A NOTE ON A RESULT OF ZHANG ABOUT MONOCHROMATIC SUMSETS OF REALS

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Abstract. We give an application of a higher-dimensional Δ-system lemma by using it in a slight modification of the proof of a recent result of Zhang about additive partition relations on the reals. This is meant to illustrate the use of the Δ-system lemma in question, and gives a slight improvement to the local version of Zhang's result.

The purpose of this note, which is not intended for publication, is to provide an exposition of a proof of a recent result of Zhang [4] in which a certain higher-dimensional Δ-system lemma used in [4] is replaced by a different higher-dimensional Δ-system lemma proven in [3]. Both lemmas involve starting with a sequence $\langle u_a \mid a \in [\mu]^n \rangle$ of sets of ordinals indexed by n-tuples from some cardinal μ, and then finding a set $H \subseteq \mu$ of some specified size such that $\langle u_a \mid a \in [H]^n \rangle$ satisfies certain uniformities. The advantage of our lemma in [3] is that, at least in the context of accessible cardinals, weaker assumptions are placed on the size of μ necessary to guarantee the existence of such a set H.

Zhang's result deals with partition relations for the additive structure $(\mathbb{R}, +)$. Given an additive structure $(A, +)$ and cardinals κ, r, the partition relation $A \rightarrow^+ (\kappa)_r$ is the assertion that, for every coloring $c : A \rightarrow r$, there is $X \in [A]^\kappa$. such that $c \restriction (X + X)$ is constant, where $X + X = \{x + y \mid x, y \in X\}$ (i.e., repetitions are allowed). Hindman, Leader, and Strauss prove in [1] that, if $2^{\aleph_0} < \aleph_\omega$, then there is $r < \omega$ such that $\mathbb{R} \not\rightarrow^+ (\aleph_0)_r$. It was then shown by Komjáth et al. [2] that, modulo a large cardinal assumption, it is consistent that $\mathbb{R} \rightarrow^+ (\aleph_0)_r$ for all $r < \omega$. This was improved by Zhang [4], who removed the large cardinal assumption and proved the following theorem.

Theorem 1 (Zhang, [4])**.** *Suppose that $\mathbb{P} = \mathrm{Add}(\omega, \beth_\omega)$ is the forcing notion to add $\beth_\omega$-many Cohen reals. Then in $V^{\mathbb{P}}$, we have $\mathbb{R} \rightarrow^+ (\aleph_0)_r$ for all $r < \omega$.*

This shows that the result of Hindman, Leader, and Strauss is at least consistently sharp in the sense that, applying Zhang's result to a model of GCH, we obtain a forcing extension in which $2^{\aleph_0} = \aleph_{\omega+1}$ and $\mathbb{R} \rightarrow^+ (\aleph_0)_r$ holds for all $r < \omega$.

An examination of Zhang's proof and the assumptions on the size of μ needed to prove the relevant Δ-system lemma shows that, for a fixed $r < \omega$, if $\mathbb{P}$ is the forcing to add at least $\beth_{4r}^+$-many Cohen, reals, then $\mathbb{R} \rightarrow^+ (\aleph_0)_r$ holds in $V^{\mathbb{P}}$. Our proof lowers this $\beth_{4r}^+$ to $\beth_{2r}^+$, thus providing a local improvement to Zhang's result. We first make a note of some of our notational conventions.

Notation 2. If X is a set and κ is a cardinal, then $[X]^\kappa = \{Y \subseteq X \mid |Y| = \kappa\}$. If a is a set of ordinals, then $\mathrm{otp}(a)$ denotes the order type of a under the natural ordering of the ordinals. We will frequently conflate sets of ordinals with increasing sequences of ordinals. So, for instance, if a is a set of ordinals and $i < \mathrm{otp}(a)$, then $a(i)$ is the unique $\eta \in a$ such that $\mathrm{otp}(a \cap \eta) = i$. If $\mathbf{m} \subseteq \mathrm{otp}(a)$, then $a[\mathbf{m}] = \{a(i) \mid i \in \mathbf{m}\}$. If a and b are sets of ordinals, then we write $a < b$ to mean that $\alpha < \beta$ for all $(\alpha, \beta) \in a \times b$.

