

A NOTE ON HIGHLY CONNECTED AND WELL-CONNECTED RAMSEY THEORY

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ABSTRACT. We study a pair of weakenings of the classical partition relation $\nu \rightarrow (\mu)_\lambda^2$ recently introduced by Bergfalk-Hrušák-Shelah and Bergfalk, respectively. Given an edge-coloring of the complete graph on ν -many vertices, these weakenings assert the existence of monochromatic subgraphs exhibiting high degrees of connectedness rather than the existence of complete monochromatic subgraphs asserted by the classical relations. As a result, versions of these weakenings can consistently hold at accessible cardinals where their classical analogues would necessarily fail. We prove some complementary positive and negative results indicating the effect of large cardinals, forcing axioms, and square principles on these partition relations. We also prove a consistency result indicating that a non-trivial instance of the stronger of these two partition relations can hold at the continuum.

1. INTRODUCTION

In this paper, we study some natural variations of the classical partition relation for pairs. Recalling the arrow notation of Erdős and Rado, given cardinals μ , ν , and λ , the expression

$$\nu \rightarrow (\mu)_\lambda^2$$

denotes the assertion that, for every coloring $c : [\nu]^2 \rightarrow \lambda$, there is a set $X \subseteq \nu$ of cardinality μ such that $c \upharpoonright [X]^2$ is constant. This can be usefully interpreted in the language of graph theory as asserting that for every edge-coloring of the complete graph K_ν on ν -many vertices using λ -many colors, there is a monochromatic subgraph of K_ν isomorphic to K_μ . With this notation, the infinite Ramsey theorem for pairs can be expressed succinctly as

$$\aleph_0 \rightarrow (\aleph_0)_k^2$$

for all $k < \omega$.

When one tries to generalize the infinite Ramsey theorem to uncountable cardinals in the most straightforward way, by replacing each instance of ‘ \aleph_0 ’ in the above expression by some fixed uncountable cardinal ‘ ν ’, one immediately runs into a statement that can only hold at large cardinals, as the assertion that ν is uncountable and $\nu \rightarrow (\nu)_2^2$ is equivalent to the assertion that ν is weakly compact.

To achieve a consistent statement at accessible uncountable cardinals, then, one must weaken the statement $\nu \rightarrow (\nu)_2^2$. One natural approach is to increase the value of the cardinal on the left side of the expression. This is precisely what was done

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by Erdős and Rado in [5]. A special case of what became known as the Erdős-Rado theorem states that, for every infinite cardinal κ , we have

$$(2^\kappa)^+ \rightarrow (\kappa^+)_\kappa^2.$$

This result is sharp, in the sense that there are colorings witnessing the negative relations $2^\kappa \not\rightarrow (3)_\kappa^2$ and $2^\kappa \not\rightarrow (\kappa^+)_2^2$. (We refer the reader to [7, §7] for proofs and further discussion of these facts.)

Another approach is to weaken the meaning of the arrow notation, in particular by replacing the requirement that the monochromatic subgraph witnessing the partition relation be complete by a weaker but still nontrivial notion of largeness. This is the approach taken by Bergfalk-Hrušák-Shelah and Bergfalk in [2] and [3], where they introduce the notions of partition relations for *highly connected* and *well-connected* subsets, respectively. It is these notions that provide the primary subject of this paper; let us turn to their definitions, beginning with the partition relation for *highly connected* subsets.

Throughout the paper, if $G = (X, E)$ is a graph and $Y \subseteq X$, then we let $G \setminus Y$ denote the graph $(X \setminus Y, E \cap [X \setminus Y]^2)$. We will also sometime write $|G|$ to mean $|X|$.

Definition 1.1. Given a graph $G = (X, E)$ and a cardinal κ , we say that G is κ -*connected* if $G \setminus Y$ is connected for every $Y \in [X]^{<\kappa}$. We say that G is *highly connected* if it is $|G|$ -connected.

Definition 1.2. (Bergfalk-Hrušák-Shelah [2]) Suppose that μ , ν , and λ are cardinals. The partition relation $\nu \rightarrow_{hc} (\mu)_\lambda^2$ is the assertion that, for every coloring $c : [\nu]^2 \rightarrow \lambda$, there is an $X \in [\nu]^\mu$ and a highly connected subgraph (X, E) of $(\nu, [\nu]^2)$ such that $c \upharpoonright E$ is constant.

As was noted in [2], a finite graph is highly connected if and only if it is complete. As a result, if μ is finite, then the relation $\nu \rightarrow_{hc} (\mu)_\lambda^2$ is simply the classical relation $\nu \rightarrow (\mu)_\lambda^2$. In the context of infinite μ , however, the \rightarrow_{hc} version of a relation can consistently hold in situations where the classical version necessarily fails. For example, the following is proven in [2].

Theorem 1.3 (Bergfalk-Hrušák-Shelah [2]). *It is consistent, relative to the consistency of a weakly compact cardinal, that*

$$2^{\aleph_1} \rightarrow_{hc} (2^{\aleph_1})_{\aleph_0}^2.$$

We now recall the partition relation for *well-connected* subsets.

Definition 1.4. (Bergfalk [3]) Suppose that μ , ν , and λ are cardinals. Given a coloring $c : [\nu]^2 \rightarrow \lambda$ and a fixed color $i < \lambda$, we say that a subset $X \subseteq \nu$ is *well-connected in color i* (with respect to c) if, for every $\alpha < \beta$ in X , there is a finite path $\langle \alpha_0, \dots, \alpha_n \rangle$ (not necessarily contained in X) such that

- $\alpha_0 = \alpha$ and $\alpha_n = \beta$;
- $\alpha_k \geq \alpha$ for all $k \leq n$; and
- $c(\alpha_k, \alpha_{k+1}) = i$ for all $k < n$.

The partition relation $\nu \rightarrow_{wc} (\mu)_\lambda^2$ is the assertion that, for every coloring $c : [\nu]^2 \rightarrow \lambda$, there are $X \in [\nu]^\mu$ and $i < \lambda$ such that X is well-connected in color i .

As usual, $\nu \not\rightarrow_{hc} (\mu)_\lambda^2$ and $\nu \not\rightarrow_{wc} (\mu)_\lambda^2$ denote the negations of the respective partition relations.

The \rightarrow_{wc} relation is a clear weakening of the classical \rightarrow relation and is in fact a weakening of \rightarrow_{hc} . Indeed, by [3, Lemma 6], given cardinals μ , ν , and λ , we have

$$(\nu \rightarrow (\mu)_\lambda^2) \Rightarrow (\nu \rightarrow_{hc} (\mu)_\lambda^2) \Rightarrow (\nu \rightarrow_{wc} (\mu)_\lambda^2).$$

We have already seen that the left implication in the above statement can consistently fail. The right implication can consistently fail as well. For example, the following is proven in [3].

Theorem 1.5 (Bergfalk [3]). *It is consistent, relative to the consistency of a weakly compact cardinal, that $\aleph_2 \rightarrow_{wc} (\aleph_2)_{\aleph_0}^2$ holds but $\aleph_2 \rightarrow_{hc} (\aleph_2)_{\aleph_0}^2$ fails.*

The model for the above theorem is Mitchell's model for the tree property at \aleph_2 from [11]. The question of the consistency of $\aleph_2 \rightarrow_{hc} (\aleph_2)_{\aleph_0}^2$ was asked in [2] and remains open.

