

- *for* $i < 2$*, we have:*
  - $(p_i, q_i) \leq (p, q)$*;*
  - $\mathsf{top}(q_i) = \mathrm{ht}_T(\vec{z})$*;*
  - $\vec{x}$ *and* $\vec{z}$ *are* $((p_i, q_i), A)$*-consistent;*
- $(p_0, q_0) \restriction \delta = (p_1, q_1) \restriction \delta$*;*
- $(p_0, q_0) \Vdash$ "$\varepsilon \notin \dot{b}$" *and* $(p_1, q_1) \Vdash$ "$\varepsilon \in \dot{b}$"*.*

Then D *is a dense open subset of* $T_{\vec{x}}$*.*

Proof. The fact that D is open follows easily from its definition. To show that D is dense, fix $\vec{y} \in T_{\vec{x}}$, and let $\alpha' = \mathrm{ht}_T(\vec{y})$. We will find $\vec{z} \in D \cap T_{\vec{y}}$. By repeatedly applying Extension (Lemma 2.22), we can find q' such that

- $(p, q') \leq_{\mathbb{M}} (p, q)$;
- $\mathsf{top}(q') = \alpha'$;
- $\vec{x}$ and $\vec{y}$ are $((p, q'), A)$-consistent.

Since $\Vdash_{\mathbb{M}}$ "$\dot{b} \notin V[\mathbb{M}_\delta]$", we can find $(p'_0, q'_0), (p'_1, q'_1) \leq_{\mathbb{M}} (p, q')$ and $\varepsilon < \lambda$ such that

(1) $(p'_0, q'_0) \restriction \delta = (p'_1, q'_1) \restriction \delta$;
(2) $(p'_0, q'_0) \Vdash_{\mathbb{M}}$ "$\varepsilon \notin \dot{b}$";
(3) $(p'_1, q'_1) \Vdash_{\mathbb{M}}$ "$\varepsilon \in \dot{b}$".

By extending the conditions if necessary, we may assume that there is a countable ordinal $\beta > \alpha'$ such that $\mathsf{top}(q'_0) = \mathsf{top}(q'_1) = \beta$. By extending p'_0 if necessary (and also correspondingly extending $p'_1 \restriction \delta$ to ensure item (1) above) and applying the Mitchellized Key Property (Lemma 3.7), we can fix $\vec{y}_0 \in T_{\vec{y}}$ such that $\mathrm{ht}_T(\vec{y}_0) = \beta$ and such that $\vec{y}$ and $\vec{y}_0$ are $((p'_0, q'_0), A)$-consistent. Let $\vec{y}_{00} = \vec{y}_0 \restriction (\alpha' + 1)$. By extending p'_1 (and $p'_0 \restriction \delta$ if necessary) and applying the Mitchellized Key Property, we can fix $\vec{y}_{10} \in T_{\vec{y}}$ such that

- $\mathrm{ht}_T(\vec{y}_{10}) = \alpha' + 1$;
- $\vec{y}_{00}$ and $\vec{y}_{10}$ have no entries in common;
- $\vec{y}$ and $\vec{y}_{10}$ are $((p'_1, q'_1), A)$-consistent.

By the same reasoning as above, we can fix $\vec{y}_1 \in T_{\vec{y}_{10}}$ such that $\mathrm{ht}_T(\vec{y}_1) = \beta$ and such that $\vec{y}_{10}$ and $\vec{y}_1$ are $((p'_1, q'_1), A)$-consistent.

Now repeatedly apply Lemma 2.23 followed by Extension (Lemma 2.22) to find $(p'_2, q'_2) \leq_{\mathbb{M}} (p, q')$ such that

- $\mathrm{ht}(q_2') = \beta$;
- $(p_2', q_2') \restriction \delta = (p_0', q_0') \restriction \delta$;
- for each $i < 2$, $\vec{y}$ and $\vec{y}_i$ are $((p_2', q_2'), A)$-consistent.

Now find $(p_2, q_2) \leq_{\mathbb{M}} (p_2', q_2')$ deciding the truth value of "$\varepsilon \in \dot{b}$" For concreteness, suppose that $(p_2, q_2) \Vdash_{\mathbb{M}}$ "$\varepsilon \in \dot{b}$"; the other case is symmetric, with (p_1', q_1') playing the role of (p_0', q_0') in the following argument. Let $\beta^* = \mathsf{top}(q_2)$. By applying the Mitchellized Key Property and extending p_2 if necessary, we can fix $\vec{z} \in T_{\vec{y}_0}$ such that $\mathrm{ht}_T(\vec{z}) = \beta^*$ and such that $\vec{y}_0$ and $\vec{z}$ are $((p_2, q_2), A)$-consistent. Finally, repeatedly apply Extension to find $(p_0, q_0) \leq_{\mathbb{M}} (p_0', q_0')$ such that

- $\mathsf{top}(q_0) = \beta^*$;
- $(p_0, q_0) \restriction \delta = (p_2, q_2) \restriction \delta$;
- $\vec{y}_0$ and $\vec{z}$ are $((p_0, q_0), A)$-consistent.

Then $\vec{z} \in D$, as witnessed by (p_0, q_0), (p_2, q_2), and ε. □

Lemma 3.11. *Suppose that*

- $(p, q) \in \mathbb{M}$ *and* $\alpha = \mathsf{top}(q)$*;*
- A *is a finite subset of* $\mathrm{dom}(q)$*,* $\vec{x}$ *is an injective* n*-tuple from* T_α *for some* $0 < n < \omega$*, and* (p, q) *is* A*-separated on* $\vec{x}$*;*
- $m < n$ *and* $\dot{E}$ *is an* $\mathbb{M}$*-name for a dense open subset of* T*.*

Let D *be the set of all* $\vec{z} \in T_{\vec{x}}$ *for which there exists* $(p^*, q^*) \leq_{\mathbb{M}} (p, q)$ *such that*

- $\mathsf{top}(q^*) = \mathrm{ht}_T(\vec{z})$*;*
- $(p^*, q^*) \Vdash_{\mathbb{M}}$ "$z_m \in \dot{E}$"*;*
- $\vec{x}$ *and* $\vec{z}$ *are* $((p^*, q^*), A)$*-consistent.*

Then D *is a dense open subset of* $T_{\vec{x}}$*.*

Proof. The fact that D is open follows easily from the definition. To show that D is dense, fix $\vec{y} \in T_{\vec{x}}$, and let $\alpha' = \mathrm{ht}_T(\vec{y})$. Fix q' exactly as in the proof of Lemma 3.10. Find $p' \leq_{\mathbb{P}} p$ such that p' decides q' on $(A, \vec{y})$. Note that, since (p, q) is A-separated on $\vec{x}$, it follows from Mitchellized Persistence (Proposition 3.5) that (p', q') is A-separated on $\vec{y}$. Now find $(p'', q'') \leq_{\mathbb{M}} (p', q')$ and $t \in T$ such that $y_m \leq_T t$ and $(p'', q'') \Vdash_{\mathbb{M}}$ "$t \in \dot{E}$". By strengthening q'' if necessary, we may assume that $\mathsf{top}(q'') \geq \mathrm{ht}(t)$, and since $\dot{E}$ is forced to be open, we can extend t to assume that we in fact have $\mathsf{top}(q'') = \mathrm{ht}(t)$; let β denote this height. Now apply the Mitchellized 1-Key Property (Lemma 3.8) to find $p^* \leq_{\mathbb{P}} p''$ and an n-tuple $\vec{z}$ from T_β such that $z_m = t$ and $\vec{z} \in T_{\vec{y}}$, and such that $\vec{y}$ and $\vec{z}$ are $((p^*, q''), A)$-consistent. Then $\vec{z} \in D$, as witnessed by (p^*, q''). □

3.4. **Approximation and the proof of Theorem A.** This subsection finally contains the proof of Theorem A. We first recall what it means for a forcing poset to satisfy the μ*-approximation property.*

Definition 3.12. Let μ be an uncountable cardinal, and let $V \subseteq W$ be two models of ZFC. We say that (V, W) has the μ*-approximation property* if, whenever $x \in W$ is a set of ordinals such that $x \cap z \in V$ for all $z \in ([\mathrm{On}]^{<\mu})^V$, it follows that $x \in V$. Working in a model V of ZFC, we say that a forcing poset $\mathbb{R}$ has the μ-approximation property if it is forced by $1_{\mathbb{R}}$ that $(V, V[\mathbb{R}])$ has the μ-approximation property.

We begin with the following preliminary result, establishing the crucial properties of the forcing extension by $\mathbb{M}$. In what follows, we let $\dot{G}$ be the canonical $\mathbb{M}$-name for the generic filter. For all $\delta < \kappa$, let $\dot{G}_\delta$ be the canonical name for the generic filter over $\mathbb{M}_\delta$ induced by $\dot{G}$. Let $\dot{\mathbb{M}}_{\delta,\kappa}$ be the canonical $\mathbb{M}_\delta$-name for the quotient forcing $\mathbb{M}/\dot{G}_\delta$.

Theorem 3.13. *Suppose that T is a free Suslin tree in V. Then*

(1) for all limit ordinals $\delta < \kappa$ (including $\delta = 0$),

$$\Vdash_{\mathbb{M}_\delta} \text{“}\dot{\mathbb{M}}_{\delta,\kappa} \textit{ has the } \omega_1\textit{-approximation property”};$$

(2) $\Vdash_{\mathbb{M}}$ “T is Suslin”.

Proof. We will work towards proving both statements simultaneously. Fix a condition $(p, q) \in \mathbb{M}$, a limit ordinal $\delta < \kappa$, a cardinal λ, an $\mathbb{M}$-name $\dot{b}$ such that

$$(p, q) \Vdash_{\mathbb{M}} \text{“}\dot{b} \subseteq \lambda \wedge \dot{b} \notin V[\dot{G}_\delta]\text{”},$$

and an $\mathbb{M}$-name $\dot{E}$ such that

$$(p, q) \Vdash_{\mathbb{M}} \text{“}\dot{E} \text{ is a dense open subset of } T\text{”}.$$

For notational simplicity, assume without loss of generality that $p = 1_{\mathbb{P}}$ (for the general case, simply work below p in $\mathbb{P}$). Fix a sufficiently large regular cardinal θ, a well-ordering $\lhd$ of $H(\theta)$, and a countable elementary submodel $N \prec (H(\theta), \in, \lhd)$ containing everything relevant (including q, $\dot{b}$, and $\dot{E}$). Let $\beta = N \cap \omega_1$. We will find $q^* \leq_{\mathbb{Q}} q$ such that

- $(1_{\mathbb{P}}, q^*) \Vdash_{\mathbb{M}}$ “$\dot{b} \cap N \notin V[\dot{G}_\delta]$”;
- $(1_{\mathbb{P}}, q^*) \Vdash_{\mathbb{M}}$ “$T_\beta \subseteq \dot{E}$”.

This will clearly suffice to prove the theorem.

Let $\mathbb{P}_N$ denote $\mathbb{P} \cap N$. Let $\langle x_n \mid n < \omega \rangle$ enumerate T_β, let $\langle \gamma_n \mid n < \omega \rangle$ enumerate $\kappa \cap N$, let $\langle \beta_n \mid n < \omega \rangle$ be an increasing sequence of ordinals cofinal in β, and let $\langle p_n \mid n < \omega \rangle$ enumerate $\mathbb{P}_N$ so that every element of $\mathbb{P}_N$ equals p_n for infinitely many $n < \omega$. For all $n < \omega$, let $A_n = \{\gamma_m \mid m \leq n\}$.

We will now recursively build objects $\langle q_n, \hat{p}_{n,0}, \hat{p}_{n,1}, X_n, \varepsilon_n \mid n < \omega \rangle$ satisfying the following requirements:

1. $\langle q_n \mid n < \omega \rangle$ is a $\leq_{\mathbb{Q}}$-decreasing sequence of conditions from $\mathbb{Q} \cap N$ with $q_0 \leq_{\mathbb{Q}} q$;
2. for all $n < \omega$, we have $A_n \subseteq \mathrm{dom}(q_n)$ and $\mathsf{top}(q_n) \geq \beta_n$;
3. for all $n < \omega$, $\hat{p}_{n,0}$ and $\hat{p}_{n,1}$ are elements of $\mathbb{P}_N$ such that $\hat{p}_{n,0}, \hat{p}_{n,1} \leq_{\mathbb{P}} p_n$ and $\hat{p}_{n,0} \restriction \delta = \hat{p}_{n,1} \restriction \delta$;
4. for all $m < n < \omega$ and $i < 2$,
$$(\hat{p}_{n,i}, q_n) \Vdash_{\mathbb{M}} \text{“}x_m \restriction \mathsf{top}(q_n) \in \dot{E}\text{”};$$
5. for all $n < \omega$, ε_n is an element of $N \cap \lambda$ such that $(\hat{p}_{n,0}, q_n) \Vdash_{\mathbb{M}}$ “$\varepsilon_n \notin \dot{b}$” and $(\hat{p}_{n,1}, q_n) \Vdash_{\mathbb{M}}$ “$\varepsilon_n \in \dot{b}$”;
6. for all $n < \omega$, X_n is a finite subset of T_β such that
   (a) $\{x_m \mid m \leq n\} \subseteq X_n \subseteq X_{n+1}$;
   (b) X_n has unique drop-downs to $\mathsf{top}(q_n)$;
   (c) for each $i < 2$, $\hat{p}_{n,i}$ decides q_n on $(A_n, X_n \restriction \mathsf{top}(q_n))$;
   (d) for each $i < 2$, $\gamma \in A_n$, $m \leq n$, and $j \in \{-1, 1\}$, there is $y \in X_n$ such that
$$\hat{p}_{n,i} \Vdash_{\mathbb{P}} \text{“}q_n(\gamma)^j(x_m \restriction \mathsf{top}(q_n)) = y \restriction \mathsf{top}(q_n)\text{”};$$

(7) for all $n < \omega$, q_n is A_n-separated on $X_n \restriction \mathsf{top}(q_n)$;
(8) for all $n < \omega$, $X_n \restriction \mathsf{top}(q_n)$ and $X_n \restriction \mathsf{top}(q_{n+1})$ are (q_{n+1}, A_n)-consistent.

For ease of notation, let $q_{-1} = q$ and $X_{-1} = \emptyset$. Fix $-1 \leq n < \omega$ and suppose that we have constructed q_n and X_n. First, apply Lemma 3.6 inside N to find $q^*_{n+1} \leq_{\mathbb{Q}} q_n$ such that

- $q^*_{n+1} \in N$;
- $\mathsf{top}(q^*_{n+1}) = \mathsf{top}(q_n) + 1$;
- $A_{n+1} \subseteq \mathrm{dom}(q^*_{n+1})$;
- q^*_{n+1} is A_{n+1}-separated on $X_n \restriction \mathsf{top}(q^*_{n+1})$;
- for all $\gamma \in A_n$, $X_n \restriction \mathsf{top}(q_n)$ and $X_n \restriction \mathsf{top}(q^*_{n+1})$ are (q^*_{n+1}, A_n)-consistent.

Fix $p^*_{n+1} \leq_{\mathbb{P}} p_{n+1}$ with $p^*_{n+1} \in N$ and an injective enumeration $\vec{x^n}$ of X_n such that (p^*_{n+1}, q^*_{n+1}) is A_{n+1}-separated on $\vec{x^n} \restriction \mathsf{top}(q^*_{n+1})$.

Claim 3.14. *Fix $0 \leq m \leq n$ and $(\bar{p}, \bar{q}) \leq_{\mathbb{M}} (p^*_{n+1}, q^*_{n+1})$ such that $(\bar{p}, \bar{q}) \in N$, $\bar{p}$ decides $\bar{q}$ on $(A_{n+1}, \vec{x^n} \restriction \mathsf{top}(\bar{q}))$, and such that $\vec{x^n} \restriction \mathsf{top}(q^*_{n+1})$ and $\vec{x^n} \restriction \mathsf{top}(\bar{q})$ are $((\bar{p}, \bar{q}), A_{n+1})$-consistent. Then there is $(\bar{\bar{p}}, \bar{\bar{q}}) \leq_{\mathbb{M}} (\bar{p}, \bar{q})$ such that*

- *$(\bar{\bar{p}}, \bar{\bar{q}}) \in N$;*
- *$(\bar{\bar{p}}, \bar{\bar{q}}) \Vdash_{\mathbb{M}}$ "$x_m \restriction \mathsf{top}(\bar{\bar{q}}) \in \dot{E}$";*
- *$\vec{x} \restriction \mathsf{top}(\bar{q})$ and $\vec{x} \restriction \mathsf{top}(\bar{\bar{q}})$ are $((\bar{\bar{p}}, \bar{\bar{q}}), A_{n+1})$-consistent.*

Proof. Fix $m^* < \omega$ such that $x_m = x^n_{m^*}$. Let D be the set of all $\vec{z} \in T_{\vec{x^n} \restriction \mathsf{top}(\bar{q})}$ for which there exists $(\bar{\bar{p}}, \bar{\bar{q}}) \leq_{\mathbb{M}} (\bar{p}, \bar{q})$ such that

- $\mathsf{top}(\bar{\bar{q}}) = \mathrm{ht}_T(\vec{z})$;
- $(\bar{\bar{p}}, \bar{\bar{q}}) \Vdash_{\mathbb{M}}$ "$z_{m^*} \in \dot{E}$";
- $\vec{x} \restriction \mathsf{top}(\bar{q})$ and $\vec{z}$ are $((\bar{\bar{p}}, \bar{\bar{q}}), A_{n+1})$-consistent.