A cardinal λ is said to be *$<\kappa$-inaccessible* if $\nu^{<\kappa} < \lambda$ for all $\nu < \lambda$.

The proof presented here is essentially the same as that in [4]; we provide details just to verify that our Δ-system lemma is sufficient to carry out the proof. We first give two definitions from [2] and [4].

Definition 3. Suppose that μ is a cardinal, $m < \omega$, $a \in [\mu]^m$, and $s : m \to \mathbb{N}$. Then $s * a$ is the function from μ to $\mathbb{N}$ defined by letting $s(a(i)) = s(i)$ for all $i < m$ and $s(\alpha) = 0$ for all $\alpha \in \mu \setminus a$. Notice that $s * a$ is then a member of $\bigoplus_{\alpha<\mu} \mathbb{N}$.

Definition 4. Suppose that $\ell \leq r < \omega$ and $2 \leq r$. Define a function $s^r_\ell : r + \ell \to \mathbb{N}$ by setting, for all $j < r + \ell$,

$$s^r_\ell(j) = \begin{cases} 2 & \text{if } j < 2\ell \\ 4 & \text{otherwise.} \end{cases}$$

We also need to recall some notation and results about higher-dimensional Δ-systems from [3].

Definition 5. Suppose that $\ell \leq r < \omega$, with $2 \leq r$. Then define a set $\mathbf{m}^r_\ell \subseteq 2r$ by letting $\mathbf{m}^r_\ell = \{2k+1 \mid k < r\} \cup \{2k \mid k < \ell\}$. Notice that $|\mathbf{m}^r_\ell| = r + \ell$. Given a set $a \in [\mathrm{On}]^{2r}$, let $a^r_\ell = a[\mathbf{m}^r_\ell]$.

Definition 6. Suppose that a and b are sets of ordinals.

(1) We say that a and b are *aligned* if $\mathrm{otp}(a) = \mathrm{otp}(b)$ and, for all $\gamma \in a \cap b$, we have $\mathrm{otp}(a \cap \gamma) = \mathrm{otp}(b \cap \gamma)$.
(2) If a and b are aligned then we let $\mathbf{r}(a, b) := \{i < \mathrm{otp}(a) \mid a(i) = b(i)\}$. Notice that, in this case, $a \cap b = a[\mathbf{r}(a, b)] = b[\mathbf{r}(a, b)]$.

Definition 7. Suppose that H is a set of ordinals, $0 < n < \omega$, and, for all $b \in [H]^n$, u_b is a set of ordinals. We call $\langle u_b \mid b \in [H]^n \rangle$ a *uniform n-dimensional Δ-system* if there is an ordinal ρ and, for each $\mathbf{m} \subseteq n$, a set $\mathbf{r}_\mathbf{m} \subseteq \rho$ satisfying the following statements.

(1) $\mathrm{otp}(u_b) = \rho$ for all $b \in [H]^n$.
(2) For all $a, b \in [H]^n$, if a and b are aligned, then u_a and u_b are aligned and, if $\mathbf{r}(a, b) = \mathbf{m}$, then $\mathbf{r}(u_a, u_b) = \mathbf{r}_\mathbf{m}$.
(3) For all $\mathbf{m}_0, \mathbf{m}_1 \subseteq n$, we have $\mathbf{r}_{\mathbf{m}_0 \cap \mathbf{m}_1} = \mathbf{r}_{\mathbf{m}_0} \cap \mathbf{r}_{\mathbf{m}_1}$.

Definition 8. Suppose that $i < \rho$ are ordinals and $a, b \in [\mathrm{On}]^\rho$. We say that a and b are *aligned above i* if $a[\rho \setminus i]$ and $b[\rho \setminus i]$ are aligned.

Definition 9. Suppose that a and b are sets of ordinals. Then the *intersection type of a and b*, denoted $\mathrm{tp}_{int}(a, b)$, is the set $\{(i, j) \in \mathrm{otp}(a) \times \mathrm{otp}(b) \mid a(i) = b(j)\}$.