In this paper, we prove some further results about the relations \rightarrow_{hc} and \rightarrow_{wc} , some of them addressing questions from [2] and [3]. We provide here a brief outline of the remainder of the paper. In Section 2, we isolate some instances in which the relations $\nu \rightarrow_{hc} (\nu)_\lambda^2$ and $\nu \rightarrow_{wc} (\nu)_\lambda^2$ necessarily hold due either to the existence of certain large cardinals or to certain forcing axioms holding. In particular, we show that, if PFA holds, then $\nu \rightarrow_{wc} (\nu)_{\aleph_0}^2$ holds for every regular cardinal $\nu \geq \aleph_2$. This is optimal in the sense that it was proven in [3] that $\aleph_1 \not\rightarrow_{wc} (3)_{\aleph_0}^2$. In Section 3, we prove some complementary negative results, in particular indicating that certain square principles imply the failure of instances of $\nu \rightarrow_{wc} (\nu)_\lambda^2$. In the process, we introduce natural square-bracket versions of the partition relations for highly connected and well-connected subsets. In Section 4, we prove the consistency, relative to the consistency of a weakly compact cardinal, of the partition relation $2^{\aleph_0} \rightarrow_{hc} [2^{\aleph_0}]_{\aleph_0, 2}^2$. This is sharp due to the fact that, as shown in [2], the negative relation $2^{\aleph_0} \not\rightarrow_{hc} (2^{\aleph_0})_{\aleph_0}^2$ is provable in ZFC. Finally, in Section 5, we point out some recent work solving open questions stated in the first draft of this paper.

Notation and conventions. If X is a set and μ is a cardinal, then $[X]^\mu = \{Y \subseteq X \mid |Y| = \mu\}$. A *graph* is a pair $G = (X, E)$, where X is a set and $E \subseteq [X]^2$. Given a function c with domain $[X]^2$, we often slightly abuse notation and write, for instance, $c(a, b)$ instead of $c(\{a, b\})$. While elements of $[X]^2$ are unordered pairs, we will sometimes care about the relative order of the elements of such a pair. In particular, if X is a set of ordinals, then we will use the notation $(\alpha, \beta) \in [X]^2$ to indicate the conjunction of the two statements $\{\alpha, \beta\} \in [X]^2$ and $\alpha < \beta$.

A *path* in a graph $G = (X, E)$ is a finite sequence $\langle x_0, \dots, x_n \rangle$ of pairwise distinct elements of X such that $\{x_i, x_{i+1}\} \in E$ for all $i < n$. We say that G is *connected* if, for all distinct $x, y \in X$, there is a path $\langle x_0, \dots, x_n \rangle$ in G such that $x_0 = x$ and $x_n = y$.

If x is a set of ordinals, then the set of *accumulation points* of x , denoted by $\text{acc}(x)$, is defined to be $\{\alpha \in x \mid \sup(x \cap \alpha) = \alpha\}$. In particular, if β is an ordinal, then $\text{acc}(\beta)$ is the set of limit ordinals below β .

2. POSITIVE RESULTS

In [3], Bergfalk asked about conditions under which the relations $\mu^+ \rightarrow_{hc} (\mu)_{\text{cf}(\mu)}^2$ and $\mu^+ \rightarrow_{wc} (\mu)_{\text{cf}(\mu)}^2$ hold, where μ is a singular cardinal. In this section, we provide two scenarios in which such relations (and more) hold. The first scenario simply involves the presence of large cardinals and produces instances of the the highly

connected partition relation. The second involves the existence of guessing models and produces instances of the well-connected partition relation. Complementary negative results appear in the next section.

Definition 2.1. Suppose that $\theta \leq \kappa$ are regular, uncountable cardinals. κ is θ -strongly compact if, for every set X and every κ -complete filter F over X , F can be extended to a θ -complete ultrafilter over X . κ is *strongly compact* if it is κ -strongly compact.

Theorem 2.2. Suppose that $\theta \leq \kappa$ are regular uncountable cardinals and κ is θ -strongly compact. Suppose moreover that λ and ν are cardinals with $\lambda < \theta$ and $\text{cf}(\nu) \geq \kappa$. Then

$$\nu \rightarrow_{hc} (\nu)_\lambda^2.$$

Proof. Fix a coloring $c : [\nu]^2 \rightarrow \lambda$. We must find a color $i < \lambda$ and a set $X \in [\nu]^\nu$ such that the graph $(X, c^{-1}(i) \cap [X]^2)$ is highly connected.

Using the fact that κ is θ -strongly compact, let U be a θ -complete ultrafilter over ν extending the κ -complete filter

$$F = \{X \subseteq \nu \mid |\nu \setminus X| < \nu\}.$$

In particular, every element of U has cardinality ν . Now, for each $\alpha < \nu$, use the θ -completeness of U to find a color $i_\alpha < \lambda$ and a set $X_\alpha \subseteq \nu \setminus (\alpha + 1)$ such that $X_\alpha \in U$ and $c(\alpha, \beta) = i_\alpha$ for all $\beta \in X_\alpha$. Now use the θ -completeness of U again to find a fixed color $i < \lambda$ and a set $X \in U$ such that $i_\alpha = i$ for all $\alpha \in X$.

Let $G = (X, c^{-1}(i) \cap [X]^2)$. We claim that X is highly connected. To this end, let $Y \in [X]^{<\nu}$, and fix $\alpha < \beta$ in $X \setminus Y$. To show that α and β are connected in $G \setminus Y$, find

$$\gamma \in (X \cap X_\alpha \cap X_\beta) \setminus Y,$$

and note that $c(\alpha, \gamma) = c(\beta, \gamma) = i$. Then $\langle \alpha, \gamma, \beta \rangle$ is a path from α to β in $G \setminus Y$. \square

Corollary 2.3. Suppose that μ is a singular limit of strongly compact cardinals. Then

$$\mu^+ \rightarrow_{hc} (\mu^+)_\lambda^2$$

for all $\lambda < \mu$.

Recall the following definition, which comes from [14] and is a generalization of a notion from [15].

Definition 2.4. Suppose that $\kappa < \chi$ are regular uncountable cardinals, $M \prec (H(\chi), \in)$, and $\kappa \leq |M| < \chi$.

- (1) Suppose that $x \in M$ and $d \subseteq x$.
 - (a) We say that d is (κ, M) -approximated if $d \cap z \in M$ for every $z \in M \cap \mathcal{P}_\kappa(H(\chi))$.
 - (b) We say that d is M -guessed if there is $e \in M$ such that $e \cap M = d \cap M$.
- (2) M is κ -guessing if every (κ, M) -approximated set is M -guessed.

Given an infinite regular cardinal θ , let (T_θ) be the statement asserting that there are arbitrarily large regular cardinals χ such that the set

$$\{M \prec (H(\chi), \in) \mid |M| = \theta^+, {}^{<\theta}M \subseteq M, \text{ and } M \text{ is } \theta^+\text{-guessing}\}$$

is stationary in $\mathcal{P}_{\theta^{++}}(H(\chi))$.

It is proven by Viale and Weiß in [15] that the Proper Forcing Axiom (PFA) implies (T_{\aleph_0}) . Trang, in [13], proves the consistency of (T_{\aleph_1}) , assuming the consistency of a supercompact cardinal. The proof uses a Mitchell-type forcing construction and is easily modified to show that, if θ is a regular cardinal and there is a supercompact cardinal above θ , then there is a θ -closed forcing extension in which (T_{θ^+}) holds.

Theorem 2.5. *Suppose that θ is a regular cardinal and (T_θ) holds. Then, for every regular cardinal $\nu > \theta^+$ and every $\lambda \leq \theta$, we have*

$$\nu \rightarrow_{wc} (\nu)_\lambda^2.$$

Proof. Fix a regular cardinal $\nu > \theta^+$, a cardinal $\lambda \leq \theta$, and a coloring $c : [\nu]^2 \rightarrow \lambda$. We must find a color $i < \lambda$ and a set $X \in [\nu]^\nu$ such that X is well-connected in color i . Given a color $i < \lambda$ and $\alpha < \beta < \nu$, we say that $\alpha <_i \beta$ if $\{\alpha, \beta\}$ is well-connected in color i . It is easily verified that, for every $i < \lambda$, (ν, \leq_i) is a tree, i.e., it is a partial order and, for all $\beta \in \nu$, the set of $<_i$ -predecessors of β is well-ordered by $<_i$ (cf. [3, Lemma 12]).