By Lemma 3.11, D is a dense open subset of $T_{\vec{x^n} \restriction \mathrm{ht}(\bar{q})}$. Since T is a free Suslin tree, there is $\eta < \omega_1$ such that every $\vec{z} \in T_{\vec{x^n} \restriction \mathrm{ht}(\bar{q})}$ of height at least η is in D. Since $D \in N$, the least such η is in N, and hence is less than β. In particular, there is some $\eta^* \in [\mathsf{top}(\bar{q}), \beta)$ such that $\vec{x^n} \restriction \eta^* \in D$. Then any condition $(\bar{\bar{p}}, \bar{\bar{q}}) \in N$ witnessing that $\vec{x^n} \restriction \eta^* \in D$ witnesses the conclusion of the claim. □

By repeatedly applying Claim 3.14, fix a condition $(\bar{p}_{n+1}, \bar{q}_{n+1}) \leq_{\mathbb{M}} (p^*_{n+1}, q^*_{n+1})$ such that

- $(\bar{p}_{n+1}, \bar{q}_{n+1}) \in N$;
- $\bar{p}_{n+1}$ decides $\bar{q}_{n+1}$ on $(A_{n+1}, \vec{x^n} \restriction \mathsf{top}(\bar{q}_{n+1}))$;
- $\vec{x^n} \restriction \mathsf{top}(q^*_{n+1})$ and $\vec{x^n} \restriction \mathsf{top}(\bar{q}_{n+1})$ are $((\bar{p}_{n+1}, \bar{q}_{n+1}), A_{n+1})$-consistent;
- for all $m \leq n$, we have $(\bar{p}_{n+1}, \bar{q}_{n+1}) \Vdash_{\mathbb{M}}$ "$x_m \restriction \mathsf{top}(\bar{q}_{n+1}) \in \dot{E}$".

Now let D' be the set of all $\vec{z} \in T_{\vec{x^n} \restriction \mathsf{top}(\bar{q}_{n+1})}$ for which there exist $(\bar{p}_{n+1,0}, \bar{q}_{n+1,0})$ and $(\bar{p}_{n+1,1}, \bar{q}_{n+1,1})$ in $\mathbb{M}$ and $\varepsilon < \lambda$ such that

- for each $i < 2$, we have $(\bar{p}_{n+1,i}, \bar{q}_{n+1,i}) \leq_{\mathbb{M}} (\bar{p}_{n+1}, \bar{q}_{n+1})$;
- $\mathsf{top}(\bar{q}_{n+1,0}) = \mathsf{top}(\bar{q}_{n+1,1}) = \mathrm{ht}_T(\vec{z})$;
- $(\bar{p}_{n+1,0}, \bar{q}_{n+1,0}) \restriction \delta = (\bar{p}_{n+1,1}, \bar{q}_{n+1,1}) \restriction \delta$;
- for each $i < 2$, $\vec{x^n} \restriction \mathsf{top}(\bar{q}_{n+1})$ and $\vec{z}$ are $((\bar{p}_{n+1,i}, \bar{q}_{n+1,i}), A_{n+1})$-consistent;
- $(\bar{p}_{n+1,0}, \bar{q}_{n+1,0}) \Vdash_{\mathbb{M}}$ "$\varepsilon \notin \dot{b}$" and $(\bar{p}_{n+1,1}, \bar{q}_{n+1,1}) \Vdash_{\mathbb{M}}$ "$\varepsilon \in \dot{b}$".

By Lemma 3.10, D' is a dense open subset of $T_{\vec{x^n} \restriction \mathsf{top}(\bar{q}_{n+1})}$, and $D' \in N$. Therefore, as in the proof of Claim 3.14, we can find $\alpha_{n+1} < \beta$ such that

- $\alpha_{n+1} \geq \max\{\beta_{n+1}, \mathsf{top}(\bar{q}_{n+1})\}$;

- $X_n \cup \{x_{n+1}\}$ has unique drop-downs to α_{n+1};
- $\vec{x^n} \restriction \alpha_{n+1} \in D'$.

Since D' and $\vec{x^n} \restriction \alpha_{n+1}$ are in N, we can fix $(\bar{p}_{n+1,0}, \bar{q}_{n+1,0})$, $(\bar{p}_{n+1,1}, \bar{q}_{n+1,1})$, and ε_{n+1} in N witnessing that $\vec{x^n} \restriction \alpha_{n+1}$ is in D'. By extending the conditions if necessary, we can assume that

- $\mathrm{dom}(\bar{q}_{n+1,0}) = \mathrm{dom}(\bar{q}_{n+1,1})$;
- $\delta \in \mathrm{dom}(\bar{p}_{n+1,0}) \cap \mathrm{dom}(\bar{p}_{n+1,1})$ and $\bar{p}_{n+1,0}(\delta) \perp \bar{p}_{n+1,1}(\delta)$;
- for each $i < 2$, $\bar{p}_{n+1,i}$ decides $\bar{q}_{n+1,i}$ on $(A_{n+1}, X_n \cup \{x_{n+1}\})$.

By Lemma 3.9 applied in N, we can find $(\hat{p}_{n+1,0}, \hat{q}_{n+1,0}), (\hat{p}_{n+1,1}, \hat{q}_{n+1,1}) \in \mathbb{M}$ and a finite set Y, all in N, such that, letting $\alpha^*_{n+1} = \alpha_{n+1} + 1$, the following hold:

- for each $i < 2$, we have $\mathrm{dom}(\hat{q}_{n+1,i}) = \mathrm{dom}(\bar{q}_{n+1,i})$ and $(\hat{p}_{n+1,i}, \hat{q}_{n+1,i}) \leq_{\mathbb{M}} (\bar{p}_{n+1,i}, \bar{q}_{n+1,i})$;
- for each $i < 2$, we have $\mathsf{top}(\hat{q}_{n+1,i}) = \alpha^*_{n+1}$;
- $(\hat{p}_{n+1,0}, \hat{q}_{n+1,0}) \restriction \delta = (\hat{p}_{n+1,1}, \hat{q}_{n+1,1}) \restriction \delta$;
- $(X_n \cup \{x_{n+1}\}) \restriction \alpha^*_{n+1} \subseteq Y \subseteq T_{\alpha^*_{n+1}}$;
- for each $i < 2$, $(\hat{p}_{n+1,i}, \hat{q}_{n+1,i})$ is A_{n+1}-separated on Y;
- for each $i < 2$, $X_n \restriction \alpha_{n+1}$ and $X_n \restriction \alpha^*_{n+1}$ are $(\hat{p}_{n+1,i}, \hat{q}_{n+1,i})$-consistent on A_{n+1};
- for all $i < 2$, $\gamma \in A_{n+1}$, $m \leq n+1$, and $j \in \{-1, 1\}$, there is $y \in Y$ such that
$$\hat{p}_{n+1,i} \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \text{“}\hat{q}_{n+1,i}(\gamma)^j(x_m \restriction \alpha^*_{n+1}) = y\text{”}.$$

For each $y \in Y \setminus \big((X_n \cup \{x_{n+1}\}) \restriction \alpha^*_{n+1}\big)$, choose a $z_y \in T_\beta$ such that $z_y \restriction \alpha^*_{n+1} = y$, and set
$$X_{n+1} = X_n \cup \{x_{n+1}\} \cup \left\{z_y \;\middle|\; y \in Y \setminus \big((X_n \cup \{x_{n+1}\}) \restriction \alpha^*_{n+1}\big)\right\}.$$

Next, working in N, fix $r \in \mathbb{Q}$ with $\mathsf{top}(r) = \alpha^*_{n+1}$ and $\mathrm{dom}(r) = \mathrm{dom}(\hat{q}_{n+1,0})$ such that $r \leq_{\mathbb{Q}} q^*_{n+1}$, r is A_{n+1}-separated on $X_{n+1} \restriction \alpha^*_{n+1}$, and such that $X_n \restriction \mathsf{top}(q^*_{n+1})$ and $X_n \restriction \alpha^*_{n+1}$ are (r, A_n)-consistent. To see that such an r can be found, first repeatedly apply Proposition 2.17 to find $r_0 \leq_{\mathbb{Q}} q^*_{n+1}$ such that $\mathsf{top}(r_0) = \alpha_{n+1}$, $\mathrm{dom}(r_0) = \mathrm{dom}(\hat{q}_{n+1,0})$ and such that $X_n \restriction \mathsf{top}(q^*_{n+1})$ and $X_n \restriction \alpha_{n+1}$ are (r_0, A_{n+1})-consistent. Then again apply Proposition 2.17 to find $r \leq_{\mathbb{Q}} r_0$ such that $\mathsf{top}(r) = \alpha^*_{n+1}$, $\mathrm{dom}(r) = \mathrm{dom}(r_0)$ and, for all $\gamma \in A_{n+1}$, we have:

- $\Vdash_{\mathbb{P}_\gamma}$ “$X_n \restriction \alpha_{n+1}$ and $X_n \restriction \alpha^*_{n+1}$ are $r(\gamma)$-consistent”;
- for all $y \in (X_{n+1} \setminus X_n) \restriction \alpha^*_{n+1}$ and all $j \in \{-1, 1\}$, we have $\Vdash_{\mathbb{P}_\gamma}$ “$r(\gamma)^j(y) \notin X_{n+1} \restriction \alpha^*_{n+1}$.

Then, recalling that q^*_{n+1} was chosen to be A_{n+1}-separated on $X_n \restriction \mathsf{top}(q^*_{n+1})$, it is readily seen that this r is as desired.

Finally, define $q_{n+1} \in \mathbb{Q}$ with $\mathsf{top}(q_{n+1}) = \alpha^*_{n+1}$ and $\mathrm{dom}(q_{n+1}) = \mathrm{dom}(\hat{q}_{n+1,0})$ by letting, for all $\gamma \in \mathrm{dom}(\hat{q}_{n+1,0})$, $q_{n+1}(\gamma)$ be a $\mathbb{P}_\gamma$-name such that

- $\hat{p}_{n+1,0} \restriction \gamma \Vdash$ “$q_{n+1}(\gamma) = \hat{q}_{n+1,0}(\gamma)$”;
- $\hat{p}_{n+1,1} \restriction \gamma \Vdash$ “$q_{n+1}(\gamma) = \hat{q}_{n+1,1}(\gamma)$ (note that this is not in conflict with the previous point since, if $\hat{q}_{n+1,0}(\gamma) \neq \hat{q}_{n+1,1}(\gamma)$, then $\gamma > \delta$, and hence $\hat{p}_{n+1,0} \restriction \gamma \perp \hat{p}_{n+1,1} \restriction \gamma$);
- if $p^* \in \mathbb{P}_\gamma$ is incompatible with both $\hat{p}_{n+1,0} \restriction \gamma$ and $\hat{p}_{n+1,1} \restriction \gamma$, then $p^* \Vdash$ “$q_{n+1}(\gamma) = r(\gamma)$”.

It is readily verified that q_{n+1}, $\hat{p}_{n+1,0}$, $\hat{p}_{n+1,1}$, X_{n+1}, and ε_{n+1} satisfy the requirements of the construction.

Claim 3.15. *$\langle q_n \mid n < \omega\rangle$ has a lower bound in $\mathbb{Q}$.*

Proof. Recall that $\alpha^*_n = \mathsf{top}(q_n)$ for all $n < \omega$, and that $\langle \alpha^*_n \mid n < \omega\rangle$ is cofinal in β. Note also that $\bigcup\{\mathrm{dom}(q_n) \mid n < \omega\} = N \cap \kappa$. For each $\gamma \in N \cap \kappa$, let n_γ be the least $n < \omega$ such that $\gamma \in \mathrm{dom}(q_n)$. For each $\gamma \in N \cap \kappa$, let $q^-(\gamma)$ be a $\mathbb{P}_\gamma$-name forced to be equal to $\bigcup\{q_n(\gamma) \mid n_\gamma \le n < \omega\}$. In particular, $q^-(\gamma)$ is forced to be an automorphism of $T_{<\beta}$.

Subclaim 3.16. *For each $\gamma \in N \cap \kappa$, each $x \in T_\beta$, and each $j \in \{-1, 1\}$,*

$$\Vdash_{\mathbb{P}_\gamma} \text{“}\exists y \in T_\beta \forall \alpha < \beta\ [q^-(\gamma)^j(x \restriction \alpha) = y \restriction \alpha]\text{”}.$$

Proof. Suppose for the sake of contradiction that there exists $\gamma \in N \cap \kappa$, $n^* < \omega$, $j \in \{-1, 1\}$, and $p^* \in \mathbb{P}_\gamma$ such that

$$p^* \Vdash_{\mathbb{P}_\gamma} \text{“}\neg(\exists y \in T_\beta \forall \alpha < \beta\ [q^-(\gamma)^j(x_{n^*} \restriction \alpha) = y \restriction \alpha])\text{”}.$$

Find $n < \omega$ such that $p^* \cap N = p_n$, $n \ge n^*$, and $\gamma \in A_n$. Recall, then, that there is $y \in X_n$ such that $\hat{p}_{n,0} \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ “$q_n(\gamma)^j(x_{n^*} \restriction \alpha^*_n) = y \restriction \alpha^*_n$”. Moreover, for all $n' > n$, we have

$$\Vdash_{\mathbb{P}_\gamma} \text{“}X_n \restriction \alpha^*_n \text{ and } X_n \restriction \alpha^*_{n'} \text{ are } q_{n'}(\gamma)\text{-consistent”}.$$

It follows that, for all $n' > n$, we have

$$\hat{p}_{n,0} \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \text{“}q_{n'}(\gamma)^j(x_{n^*} \restriction \alpha^*_{n'}) = y \restriction \alpha^*_{n'}\text{”},$$

and hence

$$\hat{p}_{n,0} \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \text{“}\forall \alpha < \beta\ [q^-(\gamma)^j(x_{n^*} \restriction \alpha) = y \restriction \alpha]\text{”}.$$

But $\hat{p}_{n,0} \restriction \gamma \in N$ and $\hat{p}_{n,0} \restriction \gamma \le p^* \cap N$. Thus, $\hat{p}_{n,0} \restriction \gamma$ and p^* are compatible in $\mathbb{P}_\gamma$, contradicting the fact that they force contradictory statements. □

We now define a lower bound q^* for $\langle q_n \mid n < \omega\rangle$ with $\mathsf{top}(q^*) = \beta$ and $\mathrm{dom}(q^*) = N \cap \kappa$ by letting, for each $\gamma \in N \cap \kappa$, $q^*(\gamma)$ be a $\mathbb{P}_\gamma$-name for the unique element of $\mathrm{Aut}(T_{\le\beta})$ extending $q^-(\gamma)$. Such a name must exist by Subclaim 3.16 and Proposition 2.14. It is readily verified that q^* thus defined is indeed a lower bound for $\langle q_n \mid n < \omega\rangle$. □

Fix a lower bound q^* for $\langle q_n \mid n < \omega\rangle$ in $\mathbb{Q}$. We first verify that $(1_{\mathbb{P}}, q^*) \Vdash_{\mathbb{M}}$ “$\dot{b} \cap N \notin V[\dot{G}_\delta]$”. To this end, let $w = N \cap \lambda$, and suppose for the sake of contradiction that there are $(p^{**}, q^{**}) \le_{\mathbb{M}} (1_{\mathbb{P}}, q^*)$ and an $\mathbb{M}_\delta$-name $\dot{c}$ such that $(p^{**}, q^{**}) \Vdash_{\mathbb{M}}$ “$\dot{b} \cap w = \dot{c}$”. Fix $n < \omega$ such that $p^{**} \cap N = p_n$. Note that both $\hat{p}_{n,0}$ and $\hat{p}_{n,1}$ are compatible with p^{**}, and recall that $\hat{p}_{n,0} \restriction \delta = \hat{p}_{n,1} \restriction \delta$. Let r_0 denote $(p^{**} \wedge \hat{p}_{n,0}) \restriction \delta$, i.e., r_0 is the greatest common lower bound of $p^{**} \restriction \delta$ and $\hat{p}_{n,0} \restriction \delta$ in $\mathbb{P}_\delta$. Find $(r_1, s_1) \le_{\mathbb{M}_\delta} (r_0, q^{**} \restriction \delta)$ deciding the truth value of the statement “$\varepsilon_n \in \dot{c}$”. Suppose without loss of generality that $(r_1, s_1) \Vdash_{\mathbb{M}_\delta}$ “$\varepsilon_n \in \dot{c}$” (the other case is symmetric). Define $(\hat{p}, \hat{q}) \in \mathbb{M}$ by setting

- $(\hat{p}, \hat{q}) \restriction \delta = (r_1, s_1)$;
- $(\hat{p}, \hat{q}) \restriction [\delta, \kappa) = (p^{**} \wedge \hat{p}_{n,0}, q^{**}) \restriction [\delta, \kappa)$.