Definition 10. Suppose that I is a set and, for all $i \in I$, u_i is a set of ordinals. Then $\mathrm{tp}(\langle u_i \mid i \in I \rangle)$ is a function from $\mathrm{otp}(\bigcup_{i \in I} u_i)$ to $\mathcal{P}(I)$ defined as follows. First, let $\bigcup_{i \in I} u_i$ be enumerated in increasing order as $\langle \alpha_\eta \mid \eta < \mathrm{otp}(\bigcup_{i \in I} u_i) \rangle$. Then, for all $\eta < \mathrm{otp}(\bigcup_{i \in I} u_i)$, let $\mathrm{tp}(\langle u_i \mid i \in I \rangle)(\eta) := \{i \in I \mid \alpha_\eta \in u_i\}$.

Intuitively, $\mathrm{tp}(\langle u_i \mid i \in I \rangle)$ completely describes the order relations that hold between entries in $\langle u_i \mid i \in I \rangle$. We will often slightly abuse notation and write, for instance, $\mathrm{tp}(u_0, u_1, u_2)$ instead of $\mathrm{tp}(\langle u_0, u_1, u_2 \rangle)$.

Definition 11. Suppose that a is a nonempty set of ordinals and $i < \mathrm{otp}(a)$.

(1) We say that an ordinal α is *i-possible* for a if the following two statements hold:
  (a) if $i > 0$, then $\alpha > a(i-1)$;
  (b) if $i + 1 < \mathrm{otp}(a)$, then $\alpha < a(i+1)$.
  Intuitively, α is i-possible for a if $a(i)$ can be replaced by α without changing the relative positions of the other elements of a.
(2) If α is i-possible for a, then $a_{i\mapsto\alpha}$ is the set $(a \setminus \{a(i)\}) \cup \{\alpha\}$, i.e., the set obtained by replacing the i^{th} element of a with α.

Definition 12. Given a regular cardinal λ, recursively define $\sigma(\lambda, n)$ for $1 \leq n < \omega$ by letting $\sigma(\lambda, 1) = \lambda$ and, given $1 \leq n < \omega$, letting $\sigma(\lambda, n+1) = \left(2^{<\sigma(\lambda,n)}\right)^+$. Note that $\sigma(\lambda, n)$ is regular for each $1 \leq n < \omega$.

The following result is the higher-dimensional Δ-system lemma from [3].

Theorem 13. *Suppose that*

- $1 \leq n < \omega$;
- $\kappa < \lambda$ *are infinite cardinals,* λ *is regular and* $<\kappa$*-inaccessible, and* $\mu = \sigma(\lambda, n)$;
- $g : [\mu]^n \to 2^{<\kappa}$;
- *for all* $b \in [\mu]^n$*, we are given a set* $u_b \in [\mathrm{On}]^{<\kappa}$.

Then there are $H \subseteq \mu$ *and* $k < 2^{<\kappa}$ *such that*

(1) $|H| = \lambda$;
(2) $g(b) = k$ *for all* $b \in [H]^n$;
(3) $\langle u_b \mid b \in [H]^n\rangle$ *is a uniform n-dimensional Δ-system.*

Moreover, if $n \geq 2$*, for all* $a, b \in [H]^n$ *and all* $k < n$*, if it is the case that a and b are aligned above k and* $a(k) = b(k)$*, then, for any ordinal* $\alpha \in H$ *that is k-possible for both a and b, we have* $\mathrm{tp}_{int}(u_a, u_b) = \mathrm{tp}_{int}(u_{a_{k\mapsto\alpha}}, u_{b_{k\mapsto\alpha}})$.

We are now ready to prove our adaptation of Zhang's result. As mentioned above, it is essentially the same as the proof from [4]. It was proven in [4] that $\mathbb{R} \to^+ (\aleph_0)_2$ holds in ZFC, so we only consider the case $r > 2$.