Using the fact that (T_θ) holds, we may fix a regular cardinal $\chi \gg \nu$, a well-ordering \triangleleft of $H(\chi)$, and an elementary submodel $M \prec (H(\chi), \in, \triangleleft, \lambda, \theta, \nu, c)$ such that

- $|M| = \theta^+$;
- $^{<\theta}M \subseteq M$; and
- M is θ^+ -guessing.

Let $\nu_M = \sup(M \cap \nu)$.

Claim 2.6. $\text{cf}(\nu_M) = \theta^+$.

Proof. Since $|M| = \theta^+$, we clearly have $\text{cf}(\nu_M) \leq \theta^+$. Suppose for sake of contradiction that $\text{cf}(\nu_M) = \mu < \theta^+$. Let A be a cofinal subset of $M \cap \nu_M$ of order type μ , and let $\langle \alpha_\eta \mid \eta < \mu \rangle$ be the increasing enumeration of A .

We first show that A is (θ^+, M) -approximated. To this end, fix $z \in M \cap \mathcal{P}_{\theta^+}(H(\chi))$. Since $A \subseteq \nu$, we may assume that $z \subseteq \nu$. Since $|z| < \theta^+$ and $\nu > \theta^+$ is regular, we know that z is bounded below ν . Since $z \in M$, it follows by elementarity that z is bounded below ν_M , so there is $\xi < \mu$ such that $z \subseteq \alpha_\xi$. But then $A \cap z \subseteq \{\alpha_\eta \mid \eta < \xi\}$, and hence $A \cap z$ is a subset of M and $|A \cap z| < \mu \leq \theta$. Since $^{<\theta}M \subseteq M$, it follows that $A \cap z \in M$. Since z was arbitrary, we have shown that A is (θ^+, M) -approximated.

Since M is θ^+ -guessing, there is $B \in M$ such that $B \cap M = A \cap M = A$. Since A is unbounded in $M \cap \nu$, it follows by elementarity that B is unbounded in ν , and hence $\text{otp}(B) = \nu$. Let $\pi : \nu \rightarrow B$ be the order-preserving bijection, and note that $\pi \in M$. Since $^{<\theta}M \subseteq M$ and $\theta \in M$, we have $\theta + 1 \subseteq M$, and in particular $\mu + 1 \subseteq M$. But then $\pi''(\mu + 1) \subseteq B \cap M = A \cap M$, contradicting the fact that $\text{otp}(A) = \mu$. \square

Since $\text{cf}(\nu_M) = \theta^+ > \lambda$, there is fixed color $i < \theta$ and an unbounded set $d_0 \subseteq M \cap \nu_M$ such that $c(\beta, \nu_M) = i$ for all $\beta \in d_0$. Note that, for all $\alpha < \beta$, both in d_0 , we have $\alpha <_i \beta$, as witnessed by the path $\langle \alpha, \nu_M, \beta \rangle$. Let d be the $<_i$ -downward closure of d_0 , i.e.,

$$d = \{\alpha < \nu \mid \text{there is } \beta \in d_0 \text{ such that } \alpha <_i \beta\}.$$

(Note that it may not be the case that $d \subseteq M$.) We claim that d is (θ^+, M) -approximated. To see this, fix $z \in M \cap \mathcal{P}_{\theta^+}(H(\chi))$. As in the proof of the claim, we may assume that $z \subseteq \nu$, and again as in the proof of the claim it follows that there is $\beta \in d_0$ such that $z \subseteq \beta$. Then, using the fact that $<_i$ is a tree ordering, we have

$$d \cap z = \{\alpha \in z \mid \alpha <_i \beta\}.$$

Since everything needed to define this latter set is in M , we have $d \cap z \in M$. Since z was arbitrary, we have shown that d is (θ^+, M) -approximated.

As M is θ^+ -guessing, there is $e \in M$ such that $e \cap M = d \cap M$. By elementarity, it follows that e is a cofinal subset of ν that is linearly ordered by $<_i$. In other words, e is well-connected in color i . Since ν is regular, we have $|e| = \nu$, so we have proven our theorem. \square

Corollary 2.7. *Suppose that PFA holds. Then, for every regular cardinal $\nu \geq \aleph_2$, we have*

$$\nu \rightarrow_{wc} (\nu)_{\aleph_0}^2.$$

3. INDEXED SQUARES AND NEGATIVE RESULTS

In this section, we use square principles to isolate situations in which instances of \rightarrow_{wc} necessarily fail. The results in this section are refinements of [3, Lemma 9]. In order to fully state our results, we introduce *square bracket* versions of the partition relations being studied.

Definition 3.1. Suppose that ν and λ are cardinals. Given a coloring $c : [\nu]^k \rightarrow \lambda$ and a collection of colors $\Lambda \subseteq \lambda$, we say that a subset $X \subseteq \nu$ is *well-connected in Λ* (with respect to c) if, for every $\alpha < \beta$ in X , there is a finite path $\langle \alpha_0, \dots, \alpha_n \rangle$ (not necessarily contained in X) such that

- $\alpha_0 = \alpha$ and $\alpha_n = \beta$;
- $\alpha_k \geq \alpha$ for all $k \leq n$; and
- $c(\alpha_k, \alpha_{k+1}) \in \Lambda$ for all $k < n$.

Definition 3.2. Suppose that μ, ν, λ , and κ are cardinals.

- (1) The partition relation $\nu \rightarrow_{hc} [\mu]_{\lambda, \kappa}^2$ (*resp.* $\nu \rightarrow_{hc} [\mu]_{\lambda, < \kappa}^2$) is the assertion that, for every coloring $c : [\nu]^2 \rightarrow \lambda$, there is an $X \in [\nu]^\mu$, a highly connected subgraph (X, E) of $(\nu, [\nu]^2)$, and a set $\Lambda \in [\lambda]^{\leq \kappa}$ (*resp.* $\Lambda \in [\lambda]^{< \kappa}$) such that $c \restriction E \subseteq \Lambda$.
- (2) The partition relation $\nu \rightarrow_{wc} [\mu]_{\lambda, \kappa}^2$ (*resp.* $\nu \rightarrow_{wc} [\mu]_{\lambda, < \kappa}^2$) is the assertion that, for every coloring $c : [\nu]^2 \rightarrow \lambda$, there are $X \in [\nu]^\mu$ and $\Lambda \in [\lambda]^{\leq \kappa}$ (*resp.* $\Lambda \in [\lambda]^{< \kappa}$) such that X is well-connected in Λ .

As usual, the negations of these partition relations will be denoted by, e.g., $\nu \not\rightarrow_{hc} [\mu]_{\lambda, \kappa}^2$. We now recall certain square principles known as *indexed square principles*.

Definition 3.3 (Cummings-Foreman-Magidor [4]). Suppose that μ is a singular cardinal. A $\square_{\mu, \text{cf}(\mu)}^{\text{ind}}$ -sequence is a sequence $\langle C_{\alpha, i} \mid \alpha \in \text{acc}(\mu^+), i(\alpha) \leq i < \text{cf}(\mu) \rangle$ such that

- (1) for all $\alpha \in \text{acc}(\mu^+)$, we have $i(\alpha) < \text{cf}(\mu)$;
- (2) for all $\alpha \in \text{acc}(\mu^+)$ and all $i(\alpha) \leq i < \text{cf}(\mu)$, $C_{\alpha, i}$ is a club in α ;
- (3) for all $\alpha \in \text{acc}(\mu^+)$ and all $i(\alpha) \leq i < j < \text{cf}(\mu)$, we have $C_{\alpha, i} \subseteq C_{\alpha, j}$;

- (4) for all $\alpha < \beta$ in $\text{acc}(\mu^+)$ and all $i(\beta) \leq i < \text{cf}(\mu)$, if $\alpha \in \text{acc}(C_{\beta,i})$, then $i(\alpha) \leq i$ and $C_{\alpha,i} = C_{\beta,i} \cap \alpha$;
- (5) for all $\alpha < \beta$ in $\text{acc}(\mu^+)$, there is i such that $i(\beta) \leq i < \text{cf}(\mu)$ and $\alpha \in \text{acc}(C_{\beta,i})$;
- (6) there is an increasing sequence $\langle \mu_i \mid i < \text{cf}(\mu) \rangle$ of regular cardinals such that
 - (a) $\sup(\{\mu_i \mid i < \text{cf}(\mu)\}) = \mu$;
 - (b) for all $\alpha \in \text{acc}(\mu^+)$ and all $i(\alpha) \leq i < \text{cf}(\mu)$, we have $|C_{\alpha,i}| < \mu_i$.