Then $(\hat{p}, \hat{q}) \Vdash_{\mathbb{M}}$ “$\varepsilon_n \in \dot{c}$”, since (r_1, s_1) forces the same. We also have $(\hat{p}, \hat{q}) \le_{\mathbb{M}} (p^{**}, q^{**})$, so $(\hat{p}, \hat{q}) \Vdash_{\mathbb{M}}$ “$\dot{b} \cap w = \dot{c}$”, and hence $(\hat{p}, \hat{q}) \Vdash_{\mathbb{M}}$ “$\varepsilon_n \in \dot{b}$”. On the other hand, we have $(\hat{p}, \hat{q}) \le_{\mathbb{M}} (\hat{p}_{n,0}, q_n)$, and hence $(\hat{p}, \hat{q}) \Vdash_{\mathbb{M}}$ “$\varepsilon_n \notin \dot{b}$”. This is a contradiction, completing the verification that $(1_{\mathbb{P}}, q^*) \Vdash_{\mathbb{M}}$ “$\dot{b} \cap N \notin V[\dot{G}_\delta]$”.

We finally verify that $(1_{\mathbb{P}}, q^*) \Vdash_{\mathbb{M}}$ “$T_\beta \subseteq \dot{E}$”. Suppose for the sake of contradiction that there are $(p^{**}, q^{**}) \le_{\mathbb{M}} (1_{\mathbb{P}}, q^*)$ and $m < \omega$ such that $(p^{**}, q^{**}) \Vdash_{\mathbb{M}}$

"$x_m \notin \dot{E}$". Fix $n < \omega$ such that $m < n$ and $p^{**} \cap N = p_n$. By construction, we know that $(\hat{p}_{n,0}, q_n) \Vdash_{\mathbb{M}}$ "$x_m \restriction \mathrm{ht}(q_n) \in \dot{E}$", and hence $(\hat{p}_{n,0}, q_n) \Vdash_{\mathbb{M}}$ "$x_m \in \dot{E}$". Let $r = \hat{p}_{n,0} \wedge p^{**}$. Then (r, q^{**}) forces both "$x_m \in \dot{E}$" and "$x_m \notin \dot{E}$". This contradiction concludes the proof of the theorem. □

Theorem 3.17. *Suppose that T is a free Suslin tree and κ is supercompact. Then, in $V[\mathbb{M}]$, the following statements hold:*

(1) $\kappa = \omega_2$;
(2) T is an almost Kurepa Suslin tree;
(3) GMP.

Proof. Given the previous results from this section, proofs of all three assertions follow standard paths; we provide details for completeness.

Let G be $\mathbb{M}$-generic over V. By a standard counting argument using the inaccessibility of κ, $\mathbb{M}$ has the κ-cc in V, and hence κ remains a cardinal in $V[G]$. Moreover, by Theorem 3.13, $\mathbb{M}$ has the ω_1-approximation property in V, and therefore $(\omega_1)^V = (\omega_1)^{V[G]}$. Thus, to prove (1), it suffices to prove that $V[G] \models$ "$|\gamma| = \omega_1$" for all uncountable $\gamma < \kappa$.

To this end, fix an uncountable $\gamma < \kappa$. By increasing it if necessary, assume that $\gamma \in \Sigma(\kappa)$. It is readily shown that the map $\pi_\gamma : \mathbb{M} \to \mathbb{P}_\gamma * \dot{\mathbb{A}}(T)$ defined by setting $\pi_\gamma(p, q) = (p \restriction \gamma, q(\gamma))$ for all $(p, q) \in \mathbb{M}$ is a projection.[2] Therefore, G induces a $\mathbb{P}_\gamma * \dot{\mathbb{A}}(T)$-generic filter $G_{0,\gamma} * G_{1,\gamma}$ over V. In $V[G_{0,\gamma}]$, we have $2^\omega \geq |\gamma|$. Therefore, by Proposition 2.26 applied in $V[G_{0,\gamma}]$, we have $|\gamma| = \omega_1$ in $V[G_{0,\gamma} * G_{1,\gamma}]$ and hence in $V[G]$ as well.

To prove (2), first apply Theorem 3.13 to conclude that T remains a Suslin tree in $V[G]$. For each $(p, q) \in G$ and $\gamma \in \mathrm{dom}(q)$, let $q(\gamma)_G$ denote the interpretation of $q(\gamma)$ in $V[G]$. For each $\gamma \in \Sigma(\kappa)$, let $g_\gamma = \bigcup\{q(\gamma)_G \mid (p, q) \in G \text{ and } \gamma \in \mathrm{dom}(q)\}$. Then g_γ is an automorphism of T. Moreover, a routine density argument shows that the sequence of automorphisms $\langle g_\gamma \mid \gamma \in \Sigma(\kappa)\rangle$ is pairwise *almost disjoint*, i.e., for all distinct $\gamma, \delta \in \Sigma(\kappa)$ and all $t \in T$, there is $t' \in T_t$ such that $g_\gamma(t') \neq g_\delta(t')$.

If we now force with T over $V[G]$, we add a generic cofinal branch b through T. In $V[G][b]$, for all $\gamma \in \Sigma(\kappa)$ let $b_\gamma = \{g_\gamma(t) \mid t \in b\}$. Then each b_γ is a cofinal branch through T and, by the fact that $\langle g_\gamma \mid \gamma \in \Sigma(\kappa)\rangle$ is pairwise almost disjoint and a routine density argument, we have $b_\gamma \neq b_\delta$ for all distinct $\gamma, \delta \in \Sigma(\kappa)$. Since T has the ccc in $V[G]$, and therefore $\kappa = \omega_2^{V[G][b]}$, it follows that T is a Kurepa tree in $V[G][b]$ and is therefore an almost Kurepa Suslin tree in $V[G]$.

It remains to verify that GMP holds in $V[G]$. To this end, working in $V[G]$, fix a regular uncountable cardinal $\theta \geq \kappa$ and a club C in $\mathscr{P}_\kappa H(\theta)$. We must find an ω_1-guessing model in C. Recall that, given a function $f : [H(\theta)]^2 \to \mathscr{P}_\kappa H(\theta)$, we let C_f denote the set

$$\{X \in \mathscr{P}_\kappa H(\theta) \mid \forall u \in [X]^2 \ f(u) \subseteq X\}.$$

Using Menas's characterization of two-cardinal club filters [18], we can fix a function $f : [H(\theta)]^2 \to \mathscr{P}_\kappa H(\theta)$ such that $C_f \subseteq C$.

Back in V, fix an elementary embedding $j : V \to M$ witnessing that κ is $|H(\theta)|$-supercompact. Note that, in M, $j(\mathbb{M})$ is defined in the same way that $\mathbb{M}$ is defined in V, but with length $j(\kappa)$, and that $j(\mathbb{M})_\kappa = \mathbb{M}$. Moreover, $j \restriction \mathbb{M}$ is the identity map

[2]Here we are adopting the convention that $q(\gamma) = 1_{\dot{\mathbb{A}}(T)}$ if $\gamma \notin \mathrm{dom}(q)$.

and, by Theorem 3.13, in $M[G]$, $j(\mathbb{M})/G = j(\mathbb{M})_{\kappa,j(\kappa)}$ has the ω_1-approximation property. Let H be $j(\mathbb{M})/G$-generic over $V[G]$. Then, in $V[G*H]$, we can lift j to $j : V[G] \to M[G*H]$. Note that, in $M[G*H]$, $j(C)$ is a club in $\mathscr{P}_{j(\kappa)}H(j(\theta))$ and $C_{j(f)} = j(C_f) \subseteq j(C)$.

Let $N = j``H(\theta)^{V[G]}$. By the closure of $M$ and the fact that $N$ is definable from $j``H(\theta)^V$ and $G*H$, it follows that $N \in M[G*H]$. Moreover, $M[G*H] \models N \in \mathscr{P}_{j(\kappa)}H(j(\theta))$. We claim that $N \in C_{j(f)}$. To this end, fix $u \in [N]^2$, and find $a, b \in H(\theta)^{V[G]}$ such that $u = j(\{a,b\})$. Then $j(f)(u) = j(f(\{a,b\}))$. Since $f(\{a,b\}) \in \mathscr{P}_\kappa H(\theta)^V[G]$ and $\mathrm{crit}(j) = \kappa$, we have $j(f(\{a,b\})) = j``f(\{a,b\}) \subseteq N$. Thus, $N \in C_{j(f)}$, and hence $N \in j(C)$.

We next argue that, in $M[G*H]$, N is an ω_1-guessing model. To this end, fix in $M[G*H]$ a $z \in N$ and a $d \subseteq z$ such that, for every $x \in \mathscr{P}_{\omega_1} z \cap N$, we have $d \cap x \in N$. Let $z' \in H(\theta)^{V[G]}$ be such that $j(z') = z$, and let $e' = \{a \in H(\theta)^{V[G]} \mid j(a) \in d\}$. Then $e' \subseteq z'$, and $e' \in M[G*H]$.

Claim 3.18. *For all $x' \in (\mathscr{P}_{\omega_1} z')^{V[G]}$, we have $e' \cap x' \in M[G]$.*

Proof. Fix $x' \in (\mathscr{P}_{\omega_1} z')^{V[G]}$. Let $x = j(x') = j``x' \in N$. By the choice of $d$, we know that $x \cap d \in N$. We can therefore find $y' \in H(\theta)^{V[G]}$ such that $x \cap d = j(y') = j``y'$. But then, unraveling the definitions, we have $e' \cap x' = y' \in V[G]$. Since $H(\theta)^{M[G]} = H(\theta)^{V[G]}$, the desired conclusion follows. □

Since $j(\mathbb{M})/G$ has the ω_1-approximation property in $M[G]$, it follows that $e' \in M[G]$; in particular, $e' \in H(\theta)^{V[G]}$. Let $e = j(e')$. Then $e \in N$ and $e \cap N = j``e' = d \cap N$. Thus, $N \in j(C)$ is indeed an ω_1-guessing model in $M[G*H]$. By elementarity, in $V[G]$ there exists an ω_1-guessing model in C, and hence GMP holds in $V[G]$. □

The following corollary will now complete the proof of Theorem A.

Corollary 3.19. *Assume* $\mathrm{Con}(\mathsf{ZFC} +$ *there exists a supercompact cardinal*)*. Then it is consistent that* GMP *holds but is destructible by a ccc forcing of cardinality* ω_1.

Proof. Start with a model V of ZFC containing a supercompact cardinal κ. Since, for example, the classical construction of a Suslin tree from $\diamondsuit$ yields a free Suslin tree, we may assume that there exists a free Suslin tree T in V. Then, by Theorem 3.17, in $V[\mathbb{M}]$, GMP holds and T is an almost Kurepa Suslin tree. In particular, T is a ccc forcing of cardinality ω_1 such that, after forcing with T, there exists a Kurepa tree. Since GMP implies the failure of the Kurepa Hypothesis, it follows that $\Vdash_T \neg\mathsf{GMP}$. □

We end this section with a sketch of the proof of Corollary 1.1.

Sketch of proof of Corollary 1.1. Suppose that κ is an inaccessible cardinal. As in the proof of Corollary 3.19, we can fix a free Suslin tree T in V, and let $\mathbb{M} = \mathbb{M}^T_\kappa$. We argue that $V[\mathbb{M}]$ satisfies the conclusion of the corollary. Given the proof of Theorem A above, it suffices to show that $\neg\mathsf{wKH}$ holds in $V[\mathbb{M}]$. To this end, let G be $\mathbb{M}$-generic over V, and let $S \in V[G]$ be a tree of height and size ω_1. We must show that, in $V[G]$, S has at most ω_1-many cofinal branches.

Since κ is inaccessible, we know that $\mathbb{M}$ has the κ-cc in V, and $\kappa = \omega_2$ in $V[G]$. We can therefore find a limit ordinal $\delta < \kappa$ such that $S \in V[G_\delta]$. Recall that, given a tree R, $[R]$ denotes the set of cofinal branches through R. Since $(2^{\omega_1})^{V[G_\delta]} < \kappa$, we have $([S])^{V[G_\delta]} < \kappa$. By Theorem 3.13, $\mathbb{M}_{\delta,\kappa}$ has the ω_1-approximation property in $V[G_\delta]$, and hence $[S]^{V[G]} = [S]^{V[G_\delta]}$. Since $\kappa = \omega_2$ in $V[G]$, it follows that S has at most ω_1-many cofinal branches in $V[G]$, as desired. □

4. Weak Kurepa trees

In [4], Cox and Krueger show that GMP implies that there are no weak Kurepa trees at ω_1. In contrast to their result, we show in [16] that $\mathsf{GMP}^{\omega_2}(\omega_2)$ does not imply that there are no weak Kurepa trees at ω_1. In fact, we construct a model in which $\mathsf{GMP}^{\omega_2}(\omega_2)$ holds and yet there exists not only a *weak* Kurepa tree, but even a Kurepa tree at ω_1. In this section we refine our result and show that the existence of a weak Kurepa tree at ω_1 is consistent with $\neg\mathsf{KH}$ and $\mathsf{GMP}^{\omega_2}(\omega_2)$, thus proving Theorem B.

4.1. **Forcing for adding a weak Kurepa tree.** We begin this section by introducing a forcing notion to add a weak Kurepa tree, which is a variation on the standard forcing for adding a Kurepa tree due to Stewart [21].

Definition 4.1. Let λ be an uncountable ordinal.[3] We define a poset $\mathbb{K}_\lambda$ which adds a tree with size and height ω_1 with $|\lambda|$-many cofinal branches. Conditions $q \in \mathbb{K}_\lambda$ are pairs (T_q, f_q) such that

- there is $\eta_q < \omega_1$ such that T_q is a normal, infinitely splitting subtree of ${}^{<\eta_q+1}\omega_1$ of size at most ω_1;
- f_q is a countable partial function from λ to $T_q \cap {}^{\eta_q}\omega_1$.

If $q_0, q_1 \in \mathbb{K}_\lambda$, then $q_1 \leq q_0$ if and only if

- $\eta_{q_1} \geq \eta_{q_0}$;
- $T_{q_1} \cap {}^{<\eta_{q_0}+1}\omega_1 = T_{q_0}$;
- $\mathrm{dom}(f_{q_1}) \supseteq \mathrm{dom}(f_{q_0})$;
- for all $\alpha \in \mathrm{dom}(f_{q_0})$, $f_{q_1}(\alpha) \supseteq f_{q_0}(\alpha)$.

We also include the pair $(\emptyset, \emptyset)$ in $\mathbb{K}_\lambda$ as $1_{\mathbb{K}_\lambda}$.

It is easy to verify that $\mathbb{K}_\lambda$ is separative, and the proof of Lemma 4.8 below will show that it adds a tree of height and size ω_1 with $|\lambda|$-many branches. Note that $\mathbb{K}_\lambda$ is not ω_2-cc: in fact, it collapses 2^{ω_1} to ω_1 since we can code subsets of ω_1 in the ground model into the levels of the generic tree added by $\mathbb{K}_\lambda$. Therefore if $\lambda \leq 2^{\omega_1}$, then the generic tree added by $\mathbb{K}_\lambda$ will not be a weak Kurepa tree.

Lemma 4.2. *Let λ be an uncountable ordinal and μ be a regular cardinal such that $\mu > 2^{\omega_1}$ and $\mu > \gamma^\omega$ for all $\gamma < \mu$. Then $\mathbb{K}_\lambda$ is μ-Knaster. In particular $\mathbb{K}_\lambda$ is $(2^{\omega_1})^+$-Knaster.*

Proof. Let a set of conditions $\{q_\alpha = (T_\alpha, f_\alpha) \in \mathbb{K}_\lambda \mid \alpha < \mu\}$ be given. Since $\mu > 2^{\omega_1}$ is regular and there are only 2^{ω_1}-many possibilities for T_α, there is a tree $T \subseteq {}^{<\eta+1}\omega_1$ of countable height and $I \subseteq \mu$ of size μ such that $T_\alpha = T$ for all $\alpha \in I$.