Theorem 14. *Suppose that* $2 < r < \omega$ *and* $\mathbb{P}$ *is the forcing to add at least* $\beth_{2r}^+$*-many Cohen reals. Then, in* $V^{\mathbb{P}}$*, we have* $\mathbb{R} \to^+ (\aleph_0)_r$.

Proof. Let $\mu = (\beth_{2r}^+)^V$, and let $\theta \geq \mu$ be a cardinal such that $\mathbb{P} = \mathrm{Add}(\omega, \theta)$. We think of conditions of $\mathbb{P}$ as being finite partial functions from θ to 2, ordered by reverse inclusion.

We identify $(\mathbb{R}, +)$ with $\left(\bigoplus_{\alpha<2^\omega} \mathbb{Q}, +\right)$. We will actually show that, in $V^{\mathbb{P}}$, we have $\bigoplus_{\alpha<\mu} \mathbb{N} \to^+ (\aleph_0)_r$. Since we have $2^\omega \geq \theta \geq \mu$ in $V^{\mathbb{P}}$, this suffices.

Fix a $\mathbb{P}$-name $\dot{c}$ for a function from $\bigoplus_{\alpha<\mu} \mathbb{N}$ to ω. We claim that the empty condition forces the existence of an infinite X such that $c \restriction (X + X)$ is constant.

For each $\ell \leq r$, let $\dot{d}_\ell$ be a $\mathbb{P}$-name for the function from $[\mu]^{r+\ell}$ to r defined by letting $\dot{d}_\ell(a) = \dot{c}(s^r_\ell * a)$ for all $a \in [\mu]^{r+\ell}$.

For each $a \in [\mu]^{2r}$, let $\mathcal{A}_a$ be a maximal antichain in $\mathbb{P}$ such that, for each $q \in \mathcal{A}_a$ and each $\ell \leq r$, q decides the value of $\dot{d}_\ell(a^r_\ell)$. Since $\mathbb{P}$ has the countable chain condition, each $\mathcal{A}_a$ is countable, so we can enumerate it (possibly with repetitions) as $\langle q_{a,m} \mid m < \omega\rangle$. Let $u_{a,m} = \mathrm{dom}(q_{a,m})$, and let $\bar{q}_{a,m} : \mathrm{otp}(u_{a,m}) \to 2$ be defined by letting $\bar{q}_{a,m}(i) = q_{a,m}(u_{a,m}(i))$ for all $i < \mathrm{otp}(u_{a,m})$. For each $\ell \leq r$, let $w_{a,m,\ell} < r$ be such that $q_{a,m} \Vdash$ "$\dot{d}_\ell(a^r_\ell) = w_{a,m,\ell}$". Let $u_a = \bigcup_{m<\omega} u_{a,m}$.

Now the map g that takes $a \in [\mu]^{2r}$ to the triple

$$\langle \langle \bar{q}_{a,m} \mid m < \omega \rangle, \langle w_{a,m,\ell} \mid m < \omega,\ \ell \leq r \rangle, \mathrm{tp}(\langle u_a \rangle ^\frown \langle u_{a,m} \mid j < \omega \rangle) \rangle$$

can easily be coded as a map from $[\mu]^{2r}$ to $2^{<\omega_1}$. Moreover, u_a is countable for all $a \in [\mu]^{2r}$, and $\mu = \beth^+_{2r} = \sigma(\beth^+_1, 2r)$. Since $\beth^+_1$ is $<\omega_1$-inaccessible, we can apply Theorem 13 to $\langle u_a \mid a \in [\mu]^{2r} \rangle$ and g to obtain $H \subseteq \mu$ of size $\beth^+_1$ and a fixed triple $\tau = \langle \langle \bar{q}_m \mid m < \omega \rangle, \langle w_{m,\ell} \mid m < \omega,\ \ell \leq r \rangle, t \rangle$ such that $g(a) = \tau$ for all $a \in [H]^{2r}$ and $\langle u_a \mid a \in [H]^{2r} \rangle$ is a uniform $2r$-dimensional Δ-system and satisfies the "moreover" clause in the statement of Theorem 13. Let $\rho < \omega_1$ be such that $\mathrm{otp}(u_a) = \rho$ for all $a \in [H]^{2r}$, and let $\langle \mathbf{r_m} \subseteq \rho \mid \mathbf{m} \subseteq 2r \rangle$ witness that $\langle u_a \mid a \in [H]^{2r} \rangle$ is a uniform $2r$-dimensional Δ-system.