$\square_{\mu, \text{cf}(\mu)}^{\text{ind}}$ is the assertion that there is a $\square_{\mu, \text{cf}(\mu)}^{\text{ind}}$ -sequence.

Definition 3.4 ([9]). Suppose that $\lambda < \mu$ are infinite regular cardinals. Then a $\square^{\text{ind}}(\mu, \lambda)$ -sequence is a sequence $\langle C_{\alpha,i} \mid \alpha \in \text{acc}(\mu), i(\alpha) \leq i < \lambda \rangle$ such that

- (1) for all $\alpha \in \text{acc}(\mu)$, we have $i(\alpha) < \lambda$;
- (2) for all $\alpha \in \text{acc}(\mu)$ and all $i(\alpha) \leq i < \lambda$, $C_{\alpha,i}$ is a club in α ;
- (3) for all $\alpha \in \text{acc}(\mu)$ and all $i(\alpha) \leq i < j < \lambda$, we have $C_{\alpha,i} \subseteq C_{\alpha,j}$;
- (4) for all $\alpha < \beta$ in $\text{acc}(\mu)$ and all $i(\beta) \leq i < \lambda$, if $\alpha \in \text{acc}(C_{\beta,i})$, then $i(\alpha) \leq i$ and $C_{\alpha,i} = C_{\beta,i} \cap \alpha$;
- (5) for all $\alpha < \beta$ in $\text{acc}(\mu)$, there is i such that $i(\beta) \leq i < \lambda$ and $\alpha \in \text{acc}(C_{\beta,i})$;
- (6) for every club D in μ and every $i < \lambda$, there is $\alpha \in \text{acc}(D)$ such that $D \cap \alpha \neq C_{\alpha,i}$.

$\square^{\text{ind}}(\mu, \lambda)$ is the assertion that there is a $\square^{\text{ind}}(\mu, \lambda)$ -sequence.

Remark 3.5. These indexed square principles follow from more familiar non-indexed square principles. For example, if μ is singular, then \square_μ implies $\square_{\mu, \text{cf}(\mu)}^{\text{ind}}$ (cf. [8]), while, if $\lambda < \mu$ are regular infinite cardinals, then $\square(\mu)$ implies $\square^{\text{ind}}(\mu, \lambda)$ (cf. [10]).

Theorem 3.6. Suppose that μ is a singular cardinal and \square_μ holds. Then

$$\mu^+ \not\rightarrow_{wc} [\mu]_{\text{cf}(\mu), < \text{cf}(\mu)}^2.$$

Proof. By Remark 3.5, \square_μ implies $\square_{\mu, \text{cf}(\mu)}^{\text{ind}}$. Therefore, we may fix a $\square_{\mu, \text{cf}(\mu)}^{\text{ind}}$ -sequence $\langle C_{\alpha,i} \mid \alpha \in \text{acc}(\mu^+), i(\alpha) \leq i < \text{cf}(\mu) \rangle$ and a sequence $\langle \mu_i \mid i < \text{cf}(\mu) \rangle$ of regular cardinals such that

- $\sup(\{\mu_i \mid i < \text{cf}(\mu)\}) = \mu$;
- for all $\alpha \in \text{acc}(\mu^+)$ and all $i(\alpha) \leq i < \text{cf}(\mu)$, we have $|C_{\alpha,i}| < \mu_i$.

Define a coloring $c : [\text{acc}(\mu^+)]^2 \rightarrow \text{cf}(\mu)$ by letting $c(\alpha, \beta)$ be the least ordinal $i < \text{cf}(\mu)$ such that $i(\beta) \leq i$ and $\alpha \in \text{acc}(C_{\beta,i})$ for all $\alpha < \beta$ in $\text{acc}(\mu^+)$.

We claim that c witnesses the negative relation $\mu^+ \not\rightarrow_{wc} [\mu]_{\text{cf}(\mu), < \text{cf}(\mu)}^2$. (More formally, the composition of c with the unique order-preserving bijection from μ^+ to $\text{acc}(\mu^+)$ will witness the negative relation.) To show this, we will prove that if $i < \text{cf}(\mu)$ is a color, $\Lambda \subseteq i$, and $X \subseteq \text{acc}(\mu^+)$ is well-connected in Λ , then $\text{otp}(X) \leq \mu_i$.

To this end, fix a color $i < \text{cf}(\mu)$, a set $\Lambda \subseteq i$, and a set $X \subseteq \text{acc}(\mu^+)$ that is well-connected in Λ .

Claim 3.7. For all $\alpha < \beta$ in X , we have $\alpha \in \text{acc}(C_{\beta,i})$.

Proof. The proof will be by induction on the minimal length of a path connecting α and β as in Definition 3.1.

Fix $\alpha < \beta$ in X . Since X is well-connected in Λ , we can fix a path $\vec{\alpha} = \langle \alpha_0, \dots, \alpha_n \rangle$ such that

- $\alpha_0 = \alpha$ and $\alpha_n = \beta$;
- $\alpha_k \geq \alpha$ for all $k \leq n$; and
- $c(\alpha_k, \alpha_{k+1}) \in \Lambda$ for all $k < n$.

Assume moreover that $\vec{\alpha}$ has minimal length among all such paths. If $n = 1$, then we will have $c(\alpha, \beta) \in \Lambda$. Therefore, there is $j \in \Lambda$ such that $\alpha \in \text{acc}(C_{\beta,j}) \subseteq \text{acc}(C_{\beta,i})$, so $\alpha \in \text{acc}(C_{\beta,i})$, as desired.

Suppose therefore that $n > 1$ and that we have established the claim for all pairs connected by paths of length less than n . In particular, it follows that $\alpha \in \text{acc}(C_{\alpha_{n-1},i})$. Note also that $c(\alpha_{n-1}, \beta) \in \Lambda$, so either $\alpha_{n-1} \in \text{acc}(C_{\beta,i})$ or $\beta \in \text{acc}(C_{\alpha_{n-1},i})$. Let $\gamma = \min(\{\alpha_{n-1}, \beta\})$. Then $C_{\alpha_{n-1},i} \cap \gamma = C_{\beta,i} \cap \gamma$, so, since $\alpha \in \text{acc}(C_{\alpha_{n-1},i})$, it follows that $\alpha \in \text{acc}(C_{\beta,i})$, as desired. \square

Now suppose for sake of contradiction that $\text{otp}(X) > \mu_i$. It follows that there is $\beta \in X$ such that $\text{otp}(X \cap \beta) = \mu_i$. But then, by the claim, we have $X \cap \beta \subseteq \text{acc}(C_{\beta,i})$, contradicting the fact that $|C_{\beta,i}| < \mu_i$. Therefore, for every color $i < \text{cf}(\mu)$, every $\Lambda \subseteq i$, and every set $X \subseteq \text{acc}(\mu^+)$ that is well-connected in Λ , we have $|X| < \mu$, and hence c witnesses $\mu^+ \not\rightarrow_{wc} [\mu]_{\text{cf}(\mu), < \text{cf}(\mu)}^2$. \square

Theorem 3.8. *Suppose that $\lambda < \mu$ are infinite regular cardinals and $\square(\mu)$ holds. Then*

$$\mu \not\rightarrow_{wc} [\mu]_{\lambda, < \lambda}^2.$$

Proof. The proof is quite similar to that of Theorem 3.6, so we only indicate its differences. By Remark 3.5, we can fix a $\square^{\text{ind}}(\mu, \lambda)$ -sequence $\langle C_{\alpha,i} \mid \alpha \in \text{acc}(\mu), i(\alpha) \leq i < \lambda \rangle$. Define a coloring $c : [\text{acc}(\mu)]^2 \rightarrow \lambda$ by letting $c(\alpha, \beta)$ be the least ordinal $i < \lambda$ such that $i(\beta) \leq i$ and $\alpha \in \text{acc}(C_{\beta,i})$.