Since $\gamma^\omega < \mu$ for all $\gamma < \mu$, there is $I' \subseteq I$ of size μ such that the set $\{\mathrm{dom}(f_\alpha) \mid \alpha \in I'\}$ forms a Δ-system with root $a \subseteq \lambda$. Since a is at most countable, there are at most $2^\omega < \mu$ many functions from a to T_η and therefore there is a

[3]In practice, λ will typically be a cardinal, but we will sometimes need to consider λ of the form $j(\kappa)$, where $j : V \to M$ is an elementary embedding with critical point κ, in which case λ is a cardinal in M but may not be a cardinal in V.

countable f from a to T_η and $J \subseteq I'$ of size μ such $f = f_\alpha \cap f_\beta$ for all $\alpha \neq \beta \in J$. Then all conditions in $\{q_\alpha \mid \alpha \in J\}$ are compatible. □

Lemma 4.3. *Let λ be an uncountable ordinal. Then $\mathbb{K}_\lambda$ is ω_1-closed.*

Proof. Let $\langle q_n \mid n < \omega\rangle$ be a decreasing sequence from $\mathbb{K}_\lambda$. To avoid trivialities, assume that the sequence $\langle \eta_{q_n} \mid n < \omega\rangle$ is strictly increasing. Let $\eta := \sup\{\eta_{q_n} \mid n < \omega\}$. We will construct a lower bound q for $\langle q_n \mid n < \omega\rangle$ such that $\eta_q = \eta$. Let $T = \bigcup_{n<\omega} T_{q_n}$. Note that T is a normal tree of height η. To define T_q, first note that there are countably many branches through T that we are obliged to extend because of the functions $\{f_{q_n} \mid n < \omega\}$. Namely, let $a = \bigcup_{n<\omega} \mathrm{dom}(f_{q_n})$ and, for each $\alpha \in a$, let

$$b_\alpha := \bigcup\{f_{q_n}(\alpha) \mid n < \omega \wedge \alpha \in \mathrm{dom}(f_{q_n})\}.$$

Then each b_α is a cofinal branch through T, and we are obliged to put b_α in T_q.

Then we need to ensure that T_q will be a normal, infinitely splitting subtree of ${}^{<\eta+1}\omega_1$; namely for every node in T we need to add a node above it on level η. For every $t \in T$ let C_t be a cofinal branch in T such that $t \in C_t$. Note that such a cofinal branch exists since T is a normal tree of countable height. Let $c_t = \bigcup C_t$, then $c_t \in {}^\eta\omega_1$ and extends t.

At the end, set $T_q := T \cup \{b_\alpha \mid \alpha \in a\} \cup \{c_t \mid t \in T\}$. Let f_q be such that $\mathrm{dom}(f_q) = a$ and, for all $\alpha \in a$, $f_q(a) = b_\alpha$. □

Lemma 4.4. *Let $\lambda < \kappa$ be uncountable ordinals. Then there is a projection from $\mathbb{K}_\kappa$ to $\mathbb{K}_\lambda$.*

Proof. We define π from $\mathbb{K}_\kappa$ to $\mathbb{K}_\lambda$ by letting $\pi(T_q, f_q) = (T_q, f_q \restriction \lambda)$ for all $q \in \mathbb{K}_\kappa$. It is routine to verify that π is order-preserving and that $\pi(1_{\mathbb{K}_\kappa}) = 1_{\mathbb{K}_\lambda}$.

Let $q \in \mathbb{K}_\kappa$ and $r \in \mathbb{K}_\lambda$ be such that $r \leq \pi(q) = (T_q, f_q \restriction \lambda)$. Define $r' \leq q$ by first letting $T_{r'} = T_r$ (note that T_r is an end-extension of T_q). Let $\mathrm{dom}(f_{r'}) = \mathrm{dom}(f_q) \cup \mathrm{dom}(f_r)$ and define

- $f_{r'}(\alpha) = f_r(\alpha)$ for every $\alpha \in \mathrm{dom}(f_r)$ and
- $f_{r'}(\alpha) = \tau$, where $\tau \in (T_{r'})_{\eta_r}$ with $\tau \supseteq f_q(\alpha)$, for every $\alpha \in \mathrm{dom}(f_q) \setminus \mathrm{dom}(f_r)$.

It is easy to check that $\pi(r') = r$. Therefore π is a projection. □

Assume that $\lambda < \kappa$ are uncountable ordinals. Let H be a $\mathbb{K}_\lambda$-generic filter, $T_H = \bigcup\{T_q \mid q \in H\}$, and let $\mathbb{K}_\kappa/H = \{r \in \mathbb{K}_\kappa \mid \pi(r) \in H\}$ be the quotient given by H and the projection π. Then $\mathbb{K}_\kappa$ is forcing equivalent to a two step iteration $\mathbb{K}_\lambda * \mathbb{K}_\kappa/H$. It is easy to see that in $V[H]$, $\mathbb{K}_\kappa/H$ is forcing equivalent to the forcing notion $\mathbb{K}_{\lambda,\kappa}$, where conditions in $\mathbb{K}_{\lambda,\kappa}$ are pairs $r = (\eta_r, f_r)$ such that f_r is a countable partial function from $\kappa \setminus \lambda$ to $(T_H)_{\eta_r}$ and $r \leq q$ if and only if $\eta_r \geq \eta_q$, $\mathrm{dom}(f_q) \subseteq \mathrm{dom}(f_r)$, and $f_q(\alpha) \subseteq f_r(\alpha)$ for all $\alpha \in \mathrm{dom}(f_q)$.

Lemma 4.5. *Let $\lambda < \kappa$ be uncountable ordinals and μ be a regular cardinal such that $\mu > 2^{\omega_1}$ and $\mu > \gamma^\omega$ for all $\gamma < \mu$. Let H be a $\mathbb{K}_\lambda$-generic filter. Then $\mathbb{K}_\kappa/H$ is μ-Knaster in $V[H]$.*

Proof. Since $\mu > 2^{\omega_1}$ is regular and $\mu > \gamma^\omega$ for all $\gamma < \mu$, then $\mathbb{K}_\lambda$ is μ-Knaster and hence μ is still a regular cardinal in $V[H]$. Moreover since $\mathbb{K}_\lambda$ is ω_1-closed, $\mu > \gamma^\omega$ still holds for all $\gamma < \mu$ in $V[H]$.

Work in $V[H]$, where $\mathbb{K}_\kappa/H$ is forcing equivalent to the forcing notion $\mathbb{K}_{\lambda,\kappa}$ isolated above. Let a set of conditions $\{q_\alpha = (\eta_\alpha, f_\alpha) \in \mathbb{K}_{\lambda,\kappa} \mid \alpha < \mu\}$ be given. Since $\mu > 2^{\omega_1}$ in V, $\mu > \omega_1$ in $V[H]$ and there are only ω_1-many possibilities for η_α; therefore there is an η and an $I \subseteq \mu$ of size μ such that $\eta_\alpha = \eta$ for all $\alpha \in I$.

Since $\gamma^\omega < \mu$ for all $\gamma < \mu$, there is $I' \subseteq I$ of size μ such that the set $\{\mathrm{dom}(f_\alpha) \mid \alpha \in I'\}$ forms a Δ-system with root $a \subseteq \lambda$. Since a is at most countable, there at most $\omega_1^\omega < \mu$ many functions from a to T_η and therefore there is a countable f from a to T_η and $J \subseteq I'$ of size μ such $f = f_\alpha \cap f_\beta$ for all $\alpha \neq \beta \in J$. Then all conditions in $\{q_\alpha \mid \alpha \in J\}$ are compatible. □

Lemma 4.6. *Let $\lambda < \kappa$ be uncountable ordinals and H be a $\mathbb{K}_\lambda$-generic filter. Then $\mathbb{K}_\kappa/H$ is ω_1-distributive in $V[H]$.*

Proof. The forcing $\mathbb{K}_\kappa$ is forcing equivalent to a two step iteration $\mathbb{K}_\lambda * \mathbb{K}_\kappa/\dot{H}$. Since $\mathbb{K}_\kappa$ is ω_1-closed by Lemma 4.3, $\mathbb{K}_\kappa/H$ cannot add new countable sequences of ordinals over $V[H]$, hence it is ω_1-distributive in $V[H]$. □

We fix the following notation: for $r \in \mathbb{K}_\kappa$ and $\lambda < \kappa$, let $r \restriction \lambda$ denote $(T_r, f_r \restriction \lambda)$, i.e. $r \restriction \lambda = \pi(r)$ for the projection π defined in Lemma 4.4.

Lemma 4.7. *Let $\lambda < \kappa$ be uncountable ordinals and $\dot{H}$ be a canonical $\mathbb{K}_\lambda$-name for the $\mathbb{K}_\lambda$-generic filter. Let $q \in \mathbb{K}_\lambda$ and $r \in \mathbb{K}_\kappa$. Then the following are equivalent.*

(i) $q \Vdash r \in \mathbb{K}_\kappa/\dot{H}$;
(ii) $q \leq r \restriction \lambda$.

Proof. To see that (i) implies (ii), note that if $q \nleq r \restriction \lambda$, then by the separativity of $\mathbb{K}_\lambda$ there is $q' \leq q$ which is incompatible with $r \restriction \lambda$, and hence q does not force r into $\mathbb{K}_\kappa/\dot{H}$. The other direction is clear, since $q \leq r \restriction \lambda$ means that $q \leq \pi(r)$. □

Lemma 4.8. *Assume* GCH *and let $\lambda > \omega_2$ be a cardinal. Let H be a $\mathbb{K}_\lambda$-generic filter over V. Then the generic tree $T_H = \bigcup\{T_q \mid q \in H\}$ is a weak Kurepa tree with λ-many cofinal branches.*

Proof. Since $\mathbb{K}_\lambda$ is ω_1-closed, ω_1 is preserved by $\mathbb{K}_\lambda$ and the generic tree $T_H = \bigcup\{T_q \mid q \in H\}$ is thus a tree with height and size ω_1. By a standard density argument, T_H has λ-many cofinal branches in $V[H]$. Since GCH holds in the ground model, $\mathbb{K}_\lambda$ is ω_3-Knaster by Lemma 4.2, hence all cardinals greater than ω_2 are preserved (recall that 2^{ω_1} is always collapsed); in particular λ is preserved. Since $\lambda > \omega_2$ in the ground model, $\lambda > \omega_1$ in $V[H]$; therefore the generic tree T_H is a weak Kurepa tree in $V[H]$. □

4.2. **The consistency of wKH with ¬KH and $\mathsf{GMP}^{\omega_2}(\omega_2)$.** Before proving the main theorem of this section, we show that no Kurepa trees exist in the generic extension by the product of the Mitchell forcing (as defined in Subsection 2.3) and the forcing for adding a weak Kurepa tree.

Theorem 4.9. *Suppose that κ is an inaccessible cardinal. Then, in the generic extension by $\mathbb{M}(\omega,\kappa) \times \mathbb{K}_\kappa$, there are no Kurepa trees.*

Proof. Let κ be an inaccessible cardinal, and assume that GCH holds.

First note that it is enough to show that there are no Kurepa trees in the generic extension by $\mathrm{Add}(\omega,\kappa) \times \mathbb{Q} \times \mathbb{K}_\kappa$, where $\mathbb{Q}$ is the term forcing of the Mitchell forcing $\mathbb{M}(\omega,\kappa)$ (see Remark 2.13 for more details; formally, $\mathbb{Q}$ is the set of all conditions

$(\emptyset, q) \in \mathbb{M}(\omega, \kappa)$, with the order inherited from $\mathbb{M}(\omega, \kappa)$). To see this, first observe that $\mathrm{Add}(\omega, \kappa) \times \mathbb{Q} \times \mathbb{K}_\kappa$ is forcing equivalent to $(\mathbb{M}(\omega, \kappa) * \mathbb{S}) \times \mathbb{K}_\kappa$ for some quotient forcing $\mathbb{S}$ by Remark 2.13(4). This in turn is forcing equivalent to $(\mathbb{M}(\omega, \kappa) \times \mathbb{K}_\kappa) * \mathbb{S}$. Moreover, $\mathbb{S}$ does not collapse any cardinals over $V[\mathbb{M}(\omega, \kappa) \times \mathbb{K}_\kappa]$, since it is easily checked that the cardinals are the same in both extensions; i.e. ω_1 and all cardinals greater or equal to κ are preserved in both extensions and cardinals between ω_1 and κ are collapsed in both extensions. Hence, any Kurepa tree in $V[\mathbb{M}(\omega, \kappa) \times \mathbb{K}_\kappa]$ remains a Kurepa tree in $V[\mathrm{Add}(\omega, \kappa) \times \mathbb{Q} \times \mathbb{K}_\kappa]$.

Let $F \times G \times H$ be $\mathbb{P} \times \mathbb{Q} \times \mathbb{K}_\kappa$-generic over V, where $\mathbb{P}$ denotes $\mathrm{Add}(\omega, \kappa)$. Assume that S is an ω_1-tree in $V[F][G][H]$. Since S has size ω_1 and $\mathbb{P} \times \mathbb{Q} \times \mathbb{K}_\kappa$ is κ-Knaster, there is a nice $\mathbb{P} \times \mathbb{Q} \times \mathbb{K}_\kappa$-name $\dot{S}$ of size less than κ for S. Since $\dot{S}$ has size less than κ, $\dot{S}$ is a $\mathbb{P}_\theta \times \mathbb{Q}_\theta \times \mathbb{K}_\theta$-name for some regular cardinal $\theta < \kappa$, where $\mathbb{P}_\theta = \mathrm{Add}(\omega, \theta)$ and $\mathbb{Q}_\theta = \{(\emptyset, q \restriction \theta) \,|\, (\emptyset, q) \in \mathbb{Q}\}$. Let F_θ denote the $\mathbb{P}_\theta$-generic over V determined by F; i.e. $F_\theta = \{p \restriction \theta \,|\, p \in F\}$, G_θ denote the $\mathbb{Q}_\theta$-generic over $V[F_\theta]$ determined by G; i.e. $G_\theta = \{(\emptyset, q \restriction \theta) \,|\, (\emptyset, q) \in G\}$, and let H_θ denote the $\mathbb{K}_\theta$-generic filter over $V[F_\theta][G_\theta]$ determined by H and π, where π is the projection from $\mathbb{K}_\kappa$ to $\mathbb{K}_\theta$ from Lemma 4.4. The ω_1-tree S is an element of $V[F_\theta][G_\theta][H_\theta]$ and has at most $(2^{\omega_1})^{V[F_\theta][G_\theta][H_\theta]}$-many cofinal branches here, which is less than κ, since κ is still inaccessible in $V[F_\theta][G_\theta][H_\theta]$. We show that the quotient forcing $\mathbb{P}^\theta \times \mathbb{Q}^\theta \times \mathbb{K}_\kappa / H_\theta$ cannot add a cofinal branch to S over $V[F_\theta][G_\theta][H_\theta]$. Here $\mathbb{P}^\theta$ denotes $\mathrm{Add}(\omega, [\theta, \kappa))$ and $\mathbb{Q}^\theta$ denotes the quotient forcing $\mathbb{Q}/G_\theta$. It will follow that S has $<\kappa = \omega_2$-many cofinal branches in $V[F][G][H]$, i.e. it is not a Kurepa tree. Since S was an arbitrary ω_1-tree, this will conclude the proof of Theorem 4.9. The proof that there are no Kurepa trees in $V[F][G][H]$ is relatively long; for easier reading, it is divided into several claims (Claims 4.11 to 4.14).

First we observe that we can express the generic extension $V[F][G][H]$ as a forcing extension by $(\mathbb{P}_\theta \times \mathbb{Q}_\theta \times \mathbb{K}_\theta) * (\mathbb{K}_\kappa / \dot{H}_\theta \times \mathbb{Q}^\theta \times \mathbb{P}^\theta)$: Since $\mathbb{P}$ is forcing equivalent to $\mathbb{P}_\theta \times \mathbb{P}^\theta$, we can view F as a filter $F_\theta \times f$ which is $\mathbb{P}_\theta \times \mathbb{P}^\theta$-generic over V. Similarly, since $\mathbb{Q}$ is forcing equivalent to $\mathbb{Q}_\theta * \dot{\mathbb{Q}}^\theta$, we can view G as a filter $G_\theta * g$ which is generic over $V[F_\theta][f]$ and lastly since $\mathbb{K}_\kappa$ is forcing equivalent to $\mathbb{K}_\theta * \mathbb{K}_\kappa / \dot{H}_\theta$, we can view H as a filter $H_\theta * h$ which is generic over $V[F_\theta][f][G_\theta][g]$. Therefore $V[F][G][H]$ is equal to the generic extension $V[F_\theta][f][G_\theta][g][H_\theta][h]$.