Fix sets $\langle A_k \mid k < r \rangle$ such that each A_k is a subset of H of order type $\omega + 1$ and $A_k < A_{k'}$ for all $k < k' < r$. Let $\alpha^k_0 = \min(A_k)$ and $\alpha^k_\omega = \max(A_k)$ for all $k < r$. We identify elements of $\prod_{k<r}[A_k]^2$ as elements of $[\mu]^{2r}$ in the obvious way.

Let G be $\mathbb{P}$-generic over V, and let c and $\langle d_\ell \mid \ell \leq r \rangle$ be the realizations of $\dot{c}$ and $\langle \dot{d}_\ell \mid \ell \leq r \rangle$, respectively, in $V[G]$. For every $a \in [H]^{2r}$, there is a unique $m_a < \omega$ such that $q_{a,m_a} \in G$. Working now in $V[G]$, we will recursively construct a matrix of ordinals $\langle \alpha^k_j \mid k < r,\ j < \omega \rangle$ such that, for each $k < r$, $\langle \alpha^k_j \mid j < \omega \rangle$ is an increasing sequence of ordinals in $A_k \setminus \{\alpha^k_\omega\}$ (note that we have already specified $\alpha^k_0 = \min(A_k)$). At the end, we will let $A^*_k = \{\alpha^k_j \mid j \leq \omega\}$. Our construction will be by recursion on the anti-lexicographic order on $r \times \omega$, i.e., we set $(k, j) < (k', j')$ iff $j < j'$ or ($j = j'$ and $k < k'$). To specify the requirements our construction will satisfy, we need some further definitions.

At the end of the construction, an element $a \in \prod_{k<r}[A^*_k]^2$ will be called *canonical* if $a = \{\alpha^0_{j_0}, \alpha^0_{j'_0}, \alpha^1_{j_1}, \alpha^1_{j'_1}, \ldots, \alpha^{r-1}_{j_{r-1}}, \alpha^{r-1}_{j'_{r-1}}\}$, where

- for each $k < r$, we have $j_k < j'_k$;
- for each $k_0 < k_1 < r$, we have $j_{k_0} < j_{k_1}$;
- for each $k < r$, we have $j_{r-1} < j'_k$;
- for each $k_0 < k_1 < r$, if $j'_{k_0} < \omega$, then $j'_{k_0} \leq j'_{k_1}$.

If $a = \{\alpha^0_{j_0}, \alpha^0_{j'_0}, \ldots, \alpha^{r-1}_{j_{r-1}}, \alpha^{r-1}_{j'_{r-1}}\} \in \prod_{k<r}[A^*_k]^2$ is canonical, then the *index* of a is the set $\{j_k \mid k < r\}$. Note that this is an element of $[\omega]^r$. In our construction, we will arrange so that, for every canonical $a \in \prod_{k<r}[A^*_k]^2$ and every $\ell \leq r$, the value of $d_\ell(a^r_\ell)$ depends only on the index of a. This will be arranged in the following way: for each canonical $a = \{\alpha^0_{j_0}, \alpha^0_{j'_0}, \ldots, \alpha^{r-1}_{j_{r-1}}, \alpha^{r-1}_{j'_{r-1}}\} \in \prod_{k<r}[A^*_k]^2$, let $\hat{a} = \{\alpha^0_{j_0}, \alpha^0_\omega, \ldots, \alpha^{r-1}_{j_{r-1}}, \alpha^{r-1}_\omega\}$. In other words, $\hat{a}$ is the canonical element of $\prod_{k<r}[A^*_k]^2$ with the same index as a and whose other elements are precisely the elements of $\{\alpha^k_\omega \mid k < r\}$. We will ensure that, for every canonical element a, we have $m_a = m_{\hat{a}}$. It will follow that $d_\ell(a^r_\ell) = d_\ell(\hat{a}^r_\ell) = w_{m_{\hat{a}},\ell}$.