We claim that c witnesses the negative partition relation in the statement of the theorem. Suppose that $i < \lambda$, $\Lambda \subseteq i$, and $X \subseteq \text{acc}(\mu)$ is well-connected in Λ . Exactly as in the proof of Theorem 3.6, we can prove that, for all $\alpha < \beta$ in X , we have $\alpha \in \text{acc}(C_{\beta,i})$. Now suppose for sake of contradiction that $|X| = \mu$, and let $D = \bigcup_{\alpha \in X} C_{\alpha,i}$. Since $C_{\beta,i}$ end-extends $C_{\alpha,i}$ for all $\alpha < \beta$ in X , it follows that D is a club in μ .

We claim that $D \cap \alpha = C_{\alpha,i}$ for all $\alpha \in \text{acc}(D)$. Indeed, if $\alpha \in \text{acc}(D)$, then, letting $\beta = \min(X \setminus (\alpha + 1))$, we have $D \cap \beta = C_{\beta,i}$, so $D \cap \alpha = C_{\beta,i} \cap \alpha$. Since $\alpha \in \text{acc}(D)$, it follows that $\alpha \in \text{acc}(C_{\beta,i})$, so, by Definition 3.4, we have $C_{\alpha,i} = C_{\beta,i} \cap \alpha = D \cap \alpha$.

D is then a counterexample to clause (6) of Definition 3.4, so it follows that $|X| < \mu$. Therefore, c witnesses $\mu \not\rightarrow_{wc} [\mu]_{\lambda, < \lambda}^2$. \square

Recall that, if μ is a regular uncountable cardinal and $\square(\mu)$ fails, then μ is weakly compact in L. As a result, we immediately obtain the following equiconsistency.

Corollary 3.9. *The following statements are equiconsistent over ZFC.*

- (1) *There exists a weakly compact cardinal.*
- (2) *There exist infinite regular cardinals $\lambda < \mu$ such that $\mu \rightarrow_{wc} [\mu]_{\lambda, < \lambda}^2$ holds.*

4. A SHARP POSITIVE RESULT AT THE CONTINUUM

In [2, Proposition 8], it is shown that, for all infinite cardinals μ and λ with $\mu \leq 2^\lambda$, we have the negative relation

$$\mu \not\rightarrow_{hc} (\mu)_\lambda^2.$$

In particular,

$$2^\lambda \not\rightarrow_{hc} (2^\lambda)_\lambda^2.$$

In [2], this was seen to be sharp in the sense that reducing the number of colors results in a consistent statement. In particular, it was shown that assuming the consistency of a weakly compact cardinal, it is consistent that, for example, the positive relation $2^{\aleph_1} \rightarrow_{hc} (2^{\aleph_1})_{\aleph_0}^2$ holds.

In this section, we show it is sharp in a different sense, namely that increasing the number of colors allowed to appear in the desired homogeneous set also results in a consistent statement. More precisely, we show that assuming the consistency of a weakly compact cardinal, the positive relation $2^\lambda \rightarrow_{hc} [2^\lambda]_{\lambda,2}^2$ consistently holds.

Theorem 4.1. *Suppose that $\lambda < \theta$ are infinite regular cardinals and θ is weakly compact. Let \mathbb{P} be the forcing to add θ -many Cohen subsets to λ . Then, after forcing with \mathbb{P} , we have*

$$2^\lambda \rightarrow_{hc} [2^\lambda]_{\lambda',2}^2$$

for all $\lambda' < 2^\lambda$.

Proof. Elements of \mathbb{P} are partial functions $p : \theta \rightarrow 2$ such that $|\text{dom}(p)| < \lambda$, ordered by reverse inclusion. Note that, in $V^\mathbb{P}$, we have $2^\lambda = \theta$, so we must show that $\theta \rightarrow_{hc} [\theta]_{\lambda',2}^2$ holds after forcing with \mathbb{P} for all $\lambda' < \theta$. Fix a cardinal $\lambda' < \theta$, a condition $p \in \mathbb{P}$ and a \mathbb{P} -name \dot{c} that is forced by p to be a name for a function from $[\theta]^2$ to λ' . We will find a condition $r \leq p$ and colors $i_0, i_1 < \lambda'$ such that r forces the existence of a highly connected subgraph (\dot{X}, \dot{E}) of $(\theta, [\theta]^2)$ such that $\dot{c}''\dot{E} = \{i_0, i_1\}$.

For all $\alpha < \beta < \theta$, fix a condition $q_{\alpha,\beta} \leq p$ and a color $i_{\alpha,\beta} < \lambda'$ such that

$$q_{\alpha,\beta} \Vdash \dot{c}(\alpha, \beta) = i_{\alpha,\beta}.$$

Without loss of generality, let us assume that $\{\alpha, \beta\} \subseteq \text{dom}(q_{\alpha,\beta})$. We will take advantage of the weak compactness of θ to find an unbounded $A \subseteq \theta$ such that the conditions $\{q_{\alpha,\beta} \mid (\alpha, \beta) \in [A]^2\}$ enjoy a certain uniformity. For notational convenience, for all $(\alpha, \beta) \in [A]^2$, let $u_{\alpha,\beta} = \text{dom}(q_{\alpha,\beta})$.

First, we can appeal to the weak compactness of θ to find ordinals $i_0 < \lambda'$ and $\xi_0 < \lambda$, a function $d_0 : \xi_0 \rightarrow 2$, and an unbounded $A_0 \subseteq \theta$ such that, for all $(\alpha, \beta) \in [A_0]^2$, we have

- (1) $i_{\alpha,\beta} = i_0$;
- (2) $\text{otp}(u_{\alpha,\beta}) = \xi_0$;
- (3) letting $u_{\alpha,\beta}$ be enumerated in increasing order as $\{\gamma_\zeta \mid \zeta < \xi_0\}$, we have that $q_{\alpha,\beta}(\gamma_\zeta) = d_0(\zeta)$ for all $\zeta < \xi_0$.

We now appeal to [2, Lemma 18] (see also [12]), which can be seen as a two-dimensional Δ -system lemma, to find an unbounded $A_1 \subseteq A_0$ such that

- (4) for all $\alpha \in A_1$, the set $\{u_{\alpha,\beta} \mid \beta \in A_1 \setminus (\alpha + 1)\}$ is a Δ -system, with root u_α^+ ;

- (5) letting $A_1^* = A_1 \setminus \{\min(A_1)\}$, for all $\beta \in A_1^*$, the set $\{u_{\alpha,\beta} \mid \alpha \in A_1 \cap \beta\}$ is a Δ -system, with root u_β^- ;
- (6) the sets $\{u_\alpha^+ \mid \alpha \in A_1\}$ and $\{u_\alpha^- \mid \alpha \in A_1^*\}$ are both Δ -systems, with roots u_\emptyset^+ and u_\emptyset^- , respectively.