Moreover, since $\mathbb{P}^\theta$ and $\mathbb{Q}_\theta * \mathbb{Q}^\theta$ are both in $V[F_\theta]$ and $G_\theta * g$ is generic over $V[F_\theta][f]$, we have $V[F_\theta][f][G_\theta][g][H_\theta][h] = V[F_\theta][G_\theta][g][f][H_\theta][h]$. Similarly, we can swap $g \times f$ and $H_\theta * h$ since both $\mathbb{Q}^\theta \times \mathbb{P}^\theta$ and $\mathbb{K}_\theta * \mathbb{K}_\kappa / H_\theta$ are in $V[F_\theta][G_\theta]$ and $H_\theta * h$ is generic over $V[F_\theta][H_\theta][g][f]$. It follows that

$$V[F_\theta][G_\theta][g][f][H_\theta][h] = V[F_\theta][G_\theta][H_\theta][h][g][f].$$

Since $\mathbb{P}^\theta = \mathrm{Add}(\omega, [\theta, \kappa))$, it is ω_1-Knaster in $V[F_\theta][G_\theta][H_\theta][h][g]$ and hence it cannot add new cofinal branches to an ω_1-tree by Fact 2.7. Therefore, any cofinal branch through S is already in $V[F_\theta][G_\theta][H_\theta][h][g]$.

Now, note that $\mathbb{Q}^\theta$ is ω_1-closed in $V[G_\theta]$ and also $\mathbb{K}_\kappa$ is ω_1-closed in $V[G_\theta]$ since $\mathbb{Q}_\theta$ and $\mathbb{K}_\kappa$ are both ω_1-closed in V; therefore $\mathbb{Q}^\theta$ is ω_1-closed in $V[G_\theta][H_\theta][h]$. The forcing $\mathbb{P}_\theta$ is ccc in $V[G_\theta][H_\theta][h]$ since it is just Cohen forcing for adding subsets of ω. Therefore we can apply Fact 2.11 to $\mathbb{P}_\theta$ and $\mathbb{Q}^\theta$ over $V[G_\theta][H_\theta][h]$ to show that $\mathbb{Q}^\theta$ cannot add new cofinal branches to an ω_1-tree over $V[F_\theta][G_\theta][H_\theta][h]$ and hence any cofinal branch through S is already in $V[F_\theta][G_\theta][H_\theta][h]$

To show that $\mathbb{K}_\kappa/H_\theta$ cannot add a cofinal branch to S over $V[F_\theta][G_\theta][H_\theta]$, we will work in $V[G_\theta]$. Note that $\mathbb{K}_\kappa/H_\theta$ is only ω_1-distributive in $V[G_\theta][H_\theta]$ and therefore we cannot use Fact 2.11. Instead, we will show that if there is a $\mathbb{P}_\theta \times \mathbb{K}_\theta * (\mathbb{K}_\kappa/\dot{H}_\theta)$-name $\dot{b}$ and a condition $(p^*, q^*, r^*) \in \mathbb{P}_\theta \times \mathbb{K}_\theta * (\mathbb{K}_\kappa/\dot{H}_\theta)$ which forces that $\dot{b}$ is a cofinal branch through $\dot{S}$ that is not in $V[G_\theta][\dot{F}_\theta][\dot{H}_\theta]$, then there is a condition $q \leq q^*$ such that (p^*, q) forces that $\dot{S}$ has an uncountable level. This is a contradiction, since we can assume that (p^*, q^*, r^*) is in $F_\theta \times H_\theta * h$ and that (p^*, q^*) forces that $\dot{S}$ is an ω_1-tree.

The proof is similar to the argument that an ω_1-closed forcing over a ccc forcing does not add cofinal branches to ω_1-trees: we will build a tree $\mathcal{T}$ of conditions in $\mathbb{K}_\kappa$ labeled by ${}^{<\omega}2$ and we will diagonalize over antichains in $\mathbb{P}_\theta$, but since we are working with the quotient $\mathbb{K}_\kappa/\dot{H}_\theta$ and we work in $V[G_\theta]$ and not in $V[G_\theta][H_\theta]$, we will guide this construction by a decreasing sequence of length ω of conditions in $\mathbb{K}_\theta$, which will force conditions from $\mathcal{T}$ into the quotient $\mathbb{K}_\kappa/\dot{H}_\theta$.

We make a natural identification and view $(\dot{S}, <_{\dot{S}})$ as a name for a tree with underlying set $\omega_1 \times \omega$, assuming that, for each $\delta < \omega_1$, the δ^{th} level of $\dot{S}$ is forced to be equal to $\{\delta\} \times \omega$. Hence the domain of the tree exists in $V[G_\theta]$, with the forcing deciding the ordering $<_{\dot{S}}$ on the tree. Going forward, for $\delta < \omega_1$, we let $\dot{S}_\delta$ denote $\{\delta\} \times \omega$.

Work in $V[G_\theta]$. Assume that $\dot{b}$ is a $\mathbb{P}_\theta \times \mathbb{K}_\theta * (\mathbb{K}_\kappa/\dot{H}_\theta)$-name and $(p^*, q^*, \dot{r}^*) \in \mathbb{P}_\theta \times \mathbb{K}_\theta * (\mathbb{K}_\kappa/\dot{H}_\theta)$ forces that $\dot{b}$ is a cofinal branch through $\dot{S}$ that is not in $V[G_\theta][\dot{F}_\theta][\dot{H}_\theta]$. By our assumption about $\dot{S}$, we can think of $\dot{b}$ as a name for an element of ${}^{\omega_1}\omega$. We can also assume that there is $r^* \in \mathbb{K}_\kappa$ such that $q^* \Vdash \dot{r}^* = r^*$. We will build by induction on ω the following objects:

- a decreasing sequence $\langle q_n \mid n < \omega \rangle$ of conditions in $\mathbb{K}_\theta$ with $q_0 \leq q^*$;
- a labeled tree $\mathcal{T} = \{r_s \mid s \in {}^{<\omega}2\}$ of conditions in $\mathbb{K}_\kappa$, with $r_s \leq r^*$ for all $s \in {}^{<\omega}2$;
- a maximal antichain A_s in $\mathbb{P}_\theta$ below p^* for all $s \in {}^{<\omega}2$;
- a strictly increasing sequence $\langle \gamma_n \mid n < \omega \rangle$ of ordinals below ω_1;

such that the following hold for all n and all $s \in {}^{<\omega}2$ of length n:

(a) $q_n = r_s \restriction \theta$; in particular $q_n \Vdash r_s \in \mathbb{K}_\kappa/\dot{H}_\theta$;
(b) for all $p \in A_s$ the conditions $(p, q_{n+1}, r_{s^\frown 0})$ and $(p, q_{n+1}, r_{s^\frown 1})$ decide $\dot{b}$ up to γ_{n+1} differently; i.e. there are $\delta \leq \gamma_{n+1}$ and $\tau_{s^\frown 0} \neq \tau_{s^\frown 1}$ in $\dot{S}_\delta$ such that $(p, q_n, r_{s^\frown 0}) \Vdash \dot{b}(\delta) = \tau_{s^\frown 0}$ and $(p, q_n, r_{s^\frown 1}) \Vdash \dot{b}(\delta) = \tau_{s^\frown 1}$;
(c) $(q_{|t|}, r_t) \leq (q_{|s|}, r_s)$ for all $s \subseteq t$ in ${}^{<\omega}2$.

Definition 4.10. Let us call a system as above, satisfying conditions (a)–(c), a *labeled system* for $\dot{b}$. We say just a labeled system if $\dot{b}$ is clear from the context.

A labeled system will be constructed below, using Claims 4.11, 4.12, and 4.13.

Claim 4.11. *For every $(p, q, r^0), (p, q, r^1) \leq (p^*, q^*, r^*)$ in $\mathbb{P}_\theta \times \mathbb{K}_\theta * (\mathbb{K}_\kappa/\dot{H}_\theta)$ and $\gamma' < \omega_1$ there are $\gamma' < \gamma < \omega_1$, $(p', q', u^0) \leq (p, q, r^0)$ and $(p', q', u^1) \leq (p, q, r^1)$ such that (p', q', u^0) and (p', q', u^1) decide $\dot{b}(\gamma)$ differently and the condition q' extends both $u^0 \restriction \theta$ and $u^1 \restriction \theta$. Moreover we can ensure that $T_{q'} = T_{u^0} = T_{u^1}$.*

Proof. Let $(p, q, r^0), (p, q, r^1) \leq (p^*, q^*, r^*)$ and $\gamma' < \omega_1$ be given. Fix for the moment a $\mathbb{P}_\theta \times \mathbb{K}_\theta$-generic $G_P \times G_K$ over $V[G_\theta]$ such that $(p, q) \in G_P \times G_K$, and work in $V[G_\theta][G_P \times G_K]$. Since $\dot{b}$ is forced by (p^*, q^*, r^*) to be a new cofinal branch through $\dot{S}$ and $(p^*, q^*) \in G_P \times G_K$, there are $\tilde{r}^0 \leq r^0$, $\tilde{r}^1 \leq r^1$ and $\gamma > \gamma'$ such that $\tilde{r}^0$ and $\tilde{r}^1$ decide $\dot{b}(\gamma)$ differently; i.e. there are $\tau^0 \neq \tau^1$ in $\dot{S}_\gamma$ such that $\tilde{r}^0 \Vdash \dot{b}(\gamma) = \tau^0$ and $\tilde{r}^1 \Vdash \dot{b}(\gamma) = \tau^1$.

Since $\tilde{r}^0$ and $\tilde{r}^1$ are in $\mathbb{K}_\kappa/G_K$, $\tilde{r}^0 \restriction \theta$ and $\tilde{r}^1 \restriction \theta$ are in G_K and hence they are compatible. Let q' be a common extension of $\tilde{r}^0 \restriction \theta$, $\tilde{r}^1 \restriction \theta$ and q such that, for each $i < 2$, we have $(q', \tilde{r}^i) \Vdash \dot{b}(\gamma) = \tau^i$ in $V[G_\theta][G_P]$. Since G_P is $\mathbb{P}_\theta$-generic over $V[G_\theta]$, there is a condition $p' \in G_P$ which forces this, and therefore $(p', q', \tilde{r}^0)$ and $(p', q', \tilde{r}^1)$ decide $\dot{b}(\gamma)$ differently over $V[G_\theta]$. Since we assume that $p \in G_P$ we can take p' such that it extends p.

Since q' extends both $\tilde{r}^0 \restriction \theta$ and $\tilde{r}^1 \restriction \theta$, $T_{q'}$ is an end-extension of both $T_{\tilde{r}^0}$ and $T_{\tilde{r}^1}$. Let us define $u^i = (T_{u^i}, f_{u^i}) \leq (T_{\tilde{r}^i}, f_{\tilde{r}^i}) = \tilde{r}^i$ for $i < 2$ as follows: set $T_{u^i} = T_{q'}$ and $\mathrm{dom}(f_{u^i}) = \mathrm{dom}(f_{q'}) \cup \mathrm{dom}(f_{\tilde{r}^i})$, and define

- $f_{u^i}(\delta) = f_{q'}(\delta)$ for every $\delta \in \mathrm{dom}(f_{q'})$ and
- $f_{u^i}(\delta) = \tau$, where $\tau \in (T_{q'})_{\eta_{q'}}$ with $\tau \supseteq f_{\tilde{r}^i}(\delta)$, for every $\delta \in \mathrm{dom}(f_{\tilde{r}^i}) \setminus \mathrm{dom}(f_{q'})$.

It is easy to see that q' still extends $u^i \restriction \theta$ for $i < 2$ and therefore it forces u^i into $\mathbb{K}_\kappa/\dot{H}_\theta$. Moreover, by the definition of u^i for $i < 2$, (q', u^i) extends $(q', \tilde{r}^i)$ and therefore (p', q', u^0) and (p', q', u^1) decide $\dot{b}(\gamma)$ differently. □

Claim 4.12. *For every $(q, r^0), (q, r^1) \leq (q^*, r^*) \in \mathbb{K}_\theta * (\mathbb{K}_\kappa/\dot{H}_\theta)$ and $\gamma' < \omega_1$ there are $\gamma' < \gamma < \omega_1$, a maximal antichain A in $\mathbb{P}_\theta$ below p^*, $(q', \bar{r}^0) \leq (q, r^0)$, and $(q', \bar{r}^1) \leq (q, r^1)$ such that for every $p \in A$, $(p, q', \bar{r}^0)$ and $(p, q', \bar{r}^1)$ decide $\dot{b}$ up to γ differently and the condition q' extends both $\bar{r}^0 \restriction \theta$ and $\bar{r}^1 \restriction \theta$.*

Proof. Let $(q, r^0), (q, r^1) \leq (q^*, r^*) \in \mathbb{K}_\theta * (\mathbb{K}_\kappa/\dot{H}_\theta)$ and $\gamma' < \omega_1$ be given. We proceed by induction of length α for some α less than ω_1 that will be determined during the construction, and we construct q', $\bar{r}^0$ and $\bar{r}^1$ such that $(q', \bar{r}^0)$ is a lower bound of a decreasing sequence $\langle (q'_\beta, r^0_\beta) \leq (q, r^0) \mid \beta < \alpha \rangle$ and $(q', \bar{r}^1)$ is a lower bound of a decreasing sequence $\langle (q'_\beta, r^1_\beta) \leq (q, r^1) \mid \beta < \alpha \rangle$. In the process, we also construct a maximal antichain $A = \{p_\beta \mid \beta < \alpha\}$ in $\mathbb{P}_\theta$, and γ will be a supremum of increasing sequence of ordinals $\langle \gamma_\beta < \omega_1 \mid \beta < \alpha \rangle$.

To begin, let p be an arbitrary condition below p^*. By Claim 4.11, there are $p_0 \leq p$, $q'_0 \leq q$, $r^0_0 \leq r^0$,$r^1_0 \leq r^1$ and $\gamma_0 > \gamma'$ such that (p_0, q'_0, r^0_0), (p_0, q'_0, r^1_0) decide $\dot{b}(\gamma_0)$ differently, q'_0 extends both conditions $r^0_0 \restriction \theta$ and $r^1_0 \restriction \theta$, and moreover $T_{q'_0} = T_{r^0_0} = T_{r^1_0}$.

Let $\alpha < \omega_1$ and assume that for all $\beta < \alpha$, we have constructed p_β, γ_β, q'_β, r^0_β, and r^1_β. If there is a condition p below p^* which is incompatible with all conditions in $\{p_\beta \mid \beta < \alpha\}$, we fix such a p and proceed as follows.

If $\alpha = \beta + 1$, by Claim 4.11, there are $p_\alpha \leq p$, $q'_\alpha \leq q'_\beta$, $r^0_\alpha \leq r^0_\beta$, $r^1_\alpha \leq r^1_\beta$ and $\gamma_\alpha > \gamma_\beta$ such that $(p_\alpha, q'_\alpha, r^0_\alpha)$, $(p_\alpha, q'_\alpha, r^1_\alpha)$ decide $\dot{b}(\gamma_\alpha)$ differently, q'_α extends both conditions $r^0_\alpha \restriction \theta$ and $r^1_\alpha \restriction \theta$, and moreover $T_{q'_\alpha} = T_{r^0_\alpha} = T_{r^1_\alpha}$.

If α is limit, we begin by constructing appropriate lower bounds of $\langle (q'_\beta, r^0_\beta) \leq (q, r^0) \mid \beta < \alpha \rangle$ and $\langle (q'_\beta, r^1_\beta) \leq (q, r^1) \mid \beta < \alpha \rangle$ such that the lower bounds will have the same first coordinate.[4] Let $q'_\beta = (T_\beta, f_\beta)$, $r^0_\beta = (S^0_\beta, g^0_\beta)$, and $r^1_\beta = (S^1_\beta, g^1_\beta)$

[4]Note that $\mathbb{K}_\theta * (\mathbb{K}_\kappa/\dot{H}_\theta)$ is ω_1-closed, therefore the lower bounds of $\langle (q'_\beta, r^0_\beta) \leq (q, r^0) \mid \beta < \alpha \rangle$ and $\langle (q'_\beta, r^1_\beta) \leq (q, r^1) \mid \beta < \alpha \rangle$ always exist; however, for our construction we need to ensure that the lower bounds have the same first coordinate.

for all $\beta < \alpha$. Note that at each step $\beta < \alpha$ of the construction we ensured that $T_\beta = S^0_\beta = S^1_\beta$, hence $T = \bigcup_{\beta<\alpha} T_\beta = \bigcup_{\beta<\alpha} S^0_\beta = \bigcup_{\beta<\alpha} S^1_\beta$

If $\mathrm{ht}(T)$ is a limit ordinal $\eta < \omega_1$ we define an end extension T' of T by one level. To define T', first note that there are countably many branches through T that we are obliged to extend because of the functions $\{f_\beta \mid \beta < \alpha\}$, $\{g^0_\beta \mid \beta < \alpha\}$, and $\{g^1_\beta \mid \beta < \alpha\}$. Namely, let $a = \bigcup_{\beta<\alpha} \mathrm{dom}(f_\beta)$ and, for each $\delta \in a$, let

$$b_\delta := \bigcup\{f_\beta(\delta) \mid \beta < \alpha \wedge \delta \in \mathrm{dom}(f_\beta)\}.$$

Then each b_δ is a cofinal branch through T, and we are obliged to put b_δ in T'.