We now describe the construction of $\langle \alpha_{k,j} \mid k < r,\ j < \omega \rangle$. We have already specified $\alpha_{k,0}$ for all $k < r$. Now fix $(k^*, j^*) \in r \times \omega$ with $j^* \geq 1$, and suppose that we have defined $\alpha_{k,j}$ for all $(k, j) < (k^*, j^*)$. For each $k < r$, let $B_k = \{\alpha_{k,j} \mid (k, j) < (k^*, j^*)\} \cup \{\alpha_{k,\omega}\}$, i.e., B_k is the portion of A^*_k that has already been specified. The notion of a canonical element of $\prod_{k<r}[B_k]^2$ is straightforwardly inherited from the

notion of a canonical element of $\prod_{k<r}[A^*_k]^2$. Our recursion hypothesis is simply that, for every canonical element a, we have $m_a = m_{\hat{a}}$.

We call a canonical element $a = \{\alpha^0_{j_0}, \alpha^0_{j'_0}, \ldots, \alpha^{r-1}_{j_{r-1}}, \alpha^{r-1}_{j'_{r-1}}\}$ of $\prod_{k<r}[B_k]^2$ *relevant* if $j'_k = \omega$ for all k with $k^* \leq k < r$. Let

$$q^* = \bigcup\{q_{a,m_a} \mid a \text{ is a relevant canonical element}\}.$$

Since there are only finitely many relevant canonical elements, we have $q^* \in G$. Also, for each relevant canonical element a and each $\alpha \in A_{k^*} \setminus (\{\alpha^{k^*}_\omega\} \cup \alpha^{k^*}_{j^*-1})$, let $a_\alpha = a_{(2k^*+1)\mapsto\alpha} = (a \setminus \{\alpha^{k^*}_\omega\}) \cup \{\alpha\}$.

Claim 15. *There is $\alpha \in A_{k^*} \setminus (\{\alpha^{k^*}_\omega\} \cup \alpha^{k^*}_{j^*-1})$ such that, for every relevant canonical element a, we have $m_{a_\alpha} = m_a$, i.e., $q_{a_\alpha,m_a} \in G$.*

Proof. Assume not. Note that, since there are only finitely many canonical relevant elements, each of which is a finite set of ordinals and hence in V, the statement of the claim is expressible in V as a statement in the forcing language for $\mathbb{P}$. Therefore, since the claim fails, we can fix a single condition $s \in G$ that forces its failure. Assume without loss of generality that $s \leq q^*$.

Let $\mathbf{m} = 2r \setminus \{2k^*+1\}$, and let $C = A_{k^*} \setminus (\{\alpha^{k^*}_\omega\} \cup \alpha^{k^*}_{j^*-1})$. For each relevant canonical element a, the set $\{u_{a_\alpha} \mid \alpha \in C\}$ is a Δ-system whose root is equal to $u_{a_\alpha}[\mathbf{r_m}]$ for each $\alpha \in C$. Since there are only finitely many relevant canonical elements a and since $\mathrm{dom}(s)$ is finite, we can therefore fix $\alpha \in C$ such that, for every relevant canonical element a, we have $(u_{a_\alpha} \setminus u_{a_\alpha}[\mathbf{r_m}]) \cap \mathrm{dom}(r) = \emptyset$. Let

$$q^{**} = s \cup \bigcup\{q_{a_\alpha,m_a} \mid a \text{ is a relevant canonical element}\}.$$

We claim that q^{**} is a condition in $\mathbb{P}$, i.e., it is actually a function. To see this, it suffices to verify the following two statements:

- For every relevant canonical element a, we have $s \parallel q_{a_\alpha,m_a}$.
- For every pair of relevant canonical elements a and b, we have $q_{a_\alpha,m_a} \parallel q_{b_\alpha,m_b}$.