It is in fact easy to see, given the above discussion, that the roots u_\emptyset^+ and u_\emptyset^- of item (6) must both be equal to the set

$$u_\emptyset := \bigcap_{(\alpha,\beta) \in [A_1]^2} u_{\alpha,\beta}.$$

By thinning out A_1 further, using the weak compactness of θ , we can assume that:

- (7) For all $(\alpha, \beta) \in [A_1^*]^2$, we have $\text{otp}(u_\alpha^+) = \text{otp}(u_\beta^+)$ and $\text{otp}(u_\alpha^-) = \text{otp}(u_\beta^-)$.
- (8) For all $(\alpha, \beta), (\gamma, \delta) \in [A_1^*]^2$, we have that $u_\alpha^+, u_\beta^-,$ and u_\emptyset “sit inside” of $u_{\alpha,\beta}$ in the same way that u_γ^+, u_δ^- , and u_\emptyset “sit inside” of $u_{\gamma,\delta}$. More formally, we have
 - (a) $\{\zeta < \xi_0 \mid u_{\alpha,\beta}(\zeta) \in u_\alpha^+\} = \{\zeta < \xi_0 \mid u_{\gamma,\delta}(\zeta) \in u_\gamma^+\}$;
 - (b) $\{\zeta < \xi_0 \mid u_{\alpha,\beta}(\zeta) \in u_\beta^-\} = \{\zeta < \xi_0 \mid u_{\gamma,\delta}(\zeta) \in u_\delta^-\}$;
 - (c) $\{\zeta < \xi_0 \mid u_{\alpha,\beta}(\zeta) \in u_\emptyset\} = \{\zeta < \xi_0 \mid u_{\gamma,\delta}(\zeta) \in u_\emptyset\}$.

By items (3) and (8) above, it follows that, if $\alpha < \beta < \gamma$ are all elements of A_1^* , then

- $q_{\alpha,\beta}$ and $q_{\alpha,\gamma}$ are compatible in \mathbb{P} ;
- $q_{\alpha,\gamma}$ and $q_{\beta,\gamma}$ are compatible in \mathbb{P} .

It also follows that, for all $\alpha \in A_1^*$, we can define a condition q_α^+ by letting $q_\alpha^+ = q_{\alpha,\beta} \restriction u_\alpha^+$ for some $\beta \in A_1 \setminus (\alpha + 1)$, and that this definition is independent of our choice of β . Similarly, for all $\beta \in A_1^*$, we can define q_β^- by letting $q_\beta^- = q_{\alpha,\beta} \restriction u_\beta^-$ for some $\alpha \in A_1 \cap \beta$, and we can define q_\emptyset by letting $q_\emptyset = q_{\alpha,\beta} \restriction u_\emptyset$ for some $(\alpha, \beta) \in [A_1]^2$. Again, these definitions are independent of our choices.

Claim 4.2. *There is an unbounded $A_1^{**} \subseteq A_1^*$ such that, for all $(\alpha, \beta) \in [A_1^{**}]^2$, we have $u_\alpha^- \cap u_\beta^+ = u_\emptyset$.*

Proof. It is clear from the definition of u_\emptyset that $u_\alpha^- \cap u_\beta^+ \supseteq u_\emptyset$ for all $(\alpha, \beta) \in [A_1^*]^2$. Define a function $f : [A_1^*]^2 \rightarrow 2$ by letting $f(\alpha, \beta) = 0$ if $u_\alpha^- \cap u_\beta^+ = u_\emptyset$ and letting $f(\alpha, \beta) = 1$ otherwise. By the weak compactness of θ , we can find an unbounded set $A_1^{**} \subseteq A_1^*$ such that f is constant on $[A_1^{**}]^2$. We claim that $f \restriction [A_1^{**}]^2 = \{0\}$. It suffices to find a single pair $(\alpha, \beta) \in [A_1^{**}]^2$ such that $f(\alpha, \beta) = 0$. Let $\alpha = \min(A_1^{**})$. We know that $\{u_\beta^+ \mid \beta \in A_1^{**}\}$ forms a Δ -system with root u_\emptyset . Since $|u_\alpha^-| < \lambda < \theta$, we can find $\beta \in A_1^{**} \setminus (\alpha + 1)$ such that $u_\beta^+ \setminus u_\emptyset$ is disjoint from u_α^- . But then $u_\alpha^- \cap u_\beta^+ = u_\emptyset$, so $f(\alpha, \beta) = 0$. It follows that A_1^{**} is as desired. \square

Fix an unbounded $A_1^{**} \subseteq A_1^*$ as given by the previous claim. It follows that, for all $(\alpha, \beta) \in [A_1^{**}]^2$, $q_\alpha^- \cup q_\beta^+$ is a condition in \mathbb{P} extending both q_α^- and q_β^+ . For $(\alpha, \beta) \in [A_1^{**}]^2$, fix a condition $r_{\alpha,\beta} \leq q_\alpha^- \cup q_\beta^+$ and a color $j_{\alpha,\beta} < \lambda'$ such that

$$r_{\alpha,\beta} \Vdash \dot{c}(\alpha, \beta) = j_{\alpha,\beta}.$$

Let $v_{\alpha,\beta} = \text{dom}(r_{\alpha,\beta})$. Repeating the above process with $\langle r_{\alpha,\beta} \mid (\alpha, \beta) \in [A_1^{**}]^2 \rangle$ in place of $\langle q_{\alpha,\beta} \mid (\alpha, \beta) \in [\theta]^2 \rangle$, we can find ordinals $i_1 < \lambda'$ and $\xi_1 < \lambda$, a function $d_1 : \xi_1 \rightarrow 2$, and an unbounded $A_2 \subseteq A_1^{**}$ such that

- (9) for all $(\alpha, \beta) \in [A_2]^2$, we have

- (a) $j_{\alpha,\beta} = i_1$;
- (b) $\text{otp}(v_{\alpha,\beta}) = \xi_1$;
- (c) letting $v_{\alpha,\beta}$ be enumerated in increasing order as $\{\gamma_\zeta \mid \zeta < \xi_1\}$, we have that $r_{\alpha,\beta}(\gamma_\zeta) = d_1(\zeta)$ for all $\zeta < \xi_1$;
- (10) for all $\alpha \in A_2$, the set $\{v_{\alpha,\beta} \mid \beta \in A_2 \setminus (\alpha + 1)\}$ is a Δ -system, with root v_α ;
- (11) the set $\{v_\alpha \mid \alpha \in A_2\}$ is a Δ -system, with root v_\emptyset ;
- (12) for all $(\alpha, \beta), (\gamma, \delta) \in [A_2]^2$, the sets v_\emptyset and v_α “sit inside” $v_{\alpha,\beta}$ in the same way that the sets v_\emptyset and v_γ “sit inside” $v_{\gamma,\delta}$.

As above, we define conditions $\langle r_\alpha \mid \alpha \in A_2 \rangle$ by letting $r_\alpha = r_{\alpha,\beta} \upharpoonright v_\alpha$ for some $\beta \in A_2 \setminus (\alpha + 1)$, and we define $r_\emptyset = r_{\alpha,\beta} \upharpoonright v_\emptyset$ for some $(\alpha, \beta) \in [A_2]^2$. Again, these definitions are independent of our choices.

We thin out our set A_2 one final time in the following way. Define a function $e : [A_2]^2 \rightarrow 2$ by letting $e(\alpha, \beta) = 0$ if $q_{\alpha,\beta}$ and r_β are compatible in \mathbb{P} , and letting $e(\alpha, \beta) = 1$ otherwise. Using the weak compactness of θ , find an unbounded $A \subseteq A_2$ such that e is constant on $[A]^2$.

Claim 4.3. $e[A]^2 = \{0\}$.

Proof. Since e is constant on $[A]^2$, it suffices to find a single pair $(\alpha, \beta) \in [A]^2$ such that $e(\alpha, \beta) = 0$.