Similarly for functions $\{g^i_\beta \mid \beta < \alpha\}$ for $i < 2$, let $c_i = \bigcup_{\beta<\alpha} \mathrm{dom}(g^i_\beta)$ and, for each $\delta \in c_i$, let

$$d^i_\delta := \bigcup\{g^i_\beta(\delta) \mid \beta < \alpha \wedge \delta \in \mathrm{dom}(g^i_\beta)\}.$$

Then each d^i_δ is a cofinal branch through T, and we are obliged to put d^i_δ in T'.

As in the proof of Lemma 4.3, for each $t \in T$, let E_t be a cofinal branch in T containing t, and let $e_t = \bigcup E_t$. Then set

$$T' = T \cup \{b_\delta \mid \delta \in a\} \cup \{d^0_\delta \mid \delta \in c_0\} \cup \{d^1_\delta \mid \delta \in c_1\} \cup \{e_t \mid t \in T\}.$$

Now let us define $\bar{q} = (T_{\bar{q}}, f_{\bar{q}})$ as follows: $T_{\bar{q}} = T'$, $\mathrm{dom}(f_{\bar{q}}) = a$ and, for all $\delta \in a$, $f_{\bar{q}}(\delta) = b_\delta$. Similarly define $u^i = (T_{u^i}, f_{u^i})$ for $i < 2$ as follows: $T_{u^i} = T'$, $\mathrm{dom}(f_{u^i}) = c^i$ and, for all $\delta \in c^i$, $f_{u^i}(\delta) = d^i_\delta$.

If $\mathrm{ht}(T)$ is a successor ordinal $\eta+1 < \omega_1$, then there is $\beta^* < \alpha$ such thet for all β between β^* and α, $T_\beta = T_{\beta^*} = T$ and hence $S^0_\beta = S^1_\beta = T_{\beta^*} = T$ for all β between β^* and α. Now, T is a normal tree and we can easily define lower bounds using T. Let us define $\bar{q} = (T_{\bar{q}}, f_{\bar{q}})$ as follows: $T_{\bar{q}} = T$, $\mathrm{dom}(f_{\bar{q}}) = \bigcup_{\beta<\alpha} \mathrm{dom}(f_\beta)$ and, for all $\delta \in \mathrm{dom}(f_{\bar{q}})$, $f_{\bar{q}}(\delta) = f_\beta(\delta)$, where $\beta \geq \beta^*$ is such that $\delta \in \mathrm{dom}(f_\beta)$. Analogously, we define $u^i = (T_{u^i}, f_{u^i})$ for $i < 2$ as follows: $T_{u^i} = T$, $\mathrm{dom}(f_{u^i}) = \bigcup_{\beta<\alpha} \mathrm{dom}(g^i_\beta)$ and, for all $\delta \in \mathrm{dom}(f_{u^i})$, $f_{u^i}(\delta) = g^i_\beta(\delta)$, where $\beta \geq \beta^*$ is such that $\delta \in \mathrm{dom}(g^i_\beta)$.

This finishes the construction of appropriate lower bounds $(\bar{q}, u^i)$ of $\langle (q'_\beta, r^i_\beta) \leq (q, r^i) \mid \beta < \alpha\rangle$ for $i < 2$. Now let γ be the supremum of $\{\gamma_\beta \mid \beta < \alpha\}$. By Claim 4.11, there are $p_\alpha \leq p$, $q'_\alpha \leq \bar{q}$, $r^0_\alpha \leq u^0$, $r^1_\alpha \leq u^1$ and $\gamma_\alpha > \gamma$ such that $(p_0, q'_\alpha, r^0_\alpha)$, $(p_0, q'_\alpha, r^1_\alpha)$ decide $\dot{b}(\gamma_\alpha)$ differently and q'_α extends both $r^0_\alpha \restriction \theta$ and $r^1_\alpha \restriction \theta$ and moreover $T_{q'_\alpha} = T_{r^0_\alpha} = T_{r^1_\alpha}$.

If there is no p below p^* which is incompatible with all conditions in $\{p_\beta \mid \beta < \alpha\}$ we stop the construction, set $A = \{p_\beta \mid \beta < \alpha\}$, and let γ be the supremum of $\langle \gamma_\beta \mid \beta < \alpha\rangle$. If $\alpha = \beta + 1$ for some $\beta < \omega$, we set $q' = q'_\beta$ and $\bar{r}^i = r^i_\beta$ for $i < 2$. If α is limit, we construct q' and $\bar{r}^i$, for $i < 2$, as $\bar{q}$ and u^i, for $i < 2$, in the limit case of the induction. It is readiliy verified that the objects thus constructed satisfy the conclusion of the claim. □

W are now ready to construct our labeled system. The construction is by induction on ω.

Let γ_0 be an arbitrary ordinal below ω_1 and let $(q_0, r_\emptyset)$ be (q^*, r^*). By Lemma 4.7, $q^* \leq r^* \restriction \theta$ since q^* forces r^* into $\mathbb{K}_\kappa / \dot{H}_\theta$ and $\mathbb{K}_\theta$ is separative. If $q^* \neq r^* \restriction \theta$, we can extend r^* appropriately to ensure condition (a); for more details, see Claim 4.13 below in the successor step of the construction.

Now fix $n < \omega$ and assume that we have constructed γ_n, q_n, and r_s for all $s \in {}^n2$. Let $\langle s_i \mid i < 2^n \rangle$ enumerate ${}^n2$. We describe how to construct γ_{n+1}, q_{n+1}, $A_{s\restriction n}$, and r_s for $s \in {}^{n+1}2$.

We proceed by induction on $2^n = m$. Let us start with s_0. By Claim 4.12, there are $q^0 \le q_n$, $r'_{s_0\frown 0}$,$r'_{s_0\frown 1} \le r_{s_0}$, a maximal antichain A_{s_0} in $\mathbb{P}_\theta$, and $\gamma^0 \ge \gamma_n$ such that for all $p \in A_{s_0}$, $(p, q^0, r'_{s_0\frown 0})$ and $(p, q^0, r'_{s_0\frown 1})$ decide $\dot{b}$ up to γ^0 differently and q^0 extends both $r'_{s_0\frown 0} \restriction \theta$ and $r'_{s_0\frown 1} \restriction \theta$.

Now fix $1 \le i < m$, and suppose that γ^{i-1} and q^{i-1} have been constructed. By Claim 4.12, there are $q^i \le q^{i-1}$, $r'_{s_i\frown 0}$, $r'_{s_i\frown 1} \le r'_{s_i}$, a maximal antichain A_{s_i} in $\mathbb{P}_\theta$, and $\gamma^i \ge \gamma^{i-1}$ such that for all $p \in A_{s_i}$, $(p, q^i, r'_{s_i\frown 0})$, $(p, q^i, r'_{s_i\frown 1})$ decide $\dot{b}$ up to γ^i differently and q^i extends both $r'_{s_i\frown 0} \restriction \theta$ and $r'_{s_i\frown 1} \restriction \theta$.

Let q_{n+1} be q^{m-1} and γ_{n+1} be γ^{m-1}. It follows by the construction that the objects q_{n+1}, γ_{n+1}, $r'_{s\frown j}$, for $j < 2$ and all $s \in {}^n2$ satisfy the desired conditions (b) and (c).

However, we have only ensured that q_{n+1} extends $r'_s \restriction \theta$ for $s \in {}^{n+1}2$, but not that they are equal as is required in condition (a). To ensure condition (a), we define appropriate extensions of r'_s for all $s \in {}^{n+1}2$.

Claim 4.13. *The objects constructed above can be extended to satisfy conditions (a)–(c) of a labeled system in Definition 4.10.*

Proof. Since q_{n+1} extends $r'_s \restriction \theta$ for all $s \in {}^{n+1}2$, $T_{q_{n+1}}$ is an end-extension of $T_{r'_s}$ for all $s \in {}^{n+1}2$. Let $\eta + 1$ be the height of $T_{q_{n+1}}$.

Now, we define extensions of r'_s for $s \in {}^{n+1}2$. For $s \in {}^{n+1}2$, let f_s be a function such that $\mathrm{dom}(f_s) = \mathrm{dom}(f_{r'_s}) \cap [\theta, \kappa)$ and for each $\alpha \in \mathrm{dom}(f_s)$, let $f_s(\alpha) \supseteq f_{r'_s}(\alpha)$ be some node of $T_{q_{n+1}}$ on level η. Set $r_s = (T_{q_{n+1}}, f_{q_{n+1}} \cup f_s)$ for $s \in {}^{n+1}2$. Clearly, r_s are conditions in $\mathbb{K}_\kappa$ for all $s \in {}^{n+1}2$ such that q_{n+1} is equal to $r_s \restriction \theta$, hence condition (a) holds. Since r_s extends r'_s for all $s \in {}^{n+1}2$, conditions (b) and (c) are still satisfied for q_{n+1}, γ_{n+1}, A_s, and $r_{s\frown j}$, for $j < 2$ and all $s \in {}^n2$. □

This completes our construction of a labeled system. Let γ be the supremum of $\langle \gamma_n \mid n < \omega \rangle$. To finish the proof of Theorem 4.9, we would like to find a lower bound q of the sequence $\langle q_n \mid n < \omega \rangle$ and a lower bound r_x of sequences $\langle r_{x\restriction n} \mid n < \omega \rangle$ for all $x \in {}^\omega2$ such that q forces r_x into the quotient $\mathbb{K}_\kappa / \dot{H}_\theta$ for every $x \in {}^\omega2$, thus ensuring that every (q, r_x) is a condition in $\mathbb{K}_\theta * (\mathbb{K}_\kappa / \dot{H}_\theta)$. If this is the case, then (p^*, q) will force over $V[G_\theta]$ that level γ of $\dot{S}$ has size 2^ω, as argued below.

We now construct the required conditions. Begin by letting $T^* = \bigcup_{n<\omega} T_{q_n}$ and $a = \bigcup_{n<\omega} \mathrm{dom}(f_{q_n})$. Let us assume that T^* is a normal tree with a limit height. If the height of T^* is a successor ordinal, then the construction of the appropriate lower bounds is analogous but simpler and we leave it as an exercise for the reader. Let $\eta < \omega_1$ denote the height of T^*. By condition (a), $T^* = \bigcup_{n<\omega} T_{r_{x\restriction n}}$ and $a = \bigcup_{n<\omega} \mathrm{dom}(f_{r_{x\restriction n}}) \cap \theta$ for all $x \in {}^\omega2$.

Note that if we want to ensure that a lower bound of $\langle q_n \mid n < \omega \rangle$ forces a lower bound of $\langle r_{x\restriction n} \mid n < \omega \rangle$, for some $x \in {}^\omega2$, into the quotient, we are obliged not only to extend all cofinal branches through T^* which are given by functions $\{f_{q_n} \mid n < \omega\}$ (recall the proof that $\mathbb{K}_\kappa$ is ω_1-closed), but also to extend all cofinal branches given by $\{f_{r_{x\restriction n}} \mid n < \omega\}$; otherwise it can happen that the lower bounds will be incompatible. Therefore if we want to ensure that a lower bound of $\langle q_n \mid n < \omega \rangle$ forces lower bounds of $\langle r_{x\restriction n} \mid n < \omega \rangle$ for $x \in {}^\omega2$ into the quotient, we are obliged to extend uncountably many cofinal branches of T^*. This can be done because the trees in the conditions can be wide.

To build suitable lower bounds, we will proceed similarly as in the proof of Lemma 4.3. First apply the construction in Lemma 4.3 to find q' which is a lower bound of $\langle q_n \mid n < \omega \rangle$ such that $T_{q'}$ has height $\eta + 1$ and $\mathrm{dom}(f_{q'}) = a$.

For each $x \in {}^{\omega}2$, let $a_x = \bigcup_{n<\omega} \mathrm{dom}(f_{r_{x\restriction n}}) \setminus \theta$. In order to ensure that q is compatible with a lower bound of $\langle r_{x\restriction n} \mid n < \omega \rangle$, we are obliged to extend cofinal branches given by $\{r_{x\restriction n} \mid n < \omega\}$. For $\alpha \in a_x$, let

$$d^x_\alpha := \bigcup \{ f_{r_{x\restriction n}}(\alpha) \mid n < \omega \wedge \alpha \in \mathrm{dom}(f_{r_{x\restriction n}}) \} \tag{1}$$

and let $D_x = \{ d^x_\alpha \mid \alpha \in a_x \}$. Moreover, let us define f_x to be a function with domain a_x such that for each $\alpha \in a_x$, $f_x(\alpha) = d^x_\alpha$.

At the end of the construction, set $T_q = T_{q'} \cup \bigcup_{x \in {}^{\omega}2} D_x$ and $f_q = f_{q'}$. Set $T_{r_x} = T_q$ and $f_{r_x} = f_{q'} \cup f_x$ for every $x \in {}^{\omega}2$.

Claim 4.14. *The following hold:*

(i) q is a condition in $\mathbb{K}_\theta$;
(ii) r_x is a condition in $\mathbb{K}_\kappa$ for all $x \in {}^{\omega}2$;
(iii) q forces r_x into the quotient $\mathbb{K}_\kappa / \dot{H}_\theta$, for all $x \in {}^{\omega}2$.

Proof. Note that T_q is a normal tree since $T_{q'}$ is normal and $T_q = T_{q'} \cup \bigcup_{x \in {}^{\omega}2} D_x$, hence q is a condition in $\mathbb{K}_\theta$ and also r_x are conditions in $\mathbb{K}_\kappa$ for all $x \in {}^{\omega}2$.

By the definition of r_x, $r_x \restriction \theta = q$ and hence q forces r_x into the quotient $\mathbb{K}_\kappa / \dot{H}_\theta$ for every $x \in {}^{\omega}2$. □

It is now straightforward to check that (p^*, q) forces over $V[G_\theta]$ that level γ of $\dot{S}$ has size 2^ω: Let $G_P \times G_K$ be a $\mathbb{P}_\theta \times \mathbb{K}_\theta$-generic over $V[G_\theta]$ which contains (p^*, q). Since q forces r_x into the quotient $\mathbb{K}/\dot{H}_\theta$ for all $x \in {}^{\omega}2$, r_x is in $\mathbb{K}/G_K$. For all $x \in {}^{\omega}2$, we take $r'_x \leq r_x$ which decides $\dot{b}(\gamma)$, i.e., $r'_x \Vdash \dot{b}(\gamma) = \tau_x$ for some $\tau_x \in \dot{S}_\gamma$. Now it is enough to show that for all $x \neq y \in {}^{\omega}2$, $\tau_x \neq \tau_y$. However, this follows from the construction of our labeled system. Let $x \neq y \in {}^{\omega}2$ be given, let $s = x \cap y$, and without loss of generality assume that x extends $s^\frown 0$ and y extends $s^\frown 1$. Now, since p^* is in G_P and A_s is a maximal antichain below p^* there is $p \in A_s$ which is in G_P. Since $(p, q_{n+1}, r_{s^\frown 0})$ and $(p, q_{n+1}, r_{s^\frown 1})$ decides $\dot{b}$ up to $\gamma_{n+1} < \gamma$ differently, there is $\delta < \gamma$ and $\tau_0 \neq \tau_1$ in $\dot{S}_\delta$ such that $(p, q_{n+1}, r_{s^\frown 0}) \Vdash \dot{b}(\delta) = \tau_0$ and $(p, q_{n+1}, r_{s^\frown 1}) \Vdash \dot{b}(\delta) = \tau_1$. Since $q \leq q_{n+1}$ and q is in G_K, (p, q_{n+1}) is in $G_P \times G_K$ and therefore $r_{s^\frown 0} \Vdash \dot{b}(\delta) = \tau_0$ and $r_{s^\frown 1} \Vdash \dot{b}(\delta) = \tau_1$ over $V[G_\theta][G_P][G_K]$. Since $r'_x \leq r_x \leq r_{s^\frown 0}$ and $r'_x \Vdash \dot{b}(\gamma) = \tau_x$, it holds $\tau_0 <_S \tau_x$. Similarly we can show that $\tau_1 <_S \tau_x$. Since $\tau_0 \neq \tau_1$ it follows that $\tau_x \neq \tau_y$. Therefore (p^*, q) forces over $V[G_\theta]$ that level γ of $\dot{S}$ has size 2^ω, hence (p^*, q) forces that $\dot{S}$ is not an ω_1-tree, which is a contradiction. This concludes the proof of Theorem 4.9. □

The following theorem will now establish Theorem B.