To verify the first statement, fix a relevant canonical element a. By our choice of α, we have $\mathrm{dom}(q_{a_\alpha,m_a}) \cap \mathrm{dom}(s) \subseteq u_{a_\alpha}[\mathbf{r_m}]$. But a_α and a are aligned, with $\mathbf{r}(a_\alpha, a) = \mathbf{m}$, so $u_{a_\alpha}[\mathbf{r_m}] = u_a[\mathbf{r_m}]$. By the fact that g is constant on $[H]^{2r}$, we have $q_{a_\alpha,m_a} \restriction u_{a_\alpha}[\mathbf{r_m}] = q_{a,m_a} \restriction u_a[\mathbf{r_m}]$. But $s \leq q_{a,m_a}$, so $s \leq q_{a_\alpha,m_a} \restriction u_{a_\alpha}[\mathbf{r_m}]$, so $s \parallel q_{a_\alpha,m_a}$.

To verify the second statement, fix a pair of relevant canonical elements, a and b. It easily follows from the definitions of "relevant" and "canonical" that a and b are aligned above $2k^*+1$. Moreover, we have $a(2k^*+1) = b(2k^*+1) = \alpha^{k^*}_\omega$. Therefore, by the "moreover" clause of Theorem 13, we have $\mathrm{tp}_{int}(u_a, u_b) = \mathrm{tp}_{int}(u_{a_\alpha}, u_{b_\alpha})$. Now suppose for sake of contradiction that $q_{a_\alpha,m_a} \perp q_{b_\alpha,m_b}$. Then there is $\gamma \in \mathrm{dom}(q_{a_\alpha,m_a}) \cap \mathrm{dom}(q_{b_\alpha,m_b})$ such that $q_{a_\alpha,m_a}(\gamma) \neq q_{b_\alpha,m_b}(\gamma)$. Fix $i_a, i_b < \rho$ such that $\gamma = u_{a_\alpha}(i_a) = u_{b_\alpha}(i_b)$. Then $(i_a, i_b) \in \mathrm{tp}_{int}(u_{a_\alpha}, u_{b_\alpha})$, so $(i_a, i_b) \in \mathrm{tp}_{int}(u_a, u_b)$, so there is δ such that $\delta = u_a(i_a) = u_b(i_b)$. By the fact that g is constant on $[H]^{2r}$, we have

$$q_{a,m_a}(\delta) = q_{a_\alpha,m_a}(\gamma) \neq q_{b_\alpha,m_b}(\gamma) = q_{b,m_b}(\delta),$$

and hence $q_{a,m_a} \perp q_{b,m_b}$. But, by assumption, we have $q_{a,m_a}, q_{b,m_b} \in G$, which is a contradiction.

This finishes the verification that q^{**} is a condition. But now note that q^{**} extends s and forces that α witnesses the truth of the claim, contradicting our choice of s. Therefore, the claim holds. $\square$

We can now let $\alpha^{k^*}_{j^*}$ be any α witnessing the truth of Claim 15. Let us verify that this maintains the recursion hypothesis. For $k < r$, let $B'_k = B_k$ if $k \neq k^*$, and let $B'_{k^*} = B_{k^*} \cup \{\alpha^{k^*}_{j^*}\}$. Fix a canonical element a of $\prod_{k<r}[B'_k]^2$. We must show that $m_a = m_{\hat{a}}$. By the recursion hypothesis, we may assume that $\alpha^{k^*}_{j^*} \in a$. Note that, for all $k < r$ with $k > k^*$, we have not yet defined $\alpha^k_{j^*}$. Therefore, by the definition of "canonical element", we must be in one of two cases:

Case 1: $\alpha^{k^*}_{j^*} = a[2k^*]$ and $k^* = r-1$. Again by the definition of "canonical element", it must be the case here that $a[2k+1] = \alpha^k_\omega$ for all $k < r$. Hence, $a = \hat{a}$, so the recursion hypothesis is trivially satisfied.