Fix $\beta \in A$ such that $|A \cap \beta| \geq \lambda$. The set $\{u_{\alpha,\beta} \mid \alpha \in A \cap \beta\}$ forms a Δ -system with root u_β^- . Moreover, by construction, we have $v_\beta \supseteq u_\beta^-$ and $r_\beta \leq q_\beta^-$. We can therefore find $\alpha \in A \cap \beta$ such that $(u_{\alpha,\beta} \setminus u_\beta^-) \cap v_\beta = \emptyset$. Then we have $q_{\alpha,\beta} \upharpoonright u_\beta^- = q_\beta^- \geq r_\beta$, so $q_{\alpha,\beta}$ and r_β are compatible in \mathbb{P} and hence $e(\alpha, \beta) = 0$. \square

Now make the following assignments:

- $r = r_\emptyset$;
- \dot{G} is the canonical \mathbb{P} -name for the \mathbb{P} -generic filter;
- \dot{X}_0 is a \mathbb{P} -name for the set $\{\alpha \in A \mid q_\alpha^+ \in \dot{G}\}$;
- \dot{X}_1 is a \mathbb{P} -name for the set $\{\beta \in A \mid r_\beta \in \dot{G}\}$;
- \dot{X} is a \mathbb{P} -name for $\dot{X}_0 \cup \dot{X}_1$;
- \dot{E} is a \mathbb{P} -name for $\dot{c}^{-1}(\{i_0, i_1\}) \cap [\dot{X}]^2$.

Notice that each $q_{\alpha,\beta} \leq p$ so also $q_\emptyset \leq p$. It similarly follows that $r \leq p$. We will end the proof by showing that r forces (\dot{X}, \dot{E}) to be a highly connected graph of cardinality θ .

Claim 4.4. $r \Vdash “|\dot{X}| = \theta”$.

Proof. We will show that $r \Vdash “|\dot{X}_0| = \theta”$, which suffices. A similar proof will in fact show that $r \Vdash “|\dot{X}_1| = \theta”$, as well.

Fix an arbitrary condition $s \leq r$ and an $\eta < \theta$. It suffices to find $\alpha \in A \setminus \eta$ such that q_α^+ and s are compatible in \mathbb{P} . Since $s \leq r$, we have $\text{dom}(s) \supseteq u_\emptyset$, and $s \leq q_\emptyset$. Recall that the set $\{u_\alpha^+ \mid \alpha \in A \setminus \eta\}$ is a Δ -system with root u_\emptyset . We can therefore find $\alpha \in A$ such that $(u_\alpha^+ \setminus u_\emptyset) \cap \text{dom}(s) = \emptyset$. Since $q_\alpha^+ \upharpoonright u_\emptyset = q_\emptyset \geq s$, it follows that q_α^+ and s are compatible in \mathbb{P} , as desired. \square

To show that r forces (\dot{X}, \dot{E}) to be highly connected, we first establish a couple of preliminary claims.

Claim 4.5. $r \Vdash “\forall(\alpha, \beta) \in [\dot{X}_0]^2 \forall \eta < \theta \exists \gamma \in \dot{X}_1 \setminus \eta [\{q_{\alpha,\gamma}, q_{\beta,\gamma}\} \subseteq \dot{G}]”$.

Proof. Fix an ordinal $\eta < \theta$, a condition $s \leq r$ and $(\alpha, \beta) \in [A]^2$ such that s forces both α and β to be in \dot{X}_0 . Without loss of generality, we can assume that $s \leq q_\alpha^+$ and $s \leq q_\beta^+$ and that $\eta > \beta$. It will suffice to find $\gamma \in A \setminus \eta$ such that the conditions s , $q_{\alpha,\gamma}$, $q_{\beta,\gamma}$, and r_γ are all pairwise compatible, since then the union of these four conditions would itself be a condition extending s and forcing γ to be as desired. Note also that, for all $\gamma \in A \setminus \eta$, we know that $q_{\alpha,\gamma}$ and $q_{\beta,\gamma}$ are compatible by the discussion following item (8) above. We also know that r_γ is compatible with each of $q_{\alpha,\gamma}$ and $q_{\beta,\gamma}$ by Claim 4.3. It therefore suffices to find $\gamma \in A \setminus \eta$ such that s is compatible with each of $q_{\alpha,\gamma}$, $q_{\beta,\gamma}$, and r_γ .

By assumption, we know that $\text{dom}(s) \supseteq u_\alpha^+ \cup u_\beta^+ \cup v_\emptyset$ and that s extends each of q_α^+ , q_β^+ , and r_\emptyset . Recall also that the sets $\{u_{\alpha,\gamma} \mid \gamma \in A \setminus \eta\}$, $\{u_{\beta,\gamma} \mid \gamma \in A \setminus \eta\}$, and $\{v_\gamma \mid \gamma \in A \setminus \eta\}$ are Δ -systems with roots u_α^+ , u_β^+ , and v_\emptyset , respectively. We can therefore find $\gamma \in A \setminus \eta$ such that each of the sets $(u_{\alpha,\gamma} \setminus u_\alpha^+)$, $(u_{\beta,\gamma} \setminus u_\beta^+)$, and $v_\gamma \setminus v_\emptyset$ is disjoint from $\text{dom}(s)$. But then we have

- $q_{\alpha,\gamma} \upharpoonright \text{dom}(s) = q_\alpha^+$;
- $q_{\beta,\gamma} \upharpoonright \text{dom}(s) = q_\beta^+$; and
- $r_\gamma \upharpoonright \text{dom}(s) = r_\emptyset$.

Therefore, since s extends each of q_α^+ , q_β^+ , and r_\emptyset , it follows that s is compatible with each of $q_{\alpha,\gamma}$, $q_{\beta,\gamma}$, and r_γ , as desired. \square

Claim 4.6. $r \Vdash \text{“}\forall \alpha \in \dot{X}_1 \ \forall \eta < \lambda \ \exists \beta \in \dot{X}_0 \setminus \eta \ [r_{\alpha,\beta} \in \dot{G}]\text{”}$.

Proof. Fix an ordinal $\eta < \theta$, a condition $s \leq r$ and $\alpha \in A$ such that $s \Vdash \text{“}\alpha \in \dot{X}_1\text{”}$. Without loss of generality, assume that $\eta > \alpha$ and $s \leq r_\alpha$, and hence $\text{dom}(s) \supseteq v_\alpha$. The set $\{v_{\alpha,\beta} \mid \beta \in A \setminus \eta\}$ is a Δ -system with root v_α ; we can therefore find $\beta \in A \setminus \eta$ such that $(v_{\alpha,\beta} \setminus v_\alpha) \cap \text{dom}(s) = \emptyset$. We know that $r_{\alpha,\beta} \upharpoonright \text{dom}(s) = r_\alpha \geq s$, so it follows that s and $r_{\alpha,\beta}$ are compatible. Recall that $r_{\alpha,\beta} \leq q_\beta^+$ and therefore forces β to be in \dot{X}_0 . Therefore, the union of s and $r_{\alpha,\beta}$ forces β to be as desired. \square

Let G be a \mathbb{P} -generic filter over V with $r \in G$. Let c , X_0 , X_1 , X , and E be the realizations of \dot{c} , \dot{X}_0 , \dot{X}_1 , \dot{X} , and \dot{E} , respectively, in $V[G]$. By the definition of \dot{E} , we know that $c \text{“}[E]^2 \subseteq \{i_0, i_1\}$. By Claim 4.4, we know that $|X| = \theta$. It thus remains to show that, for all $Y \in [X]^{<\theta}$, the graph $(X \setminus Y, E \cap [X \setminus Y]^2)$ is connected.

Fix $Y \in [X]^{<\theta}$, and let $Z = X \setminus Y$. Also fix $(\alpha, \beta) \in [Z]^2$. Since θ is regular and $|Y| < |X| = \theta$, there is $\eta < \theta$ such that $Y \subseteq \eta$, and hence $X \setminus \eta \subseteq Z$. By increasing η if necessarily, we may assume that $\beta < \eta$. There are now a number of cases, not necessarily mutually exclusive, to consider.

Case 1: $\alpha, \beta \in X_0$. In this case, Claim 4.5 implies that there is $\gamma \in X_1 \setminus \eta$ such that $\{q_{\alpha,\gamma}, q_{\beta,\gamma}\} \subseteq G$. It follows that $c(\alpha, \gamma) = c(\beta, \gamma) = i_0$, so $\langle \alpha, \gamma, \beta \rangle$ is a path from α to β in $(Z, E \cap [Z]^2)$.