Theorem 4.15. *Suppose that there is a supercompact cardinal κ. Then there is a forcing extension in which*

(1) $2^\omega = \omega_2 = \kappa$;
(2) $\mathsf{GMP}^{\omega_2}(\omega_2)$ holds;
(3) there are no Kurepa trees, but there is a weak Kurepa tree.

Proof. For simplicity, assume that GCH holds in V. The generic extension is obtained by forcing with a product of the Mitchell forcing (as defined in Subsection 2.3) and the forcing for adding a weak Kurepa tree with κ-many branches: $\mathbb{M}(\omega,\kappa)\times\mathbb{K}_\kappa$, which we denote by $\mathbb{M}\times\mathbb{K}$ for simplicity. Recall from Subsection 2.3 that the set A we are using to define $\mathbb{M}(\omega,\kappa)$ is the set of inaccessible cardinals below κ. Let $G\times H$ be $\mathbb{M}\times\mathbb{K}$-generic over V. The Mitchell forcing preserves ω_1 and all cardinal greater than or equal to κ and forces $2^\omega=\kappa=\omega_2$. Since $\mathbb{K}$ is κ-Knaster in V, $\mathbb{K}$ is still κ-Knaster in $V[G]$ by Fact 2.9, and therefore it preserves all cardinals greater or equal to κ. It also preserves ω_1 over $V[G]$, since it is ω_1-closed in V and therefore it is ω_1-distributive in $V[G]$ by the standard product analysis of the Mitchell forcing $\mathbb{M}$. This finishes the proof of (1).

There is a weak Kurepa tree T in $V[H]$ with κ-many cofinal branches by the definition of $\mathbb{K}$. Since ω_1 is preserved by $\mathbb{K}\times\mathbb{M}$, T is still an ω_1-tree with $\kappa=\omega_2$-many cofinal branches in $V[H][G]=V[G][H]$.

We now establish $\mathsf{GMP}^{\omega_2}(\omega_2)$. Note that by Fact 2.6, $\mathsf{GMP}^{\omega_2}(\omega_2)$ is equivalent to $\mathsf{ISP}^{\omega_2}(\omega_2)$. To show that $\mathsf{ISP}^{\omega_2}(\omega_2)$ holds in $V[G][H]$ we use the lifting argument and analysis of the quotient as in [5]. Fix a cardinal $\lambda>\kappa$; by increasing it if necessary, assume that $\lambda^{<\lambda}=\lambda$. We will establish $\mathsf{ISP}^{\omega_2}(\omega_2,\lambda)$ in $V[G][H]$. To begin, let

$$j:V\to M \tag{2}$$

be a supercompact elementary embedding with critical point κ given by a normal ultrafilter U on $\mathscr{P}_\kappa(\lambda)$; i.e. $M\cong\mathrm{Ult}(V,U)$. We lift the elementary embedding j in two steps. Note that $j(\mathbb{M}(\omega,\kappa))=\mathbb{M}(\omega,j(\kappa))$ and recall that there is a projection from $\mathbb{M}(\omega,j(\kappa))$ to $\mathbb{M}(\omega,\kappa)$. Let us denote the quotient $\mathbb{M}(\omega,j(\kappa))/G$ by $Q_{\mathbb{M}}$ and let g be $Q_{\mathbb{M}}$-generic over $V[G][H]$. Then we can lift in $V[G][g]$ the embedding to $j:V[G]\to M[G][g]$. Now, $j(\mathbb{K}_\kappa)=\mathbb{K}_{j(\kappa)}$ and there is a projection from $\mathbb{K}_{j(\kappa)}$ to $\mathbb{K}_\kappa$ by Lemma 4.4. Let us denote the quotient $\mathbb{K}_{j(\kappa)}/H$ by $Q_{\mathbb{K}}$ and let h be $Q_{\mathbb{K}}$-generic over $V[G][H][g]$. We can lift the embedding in $V[G][H][g][h]$ further to

$$j:V[G][H]\to M[G][H][g][h]. \tag{3}$$

Let $D=\langle d_x\,|\,x\in\mathscr{P}_\kappa(\lambda)^{V[G][H]}\rangle$ be a κ-slender list in $V[G][H]$. We want to show that there is an ineffable branch b through D in $V[G][H]$. Let us consider the image of D under j:

$$j(\langle d_x\,|\,x\in\mathscr{P}_\kappa(\lambda)^{V[G][H]}\rangle)=\langle d'_y\,|\,y\in\mathscr{P}_{j(\kappa)}(j(\lambda))^{M[G][H][g][h]}\rangle.$$

The set $j``\lambda$ is a subset of $j(\lambda)$ of size $<j(\kappa)$. We define our ineffable branch $b:\lambda\to 2$ in $V[G][H][g][h]$ as follows:[5]

$$\text{for } \alpha<\lambda,\ b(\alpha)=d'_{j''\lambda}(j(\alpha)). \tag{4}$$

Before we prove the following claim, let us state some simple properties of the lifted embedding j. Since we assume $\lambda^{<\lambda}=\lambda$, $|H(\lambda)|=\lambda$, and this still holds in $V[G][H]$, i.e. $|H(\lambda)^{V[G][H]}|=\lambda$. In (3) we lifted an embedding generated by a normal ultrafilter U on $\mathscr{P}_\kappa(\lambda)$ and therefore $j``\lambda$ is an element of $j(C)$ for every club C in $\mathscr{P}_\kappa(\lambda)^{V[G][H]}$; since there is a bijection between λ and $H(\lambda)^{V[G][H]}$, there

[5] We identify subsets of λ and $j''\lambda$ with their characteristic functions to make this formally correct.

is a correspondence between $\mathscr{P}_\kappa \lambda^{V[G][H]}$ and $\mathscr{P}_\kappa H(\lambda)^{V[G][H]}$, and in particular it holds for every club $C \in V[G][H]$ in $\mathscr{P}_\kappa H(\lambda)^{V[G][H]}$ that

$$j``H(\lambda)^{V[G][H]} \in j(C). \tag{5}$$

Claim 4.16. *For each $x \in \mathscr{P}_\kappa \lambda^{V[G][H]}$, $b \restriction x \in M[G][H]$.*

Proof. By slenderness of D, we can fix a club C in $\mathscr{P}_\kappa H(\lambda)^{V[G][H]}$ such that for every $N \in C$, $N \prec H(\lambda)^{V[G][H]}$, and for every $x \in N$ of size ω_1, we have $d_{N\cap\lambda} \cap x \in N$.

Let $x \in \mathscr{P}_\kappa \lambda^{V[G][H]}$ be arbitrary. By the κ-cc of the whole forcing, it holds that

$$x \in H(\lambda)^{V[G][H]} = H(\lambda)^{M[G][H]}$$

and therefore $j``x = j(x) \in j``H(\lambda)^{M[G][H]}$. Let us denote $H(\lambda)^{M[G][H]}$ by N. Notice that $j``N$ is an elementary submodel of $j(H(\lambda)^{V[G][H]})$ by a general model-theoretic argument, or by invoking (5). Since $j``H(\lambda)^{M[G][H]} \in j(C)$, we have $d'_{(j``N)\cap j(\lambda)} \restriction j``x = d'_{j``\lambda} \restriction j``x \in j``N$. It follows that there is a function f in $N \subseteq M[G][H]$ with domain x such that for every $\alpha \in x$,

$$f(\alpha) = d'_{j``\lambda}(j(\alpha)).$$

By the definition of b, $f = b \restriction x$, and the proof is finished. □

Claim 4.17. *b is in $M[G][H]$.*

Proof. We show that b – which is approximated in $M[G][H]$ on sets in $\mathscr{P}_\kappa \lambda^{V[G][H]} = \mathscr{P}_\kappa \lambda^{M[G][H]}$ by Claim 4.16 – cannot be added by $Q_{\mathbb{K}} \times Q_{\mathbb{M}}$ over $M[G][H]$ and therefore it is already in $M[G][H]$.

We first argue that $Q_{\mathbb{K}}$ is ω_1-distributive and ω_2-Knaster over $M[G][H]$. In M, $j(\mathbb{K}) \simeq \mathbb{K} * Q_{\mathbb{K}}$ is ω_1-closed and therefore by the product analysis of $\mathbb{M}$, it is ω_1-distributive in $M[G]$ and therefore $Q_{\mathbb{K}}$ is ω_1-distributive in $M[G][H]$. Regarding the ω_2-Knaster property, first note that $Q_{\mathbb{K}}$ is κ-Knaster in $M[H]$ by Lemma 4.5. Since $\mathbb{M}$ is κ-Knaster in M it is still κ-Knaster in $M[H]$ by Fact 2.9 and therefore $Q_{\mathbb{K}}$ is κ-Knaster in $M[H][G] = M[G][H]$ by Fact 2.9. Since $Q_{\mathbb{K}}$ is κ-Knaster in $M[G][H]$ it has the $\kappa = \omega_2$-approximation property and cannot add b over $M[G][H]$.

Now, we will show that over $M[G][H][h]$, $Q_{\mathbb{M}}$ also cannot add b. In $M[G][H][h]$, $\kappa = \omega_2 = 2^\omega$ and $Q_{\mathbb{M}}$ is forcing equivalent to $\mathrm{Add}(\omega, \kappa^\dagger) * \mathbb{M}^\kappa$, where $\kappa^\dagger$ is the first inaccessible above κ. The branch b cannot be added by $\mathrm{Add}(\omega, \kappa^\dagger)$ since this forcing is ω_1-Knaster.

In $M[G][H][h][\mathrm{Add}(\omega, \kappa^\dagger)]$, $2^\omega = \kappa^\dagger > (\omega_2)^{V[G][H]}$ and D is a list with width at most $((2^{\omega_1})^+)^{V[G][H]}) = (\omega_3)^{V[G][H]}$. By a standard product analysis (see Section 2.3), $\mathbb{M}^\kappa$ is a projection of a product of the form $\mathrm{Add}(\omega, j(\kappa)) \times \mathbb{Q}^*_\kappa$, where $\mathbb{Q}^*_\kappa$ is ω_1-closed. Again the branch b cannot be added by $\mathrm{Add}(\omega, j(\kappa))$ since this forcing is ω_1-Knaster. Since D has width $(\omega_3)^{V[G][H]} < 2^\omega$ in $M[G][H][h][\mathrm{Add}(\omega, \kappa^\dagger)]$, we can apply Fact 2.12 to $\mathrm{Add}(\omega, j(\kappa))$ as P and $\mathbb{Q}^*_\kappa$ as Q over the model $M[G][H][h][\mathrm{Add}(\omega, \kappa^\dagger)]$. Therefore, the forcing $\mathbb{Q}^*_\kappa$ cannot add b over the model

$$M[G][H][h][\mathrm{Add}(\omega, \kappa^\dagger)][\mathrm{Add}(\omega, j(\kappa))],$$

hence the cofinal branch b cannot be added by $\mathbb{M}^\kappa$ over $M[G][H][h][\mathrm{Add}(\omega, \kappa^\dagger)]$. □

Claim 4.18. *The function b is an ineffable branch through D.*

Proof. We need to show that the set $S = \{x \in \mathscr{P}_\kappa(\lambda) \mid b \restriction x = d_x\}$ is stationary, hence it is enough to show that $j``\lambda$ is in $j(S) = \{y \in \mathscr{P}_{j(\kappa)}(j(\lambda)) \mid j(b) \restriction y = d'_y\}$. However this follows from the definition of $b$: $j(b) \restriction j``\lambda = j``b = d'_{j``\lambda}$. □

This finishes the proof of (2). By Theorem 4.9, there are no Kurepa trees in the generic extension $V[G][H]$ and hence we have (3), thus finishing the proof of Theorem 4.15. □

Remark 4.19. If one starts with large cardinals weaker than supercompact, one can obtain variations on Theorem 4.15 with item (2) weakened. For example, a straightforward adaptation of the proof of Theorem 4.15 shows that, if κ is a weakly compact cardinal and $G \times H$ is $\mathbb{M}(\omega, \kappa) \times \mathbb{K}_\kappa$-generic over V, then $V[G][H]$ satisfies $\mathsf{TP}(\omega_2) + \mathsf{wKH} + \neg\mathsf{KH}$.

5. Failure of Kurepa hypothesis and weak Kurepa hypothesis

In this section, we survey what is known about the (in)destructibility of $\neg\mathsf{KH}$ and $\neg\mathsf{wKH}$ and provide a direct proof that $\neg\mathsf{wKH}$ is always preserved by σ-centered forcing. To begin, observe that, since every Kurepa tree is a weak Kurepa tree, $\neg\mathsf{wKH}$ implies $\neg\mathsf{KH}$. In contrast to $\neg\mathsf{wKH}$, which implies the failure of CH, $\neg\mathsf{KH}$ is consistent with CH. The consistency of both principles follows from the consistency of an inaccessible cardinal. The classical model of $\neg\mathsf{KH}$ is the extension by the Levy collapse $\mathrm{Coll}(\omega_1, <\kappa)$, where κ is inaccessible in the ground model; the classical model of $\neg\mathsf{wKH}$ is the extension by $\mathbb{M}(\omega, \kappa)$, where κ is inaccessible in the ground model.

Assume now that κ is an inaccessible cardinal. An argument by Todorcevic [23] shows that in the extension by $\mathbb{M}(\omega, \kappa)$, $\neg\mathsf{wKH}$ is indestructible under all ccc forcings of cardinality ω_1. An analogous result holds for $\neg\mathsf{KH}$ in the extension by the Levy collapse $\mathrm{Coll}(\omega_1, <\kappa)$.

However, by a result of Jensen, $\square_{\omega_1}$ implies that there is a ccc forcing which adds a Kurepa tree (cf. [11]). Therefore, if κ is an inaccessible cardinal which is not Mahlo, then there is a ccc forcing which adds a Kurepa tree in the generic extensions by both $\mathrm{Coll}(\omega_1, <\kappa)$ and $\mathbb{M}(\omega, \kappa)$. By the previous paragraph, this ccc forcing must have size greater than ω_1.

With respect to $\neg\mathsf{KH}$, the assumption of $\square_{\omega_1}$ is necessary by a result of Jensen and Schlechta [11]. They show that if κ is Mahlo, then $\neg\mathsf{KH}$ is indestructible under all ccc forcings in the extension by $\mathrm{Coll}(\omega_1, <\kappa)$. The analogous result remains open for Mitchell forcing $\mathbb{M}(\omega, \kappa)$ and $\neg\mathsf{wKH}$, where κ is a Mahlo cardinal in the ground model.

Returning to ccc forcing of size ω_1: in [13], Krueger and the second author show that, assuming the consistency of only an inaccessible cardinal, it is consistent with CH that the Kurepa Hypothesis fails and yet there exists an almost Kurepa Suslin tree. In particular, there is a ccc forcing of size ω_1 which adds a Kurepa tree. As Corollary 1.1 shows, assuming the consistency of an inaccessible cardinal, it is consistent that $\neg\mathsf{wKH}$ holds while an almost Kurepa Suslin tree exists.

In particular, both $\neg\mathsf{KH}$ and $\neg\mathsf{wKH}$ are indestructible under all ccc forcings of size ω_1 in their classical models, but for both, there are models in which they are destructible by ccc forcing of size ω_1.

So far, we have considered only ccc forcings. What do we know about forcings satisfying a stronger property? In [8], Honzik and authors showed that GMP implies that $\neg$wKH is preserved by any σ-centered forcing. That is, if V is a transitive model satisfying GMP, $\mathbb{P} \in V$ is σ-centered, and G is $\mathbb{P}$-generic over V, then $V[G]$ satisfies $\neg$wKH. In particular, $\neg$wKH is preserved over models of GMP by adding any number of Cohen reals. Since the argument requires only a guessing model property for small sets, which is equivalent to $\neg$wKH, it follows that $\neg$wKH is always preserved by σ-centered forcings.

Below, we provide a direct proof that $\neg$wKH is preserved by any σ-centered forcing without referencing the guessing model property. First, let us recall the definition of σ-centered forcing.

Definition 5.1. Let $\mathbb{P}$ be a forcing. We say that $\mathbb{P}$ is *σ-centered* if $\mathbb{P}$ can be written as the union of a family $\{\mathbb{P}_n \subseteq \mathbb{P} \mid n < \omega\}$ such that for every $n < \omega$:

$$\text{for every } p, q \in \mathbb{P}_n \text{ there exists } r \in \mathbb{P}_n \text{ with } r \leq p, q. \tag{6}$$

It follows that $\mathbb{P}$ can be written as a union of ω-many filters if we close each $\mathbb{P}_n$ upwards.