Case 2: $\alpha^{k^*}_{j^*} = a[2k^*+1]$. Here, it must be the case that $a[2k+1] = a^k_\omega$ for all $k < r$ with $k > k^*$. Let $b = a_{(2k^*+1)\mapsto\alpha^{k^*}_\omega}$, and, for notational simplicity, let $\alpha = \alpha^{k^*}_{j^*}$. Then b is a relevant canonical element of $\prod_{k<r}[B_k]^2$. Notice that $a = b_\alpha$, so by our choice of α, we have $m_a = m_b$. By our recursion hypothesis, we have $m_b = m_{\hat{b}}$. But $\hat{b} = \hat{a}$, so $m_a = m_{\hat{a}}$.

We have thus maintained our recursion hypothesis and can move on to the next step of the construction. This therefore completes our construction of $\langle A^*_k \mid k < r\rangle$.

The rest of the proof is exactly as in [4], but we provide a sketch for completeness. By our construction of $\langle A^*_k \mid k < r\rangle$, for each $\ell \leq r$ we have a well defined function $f_\ell : [\omega]^r \to r$ such that, for each $y \in [\omega]^r$ and each canonical $a \in \prod_{k<r} A^*_k$, if the index of a is y, then $d_\ell(a^r_\ell) = f_\ell(y)$. By Ramsey's theorem, there is an infinite $Y \subseteq \omega$ such that each f_ℓ is constant on $[Y]^r$, say with value $\varepsilon_\ell < r$. By throwing away the elements of $\omega \setminus Y$ and reindexing, we may assume for notational simplicity that $Y = \omega$, i.e., for every canonical $a \in \prod_{k<r} A^*_k$ and every $\ell \leq r$, we have $d_\ell(a^r_\ell) = \varepsilon_\ell$.

By the pigeonhole principle, there are $\ell_0 < \ell_1 \leq r$ such that $\varepsilon_{\ell_0} = \varepsilon_{\ell_1} =: \varepsilon$. For all $j < \omega$, define $a_j \in \prod_{k<\ell_0}[A^*_k]^2 \times \prod_{\ell_0 \leq k < r} A^*_k$ by specifying that a_j contains the following:

- $\left\{\alpha^k_k, \alpha^k_\omega\right\}$ for each $k < \ell_0$;
- $\left\{\alpha^k_{k+(j+1)r}\right\}$ for $\ell_0 \leq k < \ell_1$;
- $\left\{\alpha^k_\omega\right\}$ for each $\ell_1 \leq k < r$.

Note that $a_j \in [H]^{r+\ell_0}$. Let $x_j = \frac{1}{2}s^r_{\ell_0} * a_j \in \bigoplus_{\alpha<\mu}\mathbb{N}$, and let $X = \{x_j \mid j < \omega\}$. We claim that $c \restriction (X+X)$ is constant with value ε. There are two things to verify.

First, we must show that $c(x_j + x_j) = \varepsilon$ for all $j < \omega$. Thus, fix $j < \omega$. Let $a = a_j \cup \{a^k_k \mid \ell_0 \leq k < r\}$. Then a is a canonical element of $\prod_{k<r}[A^*_k]^2$ and $a^r_{\ell_0} = a_j$. Therefore, we have

$$c(x_j + x_j) = c(s^r_{\ell_0} * a_j) = d_{\ell_0}(a^r_{\ell_0}) = \varepsilon_{\ell_0} = \varepsilon,$$

as desired.

Next, we must show that $c(x_j + x_{j'}) = \varepsilon$ for all $j < j' < \omega$. Thus, fix $j < j' < \omega$. Let $a = a_j \cup a_{j'} \cup \{a^k_{k+(j+1)r} \mid \ell_1 \leq k < r\}$. The following facts are easily verified.

- a is a canonical element of $\prod_{k<r}[A^*_k]^2$.
- $a^r_{\ell_1} = a_j \cup a_{j'}$.
- $x_j + x_{j'} = s^r_{\ell_1} * (a_j \cup a_{j'})$.

As a result, we have

$$c(x_j + x_{j'}) = c(s^r_{\ell_1} * (a_j \cup a_{j'})) = d_{\ell_1}(a^r_{\ell_1}) = \varepsilon_{\ell_1} = \varepsilon.$$

We have thus shown that $c \restriction (X + X)$ is constant with value ε, thus finishing the proof. □

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