Case 2: $\alpha \in X_0$ and $\beta \in X_1$. By Claim 4.6, we can find $\gamma \in X_0 \setminus \eta$ such that $r_{\beta,\gamma} \in G$. Then, by Claim 4.5, we can find $\delta \in X_1 \setminus (\gamma + 1)$ such that $\{q_{\alpha,\delta}, q_{\gamma,\delta}\} \subseteq G$. It follows that $c(\alpha, \delta) = c(\gamma, \delta) = i_0$ and $c(\beta, \gamma) = i_1$, so $\langle \alpha, \delta, \gamma, \beta \rangle$ is a path from α to β in $(Z, E \cap [Z]^2)$.

Case 3: $\alpha \in X_1$ and $\beta \in X_0$. This is symmetric to Case 2.

Case 4: $\alpha, \beta \in X_1$. By Claim 4.6, we can first find $\gamma \in X_0 \setminus \eta$ such that $r_{\alpha,\gamma} \in G$ and then $\delta \in X_0 \setminus (\gamma + 1)$ such that $r_{\beta,\delta} \in G$. Then, by Claim 4.5, we can find

$\epsilon \in X_1 \setminus (\delta + 1)$ such that $\{q_{\gamma, \epsilon}, q_{\delta, \epsilon}\} \subseteq G$. It follows that $c(\alpha, \gamma) = c(\beta, \delta) = i_1$ and $c(\gamma, \epsilon) = c(\delta, \epsilon) = i_0$, so $\langle \alpha, \gamma, \epsilon, \delta, \beta \rangle$ is a path from α to β in $(Z, E \cap [Z]^2)$.

This exhausts all possible cases, so we have shown that, in $V[G]$, (X, E) is highly connected, thus finishing the proof. \square

Remark 4.7. We have seen that Theorem 4.1 is sharp in the sense that the conclusion cannot be improved to $2^\lambda \rightarrow_{hc} (2^\lambda)_\lambda^2$. It is also sharp in the sense that the “ hc ” subscript cannot be dropped. For example, it is easily seen that the coloring $\Delta : [\lambda 2]^2 \rightarrow \lambda$ defined by letting $\Delta(f, g)$ be the least $i < \lambda$ such that $f(i) \neq g(i)$ witnesses the negative square bracket relation

$$2^\lambda \not\rightarrow [\aleph_0]_{\lambda, < \aleph_0}^2,$$

and, more generally, if $\nu \leq \lambda$, $\mu \leq 2^\lambda$, and $2^\chi < \mu$ for all $\chi < \nu$, then Δ witnesses

$$2^\lambda \not\rightarrow [\mu]_{\lambda, < \nu}^2.$$

We end this section by noting the following equiconsistency that results from a special case of the preceding theorem.

Corollary 4.8. *The following statements are equiconsistent over ZFC.*

- (1) *There exists a weakly compact cardinal.*
- (2) $2^{\aleph_0} \rightarrow_{hc} [2^{\aleph_0}]_{\aleph_0, 2}^2$.

Proof. The implication from the consistency of (1) to that of (2) follows directly from Theorem 4.1. For the reverse implication, if $\mu := \text{cf}(2^{\aleph_0})$ is not weakly compact in L , then $\square(\mu)$ holds, and hence we can fix a coloring $c : [\mu]^2 \rightarrow \omega$ witnessing $\mu \not\rightarrow_{wc} [\mu]_{\aleph_0, < \aleph_0}^2$. Now let $\langle \nu_\eta \mid \eta < \mu \rangle$ be an increasing sequence of ordinals, cofinal in 2^{\aleph_0} . For each $\alpha < 2^{\aleph_0}$, let η_α be the least $\eta < \mu$ such that $\alpha \leq \nu_\eta$, and define a coloring $d : [2^{\aleph_0}]^2 \rightarrow \omega$ by setting

$$d(\alpha, \beta) = \begin{cases} c(\eta_\alpha, \eta_\beta) & \text{if } \eta_\alpha \neq \eta_\beta \\ 0 & \text{if } \eta_\alpha = \eta_\beta \end{cases}.$$

Now if $X \in [2^{\aleph_0}]^{2^{\aleph_0}}$, $\Lambda \subseteq \omega$, and X is well-connected in Λ with respect to d , then $Y := \{\eta_\alpha \mid \alpha \in X\} \in [\mu]^\mu$ is well-connected in Λ with respect to c . Therefore, d witnesses $2^{\aleph_0} \not\rightarrow_{wc} [2^{\aleph_0}]_{\aleph_0, < \aleph_0}^2$ and hence, *a fortiori*, $2^{\aleph_0} \not\rightarrow_{hc} [2^{\aleph_0}]_{\aleph_0, 2}^2$. \square

5. RECENT WORK

In the first draft of this paper, we included here the following two questions, which were open at the time, about whether certain positive instances of the highly connected partition relation are consistent (relative to the consistency of the existence of large cardinals). These questions were originally asked in [2] and [3], respectively.

Question 5.1 (Bergfalk-Hrušák-Shelah [2]). *Is $\aleph_2 \rightarrow_{hc} (\aleph_2)_{\aleph_0}^2$ consistent?*

Question 5.2 (Bergfalk [3]). *Is $\aleph_{\omega+1} \rightarrow_{hc} (\aleph_\omega)_{\aleph_0}^2$ consistent? If so, what about $\aleph_{\omega+1} \rightarrow_{hc} (\aleph_{\omega+1})_{\aleph_0}^2$?*

Both of these questions have positive answers. In a forthcoming work [6], Hrušák, Shelah, and Zhang prove that, if κ is measurable then, in the forcing extension by the Lévy collapse $\text{Coll}(\omega_1, < \kappa)$, $\aleph_2 \rightarrow_{hc} (\aleph_2)_{\aleph_0}^2$ holds. They also prove that if,

moreover, κ is $\kappa^{+\omega+1}$ -supercompact, then $\aleph_{\omega+1} \rightarrow_{hc} (\aleph_{\omega+1})_{\aleph_0}^2$ also holds in the extension by $\text{Coll}(\omega_1, <\kappa)$.

A more open-ended, speculative question involves generalizations of the partition relations being studied to higher dimensions. In the case of the classical partition relation $\nu \rightarrow (\mu)_\lambda^2$, it is clear how to generalize to $\nu \rightarrow (\mu)_\lambda^k$ for $k > 2$. In the case of $\nu \rightarrow_{hc} (\mu)_\lambda^2$ or $\nu \rightarrow_{wc} (\mu)_\lambda^2$, however, such a generalization would require isolating the correct definition(s) of “highly connected” and “well-connected” in the context of k -uniform hypergraphs. There are a number of different approaches one might take to this generalization, but it is presently unclear, at least to us, which, if any, of these approaches yields an interesting theory of higher-dimensional partition relations. We therefore ask the following deliberately vague problem.

Problem 5.3. *Isolate the correct definition(s) for a generalization (or generalizations) of highly connected or well-connected Ramsey theory to higher dimensions.*

In connection with Problem 5.3, we want to highlight recent work of Bannister, Bergfalk, Moore, and Todorćević [1] in which they introduce a partition hypothesis $\text{PH}_n(\mathbb{P}, \lambda)$ for a given $0 < n < \omega$, a directed quasi-order \mathbb{P} , and a cardinal λ . $\text{PH}_n(\mathbb{P}, \lambda)$ is a positive $(n+1)$ -dimensional partition relation yielding information about colorings of the form $c : [\mathbb{P}]^{n+1} \rightarrow \lambda$. For cardinals ν and λ , the principle $\text{PH}_1(\nu, \lambda)$ is related to the relations $\nu \rightarrow_{hc} (\nu)_\lambda^2$ and $\nu \rightarrow_{wc} (\nu)_\lambda^2$, and directly implies the latter, so this work can be seen in part as one possible approach to Problem 5.3. We feel that there remains much fruitful work to be done in further study of these and related partition principles.

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