Some definitions of σ-centeredness require just the compatibility of the conditions, with a witness not necessarily in $\mathbb{P}_n$. The condition (6) in this case reads:

$$\begin{aligned} &\text{for every } k < \omega \text{ and every sequence } p_0, p_1, \dots, p_{k-1} \\ &\qquad \text{of conditions in } \mathbb{P}_n \text{ there exists } r \in \mathbb{P} \text{ with } r \leq p_i \text{ for every } 0 \leq i < k. \end{aligned} \tag{7}$$

The conditions (6) and (7) are not in general equivalent (see Kunen [15], before Exercise III.3.27), but the distinction is not so important for us because the common forcings such as the Cohen forcing and the Prikry forcings are all centered in the stronger sense of (6). Also note that the conditions are equivalent for Boolean algebras: the definition (7) means that each $\mathbb{P}_n$ is a system with FIP (finite intersection property), and as such can be extended into a filter.

Theorem 5.2. *The failure of the weak Kurepa Hypothesis is preserved by any σ-centered forcing. That is, if V is a transitive model satisfying $\neg$wKH, $\mathbb{P} \in V$ is σ-centered, and G is $\mathbb{P}$-generic over V, then $V[G]$ satisfies $\neg$wKH.*

Proof. Fix a forcing notion $\mathbb{P} = \bigcup_{n<\omega} \mathbb{P}_n$, where each $\mathbb{P}_n$ is as in the definition of a σ-centered forcing; i.e., for every $n < \omega$ and every $p, q \in \mathbb{P}_n$, there exists $r \in \mathbb{P}_n$ such that $r \leq p, q$. Let G be $\mathbb{P}$-generic over V. Suppose that $\dot{T}$ be a $\mathbb{P}$-name for a weak Kurepa tree T, and assume for simplicity that $1_{\mathbb{P}}$ forces this. We also assume that the underlying set of T is ω_1.

We will define a weak Kurepa tree S in the ground model, which will complete the proof of the theorem. The underlying set of S will be $\omega \times \omega_1$; we define the ordering $<_S$ as follows: for $(n, \alpha), (m, \beta) \in S$, let $(n, \alpha) <_S (m, \beta)$ if and only if $n = m$ and there is some $p \in \mathbb{P}_n$ which forces $\alpha <_{\dot{T}} \beta$. Note that $<_S$ is a strict partial order: antisymmetry follows because any two conditions in $\mathbb{P}_n$ are compatible, and transitivity uses the fact that the compatibility of conditions in $\mathbb{P}_n$ is witnessed by a condition also in $\mathbb{P}_n$.

Now, we show that S is indeed a weak Kurepa tree. Clearly, S has size ω_1. To see that S has height ω_1, first we prove that it cannot have height greater than ω_1. If so, then there is (n, γ) in S_{ω_1} for some $(n, \gamma) \in S$; this means that $\mathrm{pred}_S((n, \gamma)) = \{(n, \alpha) <_S (n, \gamma) \mid (n, \alpha) \in S\}$ has order type ω_1. Let $I = \{\alpha < \omega_1 \mid (n, \alpha) \in \mathrm{pred}_S((n, \gamma))\}$. Then from the definition of $<_S$, it follows that for each $\alpha \in I$, there is $p_\alpha \in \mathbb{P}_n$ which forces that $\alpha <_{\dot{T}} \gamma$. Since $\mathbb{P}$ is ccc, there is $\alpha \in I$

such that $p_\alpha \Vdash \{\beta \in I \mid p_\beta \in \dot{G}\}$ has size ω_1, where $\dot{G}$ is a name for the generic filter over $\mathbb{P}$. Then we can see that p_α forces that γ has ω_1-many predecessors and hence has height at least ω_1 in T, which is a contradiction since $\dot{T}$ is forced by $1_{\mathbb{P}}$ to be a tree of height ω_1.

To see that S does not have countable height, let $\dot{b}$ be a name for a cofinal branch in T. Then, since $\dot{b}$ is forced to be a cofinal branch, in particular uncountable, there is n such that the set $I = \{\alpha < \omega_1 \mid \exists p \in \mathbb{P}_n (p \Vdash \alpha \in \dot{b})\}$ is uncountable. It is easy to see that for any $\alpha, \beta \in I$, (n, α) and (n, β) are comparable in S; hence, $d = \{(n, \alpha) \mid \alpha \in I\}$ is an uncountable chain through S. Therefore, S cannot have countable height, which, combined with the previous paragraph, gives that S has height exactly ω_1.

Moreover, we can show that d as in the previous paragraph is a cofinal branch; i.e., we prove that d is a maximal chain. Let (n, γ) be a node of S such that it is comparable with all (n, α) in d. Then since S has height ω_1, (n, γ) cannot be above all nodes in d; therefore there is $(n, \alpha) \in d$ such that $(n, \gamma) <_S (n, \alpha)$. By the definition of $<_S$, there is $p \in \mathbb{P}_n$ which forces $\gamma <_{\dot{T}} \alpha$ and by the definition of d, there is $q \in \mathbb{P}_n$ which forces that $\alpha \in \dot{b}$. Since both p and q are in $\mathbb{P}_n$, there is $r \in \mathbb{P}_n$ below both p and q, hence r forces that $\gamma \in \dot{b}$. Since $r \in \mathbb{P}_n$, $(n, \gamma) \in d$.

A similar argument shows that S has at least ω_2 many cofinal branches. Let $\langle \dot{b}_\alpha \mid \alpha < \omega_2 \rangle$ be a sequence of $\mathbb{P}$-names for cofinal branches through $\dot{T}$ such that, for all $\alpha < \beta < \omega_2$, $1_{\mathbb{P}} \Vdash \dot{b}_\alpha \neq \dot{b}_\beta$. For each $\alpha < \omega_2$, pick an uncountable subset $I_\alpha \subseteq \omega_1$ and $n_\alpha < \omega$ as in the previous paragraphs. Then $d_\alpha = \{(n_\alpha, \gamma) \mid \gamma \in I_\alpha\}$ is a cofinal branch through S. Since there are ω_2-many such branches, there exists $J \subseteq \omega_2$ of size ω_2 and some $n < \omega$ such that $n_\alpha = n$ for all $\alpha \in J$.

To finish the proof, we show that the cofinal branches $\{d_\alpha \mid \alpha \in J\}$ are pairwise distinct. Let $\alpha \neq \beta \in J$ and assume for a contradiction that $d_\alpha = d_\beta$. This in particular means that each $\gamma \in I_\alpha$ is in I_β; therefore, for each $\gamma \in I_\alpha$, there is $p_\gamma \in \mathbb{P}_n$ which forces that $\gamma \in \dot{b}_\beta$. Again, since $\mathbb{P}$ is ccc there is $\gamma \in I_\alpha$ such that $p_\gamma \Vdash \{\delta \in I_\alpha \mid p_\delta \in \dot{G}\}$ has size ω_1, where $\dot{G}$ is a name for a generic filter over $\mathbb{P}$. If $p_\gamma \in G$, then in $V[G]$, it holds that $b_\alpha = b_\beta$. This is a contradiction with the assumption that $1_{\mathbb{P}} \Vdash \dot{b}_\alpha \neq \dot{b}_\beta$. □

Let us note that the status of the corresponding question regarding ¬KH remains open, even for the forcing adding a single Cohen real. See the open questions in the next section.

6. Open questions

We show in Theorem A that GMP is consistently destructible by a ccc forcing, and even by one of size ω_1. The corresponding question for $\mathsf{TP}(\omega_2)$, asked already in [24] and [9], remains open. We restate it here.

Question 6.1. *Is it consistent that* $\mathsf{TP}(\omega_2)$ *holds and is indestructible under ccc forcing? In the opposite direction, is it consistent that* $\mathsf{TP}(\omega_2)$ *holds and there is a ccc forcing that adds an* ω_2*-Aronszajn tree?*

A natural first place to look in an attempt to answer Question 6.1 is our model $V[\mathbb{M}]$ for Theorem A. Note that, in that model GMP fails in the extension after forcing with an almost Kurepa Suslin tree T, simply because T becomes a Kurepa tree and GMP implies there are no Kurepa trees. This is not the case for $\mathsf{TP}(\omega_2)$ or

$\mathsf{GMP}^{\omega_2}(\omega_2)$. By a result of Cummings [5], it is consistent from a weakly compact cardinal that TP and KH hold together, and by a result of the authors [16], it is consistent from a supercompact cardinal that $\mathsf{GMP}^{\omega_2}(\omega_2)$ and KH hold together.

Question 6.2. *Suppose that T is a free Suslin tree and κ is a supercompact cardinal. Let $V[\mathbb{M}]$ be our model for Theorem A from Section 3. Does $\mathsf{TP}(\omega_2)$, or even $\mathsf{GMP}^{\omega_2}(\omega_2)$, hold in $V[\mathbb{M}][T]$?*

As mentioned above, it was proven in [8] that GMP is always preserved by adding any number of Cohen reals, providing one direction in which our Theorem A is sharp. By Theorem 5.2, the same is true for $\neg\mathsf{wKH}$, a consequence of GMP. However, the status of the corresponding question regarding $\neg\mathsf{KH}$ and $\mathsf{TP}(\omega_2)$, two other consequences of GMP, remains open, even for the forcing to add a single Cohen real.

Question 6.3. *Is it consistent that $\neg\mathsf{KH}$ holds but is destructible under adding a single Cohen real? Is it consistent that $\mathsf{TP}(\omega_2)$ holds but is destructible under adding a single Cohen real?*

Appendix A.

In this appendix, we fulfill a promise made in Remark 3.1 by providing an example of a two-step iteration of the form $\mathrm{Add}(\omega,1) * \dot{\mathbb{Q}}$ such that $\dot{\mathbb{Q}}$ is forced to be totally proper, and hence ω_1-distributive, and yet forcing with the associated term forcing $\mathbb{Q}$ over V collapses ω_1.

Let T be a normal coherent Aronszajn tree consisting of finite-to-one functions into ω. More precisely:

- $T \subseteq {}^{<\omega_1}\omega$ is $\subseteq$-downward closed, and $(T, \subseteq)$ is a normal Aronszajn tree;
- every element of T is a finite-to-one function;
- for all $\alpha < \omega_1$ and all $s, t \in T_\alpha$, we have $s =^* t$, i.e., the set $\{\eta < \alpha \mid s(\alpha) \neq t(\alpha)\}$ is finite.

It is straightforward to construct such Aronszajn trees in ZFC. Given a real $r \in {}^\omega\omega$, let T^r be the subtree of ${}^{<\omega_1}\omega$ defined by setting $T^r = \{r \circ t \mid t \in T\}$; note that this definition continues to make sense for reals r existing only in outer models of V. The following is well-known (see, e.g., the end of [22]).

Theorem A.1. *Suppose that $r \in {}^\omega\omega$ is Cohen-generic over V. Then, in $V[r]$, T^r is a Suslin tree.*

Let $\mathbb{P} = \mathrm{Add}(\omega, 1)$ be the forcing to add a single Cohen real. In this section, we think of the underlying set of $\mathbb{P}$ as ${}^{<\omega}\omega$, ordered by reverse inclusion. Let $\dot{G}$ be the canonical $\mathbb{P}$-name for the generic filter, and let $\dot{r}$ be a $\mathbb{P}$-name for $\bigcup \dot{G}$. By Theorem A.1, we have $\Vdash_{\mathbb{P}}$ "$T^{\dot{r}}$ is Suslin". In particular, considering $T^{\dot{r}}$ as a $\mathbb{P}$-name for a forcing notion (recall that the forcing order for a tree is the opposite of the tree order), we have $\Vdash_{\mathbb{P}}$ "$T^{\dot{r}}$ is ccc and totally proper". In particular, the two-step iteration $\mathbb{P} * T^{\dot{r}}$ is ccc. Let $\mathbb{Q}$ denote the term forcing associated with this two-step iteration. In particular, conditions in $\mathbb{Q}$ are all $\mathbb{P}$-names forced by $1_{\mathbb{P}}$ to be elements of $T^{\dot{r}}$, and, given $\dot{s}, \dot{t} \in \mathbb{Q}$, we set $\dot{t} \leq_{\mathbb{Q}} \dot{s}$ iff $1_{\mathbb{P}}$ forces that $\dot{t}$ extends $\dot{s}$ as a forcing condition, i.e., iff $1_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\dot{s} \leq_{T^{\dot{r}}} \dot{t}$".

Proposition A.2. *Forcing with $\mathbb{Q}$ over V collapses ω_1.*

Proof. Let H be $\mathbb{Q}$-generic over V, and suppose for the sake of contradiction that $(\omega_1)^V$ remains uncountable in $V[H]$. In particular, since T is a coherent Aronszajn tree consisting of finite-to-one functions, T remains Aronszajn in $V[H]$. In V, for each $\alpha < \omega_1$, let D_α be the set of $\dot{t} \in \mathbb{Q}$ such that $\Vdash_{\mathbb{P}}$ "$\mathrm{ht}_{T^{\dot{r}}}(\dot{t}) \geq \alpha$". By a straightforward argument using the normality of T and the maximum principle, each D_α is a dense, open subset of $\mathbb{Q}$; therefore, in $V[H]$, for each $\alpha < \omega_1$, we can fix a $\dot{t}_\alpha \in H \cap D_\alpha$. For each $\alpha < \omega_1$, fix a $p_\alpha \in \mathbb{P}$, a $\beta_\alpha \in [\alpha, \omega_1)$, and an $s_\alpha \in T_{\beta_\alpha}$ such that, in V, we have $p_\alpha \Vdash_{\mathbb{P}}$ "$\dot{t}_\alpha = \dot{r} \circ s_\alpha$". Since ω_1 is preserved in $V[H]$, we can fix in $V[H]$ a single $p \in \mathbb{P}$ and a stationary $A \subseteq \omega_1$ such that

- for all $\alpha \in A$, we have $p_\alpha = p$;
- for all $\alpha < \alpha'$, both in A, we have $\beta_\alpha < \alpha'$.

In particular, it follows that, for all $\alpha < \alpha'$, both in A, we have, in V, $p \Vdash_{\mathbb{P}}$ "$\dot{r} \circ s_\alpha \subseteq \dot{r} \circ s_{\alpha'}$". Let $k = \mathrm{dom}(p)$, and, for each $\alpha \in A$, let $B_\alpha = \{\eta < \beta_\alpha \mid s_\alpha(\eta) < k\}$; note that $B_\alpha \in [\beta_\alpha]^{<\omega}$.

Claim A.3. *For all $\alpha < \alpha'$, both in A, we have $B_\alpha \subseteq B_{\alpha'}$.*

Proof. Suppose for the sake of contradiction that $\alpha < \alpha'$ are both in A and $\eta \in B_\alpha \setminus B_{\alpha'}$. Let $j' = s_{\alpha'}(\eta)$ and $j = s_\alpha(\eta)$, noting that $j < k \leq j'$. Find $p' \leq_{\mathbb{P}} p$ such that $p'(j') \neq p(j)$. Then $p' \Vdash_{\mathbb{P}}$ "$(\dot{r} \circ s_\alpha)(\eta) \neq (\dot{r} \circ s_{\alpha'})(\eta)$", contradicting the fact that $p \Vdash_{\mathbb{P}}$ "$\dot{r} \circ s_\alpha \subseteq \dot{r} \circ s_{\alpha'}$". □

Since B_α is finite for each $\alpha \in A$ and $\langle B_\alpha \mid \alpha \in A \rangle$ is $\subseteq$-increasing, by removing an initial segment of A if necessary, we can assume that there is $B \in [\omega_1]^{<\omega}$ such that $B_\alpha = B$ for all $\alpha \in A$. Now find an uncountable $A^* \subseteq A$ and a fixed function $f : B \to \omega$ such that $s_\alpha \restriction B = f$ for all $\alpha \in A^*$.

Claim A.4. *For all $\alpha < \alpha'$, both in A^*, we have $s_\alpha = s_{\alpha'} \restriction \beta_\alpha$.*

Proof. Fix $\alpha < \alpha'$ in A^* and $\eta < \beta_\alpha$. If $\eta \in B$, then $s_\alpha(\eta) = f(\eta) = s_{\alpha'}(\eta)$. If $\eta \notin B$, then we have $s_\alpha(\eta), s_{\alpha'}(\eta) \geq k$. Suppose for the sake of contradiction that $s_\alpha(\eta) = j \neq j' = s_{\alpha'}(\eta)$. Find $p' \leq_{\mathbb{P}} p$ such that $p'(j) \neq p'(j')$. Then $p' \Vdash_{\mathbb{P}}$ "$(\dot{r} \circ s_\alpha)(\eta) \neq (\dot{r} \circ s_{\alpha'})(\eta)$", contradicting the fact that $p \Vdash_{\mathbb{P}}$ "$\dot{r} \circ s_\alpha \subseteq \dot{r} \circ s_{\alpha'}$". □

Now the claim implies that the downward closure of $\{s_\alpha \mid \alpha \in A^*\}$ is a cofinal branch through T in $V[H]$, contradicting the fact that T remains an Aronszajn tree in $V[H]$. □